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REDUCIBILITY MOD p OF INTEGRAL CLOSED SUBSCHEMES IN PROJECTIVE SPACES — AN APPLICATION OF ARITHMETIC BÉZOUT

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Abstract. In [4], we showed that we can improve results by Emmy Noether and Alexander Ostrowski ([8]) concerning the reducibility modulo p of absolutely irreducible polynomials with integer coefficients by giving the problem a geometric turn and using an arithmetic Bézout theorem ([2]). This paper is a generalization of [4], where we show that combining the methods of [4] with the theory of Chow forms leads to similar results for integral, closed subschemes of arbitrary codimension in $\mathbf{P}^s_{\mathbb{Z}}$.

Introduction. Let K be a number field with ring of integers R, and Z a flat, integral, closed subscheme of dimension $r+1$ and degree d in $\mathbf{P}^s_R$ $(s, d \geq 2)$, with absolutely irreducible generic fiber. One can show that the fiber $Z_{k(p)}$ is also absolutely irreducible for all but finitely many prime ideals p of R (e.g., [5, Theorem 9.7.7] and [6, Theorem 4.10]).

We would like to bound the (product of the) norms of the prime ideals p of R for which the fiber $Z_{k(p)}$ is not absolutely irreducible in terms of the projective height of Z, as defined in [2]. In this paper, using arithmetic intersection theory, we solve for any fixed $n < d$ the analogous problem obtained by replacing "absolutely irreducible" by "is not a union of two closed subschemes of degrees n and $d-n$, respectively". To prove this theorem, we use Chow forms, and translate the problem to bounding the height of an intersection in some projective space. Thus, the proof becomes a straightforward application of an arithmetic Bézout Theorem for non-proper intersections given in [2], which reduces it to bounding degrees and heights of specific cycles in terms of the data provided.

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Some notation. Given a ring R as above, and a locally free $\mathcal{O}_{\mathrm{Spec}\,(R)}$-module $\mathcal{E}$ of finite rank $s+1$ $(s \geq 0)$, let $\mathbf{P}(\mathcal{E}) = \mathbf{Proj}_{\mathrm{Spec}\,(R)}(\mathrm{Sym}(\mathcal{E}^\vee))$ be the associated *space of lines*, where $\mathcal{E}^\vee$ denotes the dual sheaf of $\mathcal{E}$, and let π denote its structural morphism. We suppose $\mathcal{E}$ endowed with a Hermitian metric h, and endow $\mathcal{E}^\vee$ with the dual metric.

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Let r be a positive integer. For $i = 0, \dots, r$, let $\mathbf{P}_i = \mathbf{P}(\mathcal{E})$, and $\mathbf{P}_i^\vee = \mathbf{P}(\mathcal{E}^\vee)$. We endow the canonical quotient line bundle $\mathcal{O}(1)$ on $\mathbf{P}_i^\vee$ with the quotient metric (cf. [2, 3.1.2.3]), and let $\overline{M_i}$ be the pullback of the resulting Hermitian line bundle $\overline{\mathcal{O}(1)}$ on $\mathbf{P}_i^\vee$ to $\prod_{i=o}^r \mathbf{P}_i^\vee$.

Finally, for $x \in \mathbb{N}$, let $F_{x,r}(\mathcal{E}) := \otimes_{i=0}^r \mathrm{Sym}^x(\mathcal{E})$

Chow divisors and forms. By [2, 4.3.1], we can associate to each non-zero algebraic cycle $Z \in Z_{r+1}(\mathbf{P}(\mathcal{E}))$ a *Chow divisor* $\mathrm{Ch}_{\mathbf{1}}(Z)$ (where $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^{r+1}$) in $Z^1(\prod_{i=0}^r \mathbf{P}_i^\vee)$, which is effective (resp. flat, resp. flat and irreducible) if such is the case for Z.

Let Z now be a non-zero effective cycle of degree x in $Z_{r+1}(\mathbf{P}(\mathcal{E}))$. Generically, the associated Chow divisor $\mathrm{Ch}_{\mathbf{1}}(Z)_K$ is the divisor of a non-zero multihomogeneous form $\phi_{\mathbf{1},Z_K}$ in

$$H^0\left(\prod_{i=0}^r (\mathbf{P}_i^\vee)_K, \otimes M_{i,K}^x\right) \cong F_{x,r}(\mathcal{E})_K,$$

called the *Chow form* of Z_K. Thus we can associate a point of $\mathbf{P}(F_{x,r}(\mathcal{E}))(K)$ to each non-zero effective cycle of degree x in $Z_{r+1}(\mathbf{P}(\mathcal{E}))$. If the class number of K is one, there exists a generalized Chow form $\phi_{\mathbf{1},Z}$ over R, for which $\mathrm{Ch}_{\mathbf{1}}(Z) = \mathrm{div}(\phi_{\mathbf{1},Z})$ in $\prod \mathbf{P}_i^\vee$. Similarly, for every point t of $\mathrm{Spec}\,(R)$, we can define Chow divisors and forms for the cycles contained in the fiber above t ([2, 4.3.2]).

If Z is moreover flat over $\mathrm{Spec}\,(R)$, we have the following result:

Proposition. *Let $Z \in Z_{r+1}(\mathbf{P}(\mathcal{E}))$ be a flat, integral, closed subscheme of $\mathbf{P}(\mathcal{E})$ of degree x, with Chow divisor $\mathrm{Ch}_{\mathbf{1}}(Z)$. Let ϕ_K be the Chow form of Z_K. Let $[\phi_K] \in \mathbf{P}(F_{x,r}(\mathcal{E}))(K)$ be the corresponding point, and P_Z its Zariski closure in $\mathbf{P}(F_{x,r}(\mathcal{E}))$. Then for every point t of $\mathrm{Spec}\,(R)$, the fiber $P_{Z,t}$ is the point of $\mathbf{P}(F_{x,r}(\mathcal{E}))_t$ corresponding to the Chow form ϕ_t of Z_t.*

Proof. It suffices to note that by construction, we have $\mathrm{Ch}_{\mathbf{1}}(Z_t) = \mathrm{Ch}_{\mathbf{1}}(Z)_t$ for every point t of $\mathrm{Spec}\,(R)$ ([2, 4.3.2]). In particular, as Z is flat, the Zariski closure of $\mathrm{div}(\phi_K) = \mathrm{Ch}_{\mathbf{1}}(Z_K)$ is $\mathrm{Ch}_{\mathbf{1}}(Z)$. □

Components of degree n. Let $d \in \mathbb{N}_{>0}$, and fix integers $1 \le n \le d-1$ and $0 \le r \le s$. Let us simplify the notation by setting $F_x := F_{x,r}(\mathcal{E})$ for every x. Consider the morphism

$$\psi : \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}) \to \mathbf{P}(F_d)$$

defined by taking the product on sections (seen as multihomogeneous forms on $\prod \mathbf{P}_i^\vee$), i.e., on sections, ψ corresponds to taking the union of two cycles of degrees n and $d-n$ in $Z_{r+1}(\mathbf{P}(\mathcal{E}))$ in order to obtain one of degree d. Let $\mathcal{W}$ denote the image of ψ.

Let Z be a flat, integral, closed subscheme of degree d in $Z_{r+1}(\mathbf{P}(\mathcal{E}))$, and let P_Z be as in the proposition. By dimension arguments and the proposition, the intersection of P_Z and $\mathcal{W}$ is either P_Z, if $Z_{\overline{K}}$ has a component (irreducible or not) of degree n, or a finite number of closed points whose images under the structural morphism $\pi : \mathbf{P}(F_d) \to \mathrm{Spec}\,(R)$ are the prime ideals $q_1, \ldots, q_v$ above which the fiber of $Z \to \mathrm{Spec}\,(R)$ has such a component.

Before stating the theorem, we note that if $\mathcal{E}$ is isomorphic to R^{s+1}, then each vector bundle F_x is free, and can be endowed, in a natural way, with a basis $\mathcal{B}_x$ ([2, p. 985]). Indeed, in this case, F_x is a space of multihomogeneous forms as described in [2, 4.3.13], whose basis is formed by the monomials. We will use this basis to identify F_x with R^{N_x+1} (where $N_x := \mathrm{rk}(F_x) - 1$).

The following theorem only deals with the trivial vector bundle, i.e. $\mathcal{E} = R^{s+1}$, endowed with the standard Hermitian metric.

Theorem. *Let $Z \in Z_{r+1}(\mathbf{P}^s_R)$ be a flat, integral, closed subscheme of $\mathbf{P}^s_R$ of dimension $r+1$ and degree d $(s, d \geq 2, r \geq 0)$, and $n \in \{1, \ldots, d-1\}$ an integer such that $Z_{\overline{K}}$ cannot be written as the union of two closed subschemes of degrees n and $d-n$, respectively. Let $q_1, \ldots, q_v$ be the distinct prime ideals of R above which the geometric fiber of Z can be written as such a union. Setting $N_{x,r,s} := \mathrm{rk}(\otimes_{i=0}^r \mathrm{Sym}^x(R^{s+1})) - 1$, we have*

$$\log \prod_{j=1}^{v} N(q_j) \;\leq\; \frac{1}{1+\delta_{n,d-n}} \binom{N_{n,r,s} + N_{d-n,r,s}}{N_{n,r,s}} h_K(Z) \;+\; C(s,d,r,n)$$

when $h_K(Z)$ tends to infinity, where h_K is the projective height associated to the standard Hermitian metric on R^{s+1}, as defined in [2, 4.1.1] *(see also* [3, 2.1.5]*), δ is the Kronecker delta function, and $C(s,d,r,n)$ is a function of $s, d, r,$ and n that can be given explicitly (see the proof).*

Remark. For the hypersurface case ($r = s$), we find a stricter bound in [4], due to the fact that horizontal hypersurfaces (which correspond to the flat integral closed subschemes here) are (directly) parametrized by a projective space, making it unnecessary to use Chow forms. The M_x used there correspond to the $N_{x,r,s}$ for $r = 0$ in this paper.

Proof. As noted before, the set $\{q_1, \ldots, q_v\}$ is the support of $\pi(P_Z \cap \mathcal{W})$ in $\mathrm{Spec}\,(R)$. In particular, $\log \prod N(q_i) = h_K(\,|P_Z \cap \mathcal{W}\,|\,)$. By the arithmetic Bézout theorem [2, 5.5.1.iii], we have

$$\begin{aligned} h_K(\,|P_Z \cap \mathcal{W}|\,) &\leq \deg_K(P_Z) h_K(\mathcal{W}) + h_K(P_Z) \deg_K(\mathcal{W}) \\ &\quad + \frac{1}{2}\,[K : \mathbb{Q}]\, \deg_K(P_Z)\, \deg_K(\mathcal{W})(M_d + 1)\log(2). \end{aligned}$$

By definition of P_Z, its degree equals one. Using the further shortened, and somewhat misleading, notation $N_x := N_{x,r,s}$, we are going to show that the other terms on the right can be bounded as follows:

$$h_K(P_Z) \leq h_K(Z) + d\,[K : \mathbb{Q}]\,(\sigma_r + (r+1)\log(s+1)), \tag{1}$$

where $\sigma_x = (1/2)(x+1)\sum_{m=2}^{x+1}(1/m)$,

$$\deg_K(\mathcal{W}) = \frac{1}{1+\delta_{n,d-n}}\binom{N_n+N_{d-n}}{N_n}, \tag{2}$$

$$\begin{aligned} h_K(\mathcal{W}) \leq & \frac{[K:\mathbb{Q}]}{1+\delta_{n,d-n}}\,\frac{N_n+N_{d-n}+1}{2}\binom{N_n+N_{d-n}}{N_n}\cdot \\ & \log\left((d+1)^{3(r+1)(s+1)}\,\frac{(N_n+1)(N_{d-n}+1)}{N_n+N_{d-n}+1}\right), \end{aligned} \tag{3}$$

leading to the result of the theorem. □

Proof of (1). Let $\{a_I\}$ be the coefficients of $P_{Z,K}$ (i.e. of the form $\phi_{\mathbf{1},Z_K}$) in the basis $\mathcal{B}_d$. Then

$$h_K(P_Z) \;=\; \sum_\sigma \log\left(\sum |a_I|^2\right)^{1/2} - \sum_p \min_I\, v_p(a_I)\,\log N(p).$$

Another height associated to $\mathcal{B}_d$ ([2, 4.3.4.1]) is

$$h_{\mathcal{B}}(P_Z) \;:=\; h_{\mathcal{B}}(\mathrm{Ch}_{\mathbf{1}}(Z)) \;=\; \sum_\sigma \log\left(\sum |a_I|\right) - \sum_p \min_I\, v_p(a_I)\,\log N(p).$$

Clearly, we have $h_K(P_Z) \leq h_{\mathcal{B}}(P_Z)$. By [2, Theorem 4.3.8, (4.3.33), and (4.1.2)],

$$h_{\mathcal{B}}(P_Z) \;\leq\; h_K(Z) \;+\; d\,[K:\mathbb{Q}]\,(\sigma_r \;+ (r+1)\log(s+1)).$$

In particular, we have $h_K(P_Z) = h_K(Z) + \mathcal{O}(1)$. □

Remark. Before giving the proofs of (2) and (3), let us note that the morphism ψ was used under the notation ϕ_n in [4], where the degree and height of its image were bounded explicitely. Here we give only sketches of the proofs of (2) and (3), the details can be found in [loc.cit.].

Proof of (2). Let f_n (resp. f_{d-n}) denote the projection from $\mathbf{P}(F_n)\times\mathbf{P}(F_{d-n})$ onto the first (resp. second) coordinate. Using intersection theory, we find

$$\deg(\psi)\deg_K(\mathcal{W}) \;=\; \deg\left(c_1\mathcal{O}_{\mathbf{P}(F_d)}(1)^{N_n+N_{d-n}}\cdot[\psi_*(\mathbf{P}(F_n)\times\mathbf{P}(F_{d-n}))]\right),$$

where $c_1\mathcal{O}_{\mathbf{P}(F_d)}(1)$ is the first Chern class of $\mathcal{O}_{\mathbf{P}(F_d)}(1)$. By the projection formula, and the fact that $\psi^*\mathcal{O}_{\mathbf{P}(F_d)}(1) = f_n^*\mathcal{O}_{\mathbf{P}(F_n)}(1)\otimes f_{d-n}^*\mathcal{O}_{\mathbf{P}(F_{d-n})}(1)$, this implies that

$$\deg_K(\mathcal{W}) \;=\; \frac{1}{1+\delta_{n,d-n}}\binom{N_n+N_{d-n}}{N_n}.$$ □

Proof of (3). As in the proof of (2), we use intersection theory, but this time with metrics. By [2, 4.1.2 and Proposition 2.3.1], we have

$$\begin{aligned} h_K(\mathcal{W}) = & [K:\mathbb{Q}]\sigma_{N_n+N_{d-n}}\deg_K(\mathcal{W}) \\ & + \frac{1}{\deg(\psi)}\widehat{\deg}\left(\widehat{c}_1\left(\psi^*\overline{\mathcal{O}_{\mathbf{P}(F_d)}(1)}\right)^{N_n+N_{d-n}+1}\;|\;\mathbf{P}(F_n)\times\mathbf{P}(F_{d-n})\right). \end{aligned}$$

The arithmetic degree on the right is the projective height of $\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})$ associated to the line bundle $\mathcal{L} := \psi^* \mathcal{O}_{\mathbf{P}(F_d)}(1)$ endowed with the pullback ρ under ψ of the standard Hermitian metric on $\mathcal{O}_{\mathbf{P}(F_d)}(1)$. We will bound this height in two steps, using a comparison of metrics on $\mathcal{L}$. First, let $(\mathcal{L}, \rho')$ denote the line bundle $\mathcal{L}$ endowed with the product metric obtained by taking the standard Hermitian metrics on $\mathcal{O}_{\mathbf{P}(F_n)}(1)$ and $\mathcal{O}_{\mathbf{P}(F_{d-n})}(1)$. The associated projective height is

$$h_{(\mathcal{L},\rho')}(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})) := \widehat{\deg}\left(\widehat{c}_1(\mathcal{L}, \rho')^{N_n + N_{d-n}+1} \,|\, \mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})\right),$$

which, by the projection formula and the decomposition of the metrized line bundle $(\mathcal{L}, \rho')$ as a (tensor)product, equals

$$[K:\mathbb{Q}]\left(\binom{N_n + N_{d-n}}{N_n}\sigma_{N_{d-n}} + \binom{N_n + N_{d-n}}{N_n}\sigma_{N_n}\right).$$

The second step in bounding the height that we want consists of comparing the norms $\|\cdot\|$ and $\|\cdot\|'$ associated to ρ, resp. ρ'. Let $\varphi : (\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))(\mathbb{C}) \to \mathbb{R}$ be defined by $(\|\cdot\|')^2 = \exp(\varphi)\|\cdot\|^2$. For each embedding $\sigma : K \hookrightarrow \mathbb{C}$, and $(a,b) = ((a_0 : \ldots : a_{N_n}), (b_0 : \ldots : b_{N_{d-n}})$ in $(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n}))_\sigma(\mathbb{C})$, we let f_a, resp. g_b, be the corresponding multihomogeneous polynomial (in $(r+1)(s+1)$ variables). We have

$$\exp(\varphi_\sigma(a,b)) = \left(\frac{L_2(f_a g_b)}{L_2(f_a)L_2(g_b)}\right)^2,$$

where the L_2-norm $L_2(f)$ of a (multi)homogeneous polynomial $f = \sum c_I X^I$ is $(\sum c_I \overline{c_I})^{1/2}$. From results of [7, 3.2], we can now deduce that

$$\sup_{(a,b)}(\varphi_\sigma(a,b)) \;\le\; 3(r+1)(s+1)\log(d+1)$$

for every $\sigma : K \hookrightarrow \mathbb{C}$. The last step consists of combining this inequality with the results of [2, Proposition 3.2.2] and [1, Lemma 2.6.ii] (see also [4]) to obtain

$$\begin{aligned} h_{(\mathcal{L},\rho)}(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})) &\le h_{(\mathcal{L},\rho')}(\mathbf{P}(F_n) \times \mathbf{P}(F_{d-n})) \\ &\quad + [K:\mathbb{Q}]\,\frac{N_n + N_{d-n}+1}{2}\,\deg(\psi)\,\deg_K(\mathcal{W})\,3\,(r+1)(s+1)\log(d+1), \end{aligned}$$

which, after some simplification, leads to the bound for $h_K(\mathcal{W})$ stated in (3). □

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