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AN IMPROVEMENT ON A THEOREM OF BEN MARTIN

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Abstract. Let π be the fundamental group of a Riemann surface of genus $g \geq 2$. The group π has a well–known presentation, as the quotient of a free group on generators $\{a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g\}$ by the one relation

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1.$$

This gives two inclusions $F \hookrightarrow \pi$, where F is the free group on g generators; we could map the generators to the a's, or to the b's. Call the images of these inclusions $F_1 \subset \pi$ and $F_2 \subset \pi$.

Given a connected, reductive group G over an algebraically closed field of characteristic 0, any representation $\pi \longrightarrow G$ restricts to two representations $f_1 : F_1 \longrightarrow G$, $f_2 : F_2 \longrightarrow G$. We prove that on a Zariski open, dense subset of the space of pairs of representations $\{f_1, f_2\}$, there exists a representation $f : \pi \longrightarrow G$ lifting them, up to (separate) conjugacy of f_1 and f_2.

Ben Martin proved this theorem, with the hypothesis that the semisimple rank of G is $> g$. We remove the hypothesis.

0. Introduction

Let G be a connected, reductive algebraic group over a field k. Assume $k = \overline{k}$ is algebraically closed, and of characteristic 0.

Recall first that, given a finitely generated group P, it is customary to write $R(P, G)$ for the variety of representations $P \longrightarrow G$. The group G acts on $R(P, G)$ by conjugation, and the quotient is usually denoted $C(P, G)$. We refer the reader to [3] and [1], for the definitions and basic properties of these varieties.

In the case where P is the fundamental group of a surface, this construction is particularly interesting. It is related to the geometry of the Riemann surface. It has been studied by many authors. We do not attempt to give an exhaustive list here. In these days of web browsers, the reader can easily produce for himself a list that is far more up–to–date than anything I could possibly offer.

Let π be the fundamental group of a Riemann surface of genus $g \geq 2$. The group π has a well–known presentation, as the quotient of a free group on generators $\{a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g\}$ by the one relation

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1.$$

Let $F_1 \subset \pi$ be the (free) subgroup generated by the a's, $F_2 \subset \pi$ be the (free) subgroup generated by the b's. The free product of F_1 and F_2 will be denoted $F_1 \coprod F_2$, and π is the quotient of $F_1 \coprod F_2$ by the one relation above.

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Restriction of representations gives an inclusion

$$R(\pi, G) \quad \subset \quad R(F_1 \coprod F_2, G) \quad = \quad R(F_1, G) \times R(F_2, G)$$

which is compatible with the conjugation action of G. We deduce a map of quotient varieties

$$C(\pi, G) \quad \subset \quad C(F_1 \coprod F_2, G) \longrightarrow C(F_1, G) \times C(F_2, G).$$

Ben Martin's theorem, in [2], says that this map $C(\pi, G) \longrightarrow C(F_1, G) \times C(F_2, G)$ is dominant, as long as the semisimple rank of G is $> g$. Ben Martin also observed that, since the dimension of the domain equals the dimension of the codomain, a dominant map has generically finite fibers. We will prove this theorem without restriction on the semisimple rank of G.

Since the main application of the theorem is to $SL(n)$ bundles on Riemann surfaces, one does not want a restriction that says $n - 1 > g$.

1. The reduction

As in Ben Martin's [2], we reduce the problem to showing that, for some suitable map $G \longrightarrow G$, the derivative at the identity $e \in G$ is an isomorphism. But the reduction here is somewhat different. I hope the reader will find my approach illuminating; see [2] for an alternative argument.

Let $Z(G)$ be the center of G, and $G' = G/Z(G)$ the adjoint group. The group G' acts by conjugation on $R(\pi, G)$, $R(F_1, G)$ and $R(F_2, G)$. And we want to show that after taking quotients, the restriction map becomes dominant. Recall that a point in $R(F_1, G) = G^g$ is a g–tuple $(a_1, a_2, \ldots, a_g)$ of elements of G, and a point in $R(F_2, G) = G^g$ is a g–tuple $(b_1, b_2, \ldots, b_g)$ of elements of G. A point in

$$R(\pi, G) \quad \subset \quad R(F_1, G) \times R(F_2, G)$$

is a pair $(a_1, a_2, \ldots, a_g) \in G^g$, $(b_1, b_2, \ldots, b_g) \in G^g$, satisfying the relation

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1.$$

Let X be the variety $G' \times R(\pi, G)$. There is a map

$$X \quad = \quad G' \times R(\pi, G) \xrightarrow{\ \phi\ } R(F_1, G) \times R(F_2, G),$$

given by the formula

$$\begin{aligned} \phi[h, (a_1, a_2, \ldots, a_g), (b_1, b_2, \ldots, b_g)] \quad = \\ [(ha_1h^{-1}, ha_2h^{-1}, \ldots, ha_gh^{-1}), (b_1, b_2, \ldots, b_g)]. \end{aligned}$$

The group $G' \times G'$ acts on $X = G' \times R(\pi, G)$, by the rule

$$\begin{aligned} (x, y)[h, (a_1, a_2, \ldots, a_g), (b_1, b_2, \ldots, b_g)] \quad = \\ [xhy^{-1}, (ya_1y^{-1}, ya_2y^{-1}, \ldots, ya_gy^{-1}), (yb_1y^{-1}, yb_2y^{-1}, \ldots, yb_gy^{-1})]. \end{aligned}$$

The group $G' \times G'$ also acts on $R(F_1, G) \times R(F_2, G)$, diagonally. That is, given a pair $(x, y) \in G' \times G'$, x acts by conjugation on $R(F_1, G)$, while y acts by conjugation on $R(F_2, G)$. The reader can easily check that, with the above definitions of the map ϕ and the actions of $G' \times G'$, the map

$$G' \times R(\pi, G) \xrightarrow{\ \phi\ } R(F_1, G) \times R(F_2, G)$$

is a morphism of $G' \times G'$–varieties. The quotient of $X = G' \times R(\pi, G)$ by $G' \times G'$ is $C(\pi, G)$, while the quotient of $R(F_1, G) \times R(F_2, G)$ by $G' \times G'$ is $C(F_1, G) \times C(F_2, G)$. We want to show the quotient map

$$C(\pi, G) \xrightarrow{\phi/\{G' \times G'\}} C(F_1, G) \times C(F_2, G)$$

to be dominant; it suffices to prove the dominance of the map ϕ. And for this, it suffices to produce one smooth point of X, at which the differential of the map ϕ is an isomorphism.

The point we will produce is of the form

$$[e, (a_1, a_2, \dots, a_g), (b_1, b_2, \dots, b_g)],$$

where e is the identity of G'. To show both that this is a smooth point of X, and that the differential is an isomorphism at it, it suffices to do the following.

Reduction 1.1. *To prove that the map ϕ is dominant, it suffices to produce a point*

$$[(a_1, a_2, \dots, a_g), (b_1, b_2, \dots, b_g)] \in R(\pi, G)$$

so that the map $\psi : G' \longrightarrow [G, G]$ given by

$$h \mapsto [ha_1h^{-1}, b_1][ha_2h^{-1}, b_2] \cdots [ha_gh^{-1}, b_g]$$

differentiates to give an isomorphism at e.

Proof. First, the map taking $h \in G'$ to

$$[(ha_1h^{-1}, ha_2h^{-1}, \dots, ha_gh^{-1}), (b_1, b_2, \dots, b_g)]$$

gives a morphism of varieties $G' \longrightarrow R(F_1, G) \times R(F_2, G)$. Our hypothesis says that, when we compose with the map

$$R(F_1, G) \times R(F_2, G) \xrightarrow{\ \theta\ } [G, G]$$

taking the point $[(a_1, a_2, \dots, a_g), (b_1, b_2, \dots, b_g)]$ to the product

$$[a_1, b_1][a_2, b_2] \cdots [a_g, b_g],$$

the differential of the composite

$$G' \longrightarrow R(F_1, G) \times R(F_2, G) \xrightarrow{\ \theta\ } [G, G]$$

is an isomorphism at e. But then the image of $e \in G'$ must be a regular point for θ, that is, the point $[(a_1, a_2, \dots, a_g), (b_1, b_2, \dots, b_g)]$ is a smooth point of $R(\pi, G) \subset R(F_1, G) \times R(F_2, G)$.

But we also know that the map $G' \longrightarrow R(F_1, G) \times R(F_2, G)$ gives an isomorphism of the tangent space at e with the normal direction to $R(\pi, G) \subset R(F_1, G) \times R(F_2, G)$. Which proves that the map

$$G' \times R(\pi, G) \xrightarrow{\ \phi\ } R(F_1, G) \times R(F_2, G)$$

induces an isomorphism on the tangent spaces, at the point

$$[e, (a_1, a_2, \dots , a_g), (b_1, b_2, \dots , b_g)].$$

□

2. The proof

Theorem 2.1. *Let $g \geq 2$. Let G be a connected, reductive algebraic group, over an algebraically closed field k of characteristic 0. Then the morphism*

$$C(\pi, G) \longrightarrow C(F_1, G) \times C(F_2, G)$$

is dominant.

Proof. Standard reduction allows us to assume that the ground field is $\mathbb{C}$, the field of complex numbers. By Reduction 1.1, it suffices to produce a point

$$[(a_1, a_2, \dots , a_g), (b_1, b_2, \dots , b_g)] \in R(\pi, G)$$

so that the map $\psi : G' \longrightarrow [G, G]$ given by

$$h \mapsto [ha_1h^{-1}, b_1][ha_2h^{-1}, b_2] \cdots [ha_gh^{-1}, b_g]$$

differentiates to give an isomorphism at e. Our point will be of the form $b_1 = a_1^{-1}$, $b_2 = a_2^{-1}$, and for all $i \geq 3$, $a_i = e = b_i$. To avoid subscripts in the notation, put $a_1 = a$, $a_2 = b$. The map above reduces to

$$h \mapsto (ha^{-1}h^{-1}ahah^{-1}a^{-1})(hb^{-1}h^{-1}bhbh^{-1}b^{-1})$$

and we want to differentiate this map at $h = e$.

Let $\mathfrak{G}$ be the Lie algebra of G'; note that the center of $\mathfrak{G}$ is trivial. Then G' acts on $\mathfrak{G}$ by the adjoint representation; for every $x \in G'$, $\mathrm{Ad}(x) : \mathfrak{G} \longrightarrow \mathfrak{G}$ is a linear map. Putting $h = \exp(tH)$ and differentiating at $t = 0$, we easily compute that the derivative of the map ψ is given by

$$-d\psi = \mathrm{Ad}(a) + \mathrm{Ad}(a^{-1}) + \mathrm{Ad}(b) + \mathrm{Ad}(b^{-1}) - 4.$$

We need to show that, for some choice of $a, b \in G'$, this is an isomorphism.

Put $a = \exp(tA)$ and $b = \exp(tB)$. Using the fact that

$$\mathrm{Ad}(\exp(H)) = \exp(\mathrm{ad}(H)),$$

the expression

$$\mathrm{Ad}(a) + \mathrm{Ad}(a^{-1}) + \mathrm{Ad}(b) + \mathrm{Ad}(b^{-1}) - 4$$

is easily computed to be

$$t^2\left[\{\mathrm{ad}(A)\}^2 + \{\mathrm{ad}(B)\}^2\right] + O(t^4).$$

But now we can use the unitary trick. Choose a maximal compact subgroup $K \subset G'$, and choose a Hermitian inner product on the vector space $\mathfrak{G}$, invariant under K. Let $\mathfrak{K}$ be the Lie algebra of K. The elements of $\mathfrak{K}$ act as skew Hermitian operators on $\mathfrak{G}$. That makes the elements of $i\mathfrak{K}$ Hermitian.

Choose two maximal tori $\mathfrak{S}$ and $\mathfrak{T}$ in $\mathfrak{K}$, with a trivial intersection. Choose $iA \in \mathfrak{S}$, so that $\mathfrak{S}$ is the centraliser of iA. Choose iB in $\mathfrak{T}$, so that $\mathfrak{T}$ is the centraliser of iB. Then the centralisers of iA and iB do not intersect. This means that $\mathrm{ad}(A)$ and $\mathrm{ad}(B)$ are Hermitian operators on the vector space $\mathfrak{G}$, whose kernels do not intersect. But then the operator

$$\{\mathrm{ad}(A)\}^2 + \{\mathrm{ad}(B)\}^2$$

is the sum of two positive semi–definite Hermitian operators, whose kernels do not intersect. Therefore the operator

$$\{\mathrm{ad}(A)\}^2 + \{\mathrm{ad}(B)\}^2$$

is positive definite. It has no kernel.

But now it is trivial that, if t is chosen sufficiently small, the operator

$$t^2\left[\{\mathrm{ad}(A)\}^2 + \{\mathrm{ad}(B)\}^2\right] + O(t^4)$$

will be positive definite, and hence invertible. That is,

$$\mathrm{Ad}(a) + \mathrm{Ad}(a^{-1}) + \mathrm{Ad}(b) + \mathrm{Ad}(b^{-1}) - 4$$

is invertible, for all suitably small t, with $a = \exp(tA)$ and $b = \exp(tB)$. $\square$

References

[1] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985), xi+117.

[2] B. M. S. Martin, *Restrictions of representations of a surface group to a pair of free subgroups*, J. Algebra **225** (2000), 231–249.

[3] R. W. Richardson, *Conjugacy classes of n–tuples in Lie algebras and algebraic groups*, Duke Math. J. **57** (1988), 1–35.

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