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CRYSTALLINE SUBREPRESENTATIONS AND NERON MODELS

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Abstract. We propose the notion of the *crystalline sub-representation functor* defined on p-adic representations of the Galois groups of finite extensions of $\mathbb{Q}_p$, with certain restrictions in the case of integral representations. By studying its right-derived functors, we find a natural extension of a formula of Grothendieck expressing the group of connected components of a Neron model of an abelian variety in terms of Galois cohomology.

Notation:
K: finite extension of $\mathbb{Q}_p$.
R: the ring of integers in K.
K_0: maximal unramified subfield of K.
k: residue field of K=residue field of K_0.
$W = W(k)$: Witt vectors of k= ring of integers in K_0.
$\bar{K}$: an algebraic closure of K.
K^u: the maximal unramified subextension of $\bar{K}/K$.
$G = \mathrm{Gal}(\bar{K}/K)$: the Galois group of $\bar{K}$ over K.
$I = \mathrm{Gal}(\bar{K}/K^u)$: the inertia subgroup of G.
l: a prime different from p.

1. Maximal Crystalline Subrepresentations

It is well known that the representations of G with coefficients in $\mathbb{Z}_p$-modules, the *p-adic representations*, have very different properties from the representations in $\mathbf{Z}_l$-modules for $l \neq p$. (Here and in the following, whenever we speak of representations, we assume that the G-action is continuous.) For example, even for a variety over K with good reduction over R, the representation of G on the p-adic étale cohomology is only rarely unramified.

On the other hand, p-adic Hodge theory has provided us with a fine classification of p-adic representations together with appropriate analogies to the l-adic case [7]. For example, the p-adic notion corresponding to an unramified l-adic representation is that of a *crystalline* representation. These are the representations that correspond via p-adic Hodge theory to weakly-admissible crystals (the correct p-adic analogue of local systems), whereas representations that are genuinely unramified correspond to the much smaller subcategory consisting of crystals of slope zero (see, for example, [1]).

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We wish to continue this analogy by presenting a new cohomology theory associated to p-adic representations of Galois groups of local fields. The definition is very natural and elementary, and is likely to be well-known to experts. However, a specific application motivated us to commit at least a short exposition to paper:

Let A be an abelian variety over K and let $\mathbf{A}$ be its Neron model over R. Let A_0 be the special fiber of $\mathbf{A}$ and A_0^0 the connected component of the identity in A_0. Finally let $\Gamma = A_0(\bar{k})/A_0^0(\bar{k})$ be the geometric points of the group of connected components of A_0. Grothendieck points out the following formula expressing the l-primary part of Γ in terms of Galois cohomology ([3] pp. 134-135):

$$\Gamma(l) = H^1(I, T_l(A))_{\text{tor}}$$

where T_l refers to the l-adic Tate module and the subscript denotes the torsion subgroup. The motivating problem is that of expressing the p part of Γ in an analogous 'cohomological' manner involving only the generic fiber.

The formula is definitely false in general if we simply substitute p for l. An easy argument using Kummer theory shows that when A is semi-stable over an absolutely unramified base, we actually have an injection

$$H^1(I, T_p(A))_{\text{tor}} \hookrightarrow \Gamma(p)$$

which is non-surjective in general. For example, we can consider the case of an elliptic curve with split semi-stable reduction and order of discriminant p. It is an easy exercise to check that in that case, the map is surjective iff the elliptic curve has an unramified point of order p which occurs exactly when its Tate parameter is a p-power in K^u. In short, the torsion in the Galois cohomology of I is not big enough to capture the p-part of the component group. (One notes, however, that there are other interpretations of the component groups using, for example, the monodromy pairing ([3] section 11).)

But notice that the Galois cohomology $H^1(I, \cdot)$ is just the first (right-)derived functor of the functor

$$(\cdot) \mapsto (\cdot)^I$$

which we view as assigning to a representation its *maximal unramified subrepresentation*. This is an example of a 'subrepresentation functor' or a 'subobject' functor, which can occur in a wide variety of contexts whenever one has suitable subcategories of categories. On the other hand, we have already remarked that the unramified objects comprise a sub-category too small for geometric applications related to p-adic representations. This motivates us to define the *crystalline subrepresentation functor*

$$\text{Crys}$$

from the category of $\mathbb{Q}_p$-representations of G to itself. Given a $\mathbb{Q}_p$ representation V of G, $\text{Crys}(V)$ is the maximal crystalline subrepresentation of V, where crystalline is defined in the usual way for finite-dimensional representations and in general, we say V is crystalline if it is a direct limit of finite-dimensional crystalline subrepresentations. Equivalently, we could say V is crystalline iff any finite dimensional subrepresentation is crystalline. This equivalence follows from the fact that the category of finite-dimensional crystalline representations is closed under sub-objects. The fact that it's also closed under quotient objects implies that there is a well-defined notion of a 'maximal' crystalline subrepresentation. The functor Crys is the natural p-adic analogue of the 'invariants under inertia' functor on l-adic representations from the point of view of subrepresentation functors. Consequently, the derived functors of Crys are natural analogues of Galois cohomology with respect to I.

To see that these notions are well-defined, we must check two things:

(1) Crys is indeed a functor: This follows from the fact that a quotient of a crystalline representation is also crystalline, so that under a map $V \rightarrow W$ of representations, the crystalline part must land in the crystalline part.

(2) Crys is left exact: The key point is that if $U \subset V$ is a subrepresentation, then $\text{Crys}(U) = U \cap \text{Crys}(V)$. The inclusion in the two directions follows from the maximality involved in the definition and the sub-object property mentioned earlier.

One could equally easily define the various 'truncated' functors $\text{Crys}_{[a,b]}$ which associates to a representation the maximal subrepresentation with Hodge-Tate weights in the interval $[a, b]$. We will concentrate mostly on the functors $\text{Crys}_{[0,h]}$ which we will abbreviate as Crys_h. It will be convenient to use the term *h-crystalline representations* for the objects in the image of this functor. It is interesting to note that Crys_0 is nothing but the old inertia-invariants functor, so that the sequence of functors Crys_0, $\text{Crys}_1, \ldots$ and their derived functors provide natural prolongations of Galois cohomology. We see also that Crys is a bit more than just an 'analogue' of the inertia invariants functor. Rather, the existence of these prolongations reflect the richer structure that p-adic representations tend to have compared to their l-adic counterparts. We propose that these derived functors are natural invariants of p-adic representations (at least as natural as Galois cohomology) and should be studied seriously. One reason for thinking so stems from the application mentioned above. For this, we need to define these functors also for integral representations. Unfortunately, here the existing techniques for making the correct definitions are rather incomplete, and we can define *only* the truncated functors Crys_i for $i \leq p - 2$.

We also need to assume that K is absolutely unramified so that $K = K_0$ and $R = W$. The foundational material we need is contained in the seminal paper of Fontaine and Laffaille [2], but the reader can find a nice summary in [10].

Let h be a natural number $\leq p - 2$. One first defines finite crystalline representations of height $\leq h$, or the finite *h-crystalline representations* , to be the essential image of the category $\mathbf{MF}^h_{R,\text{tor}}$ (the finite-length filtered ϕ-modules of height $\leq h$) under the fully-faithful functor

$$M \mapsto V^*_{\text{crys}}(M) := \text{Hom}_{\mathbf{MF}_R}(M, A_{\text{crys},\infty})$$

Next, one defines a finite-type $\mathbb{Z}_p$-module L with G-action to be $h-$crystalline if $L = \varprojlim L_i$ where the L_i are finite-length h-crystalline representations. The fact that $h-$crystalline representations are closed under sub- and quotient objects follows, for example, from Wach's theorem [10] equating h-crystalline with subquotients of h-crystalline $\mathbb{Q}_p$-representations. In particular, this implies that a finite-type $\mathbb{Z}_p$ representation L is h-crystalline iff L/p^nL is $h-$crystalline for all n (which is the definition of [6]), and when L is free, iff

$$L = V^*_{\mathrm{crys}}(M) := \mathrm{Hom}_{\mathbf{MF}_R}(M, A_{\mathrm{crys}})$$

for an object M of $\mathbf{MF}^h_R$ (the finitely generated free filtered ϕ-modules of height $\leq h$) ([10] 2.2.2).

Now for an arbitrary $\mathbb{Z}_p[G]$-module V, we define it to be crystalline if $V = \varinjlim V_i$ where the V_i are subrepresentations of finite-type.

We need to check that this definition is consistent with the existing one for $\mathbb{Q}_p$-representations. Since we defined it for the infinite-dimensional case using limits from finite dimensions, we need only check it for finite-dimensional representations. So assume that V is $h-$crystalline in the old sense. Then V=$\mathrm{Hom}_{MF_K}(\Delta, B_{\mathrm{crys}})$ for some Δ in $\mathbf{MF}^h_K$ [2] (remarque 8.5 and 8.13 (c)). Since Δ is B_{crys}-admissible, in particular, weakly admissible, one can find a strongly divisible lattice $M \subset \Delta$ which is an object of $\mathbf{MF}^h_R$. So we get $V = L \otimes \mathbb{Q}_p$ where $L = \mathrm{Hom}_{\mathbf{MF}_R}(M, A_{\mathrm{crys}})$. Now, L is $h-$crystalline and $V = \varinjlim L[1/p^n]$ while $L[1/p^n] \simeq L$ (via multiplication by p^n) is h-crystalline. So V is h-crystalline in the new sense. In the other direction, assume $V = \varinjlim L_i$ for $h-$crystalline submodules L_i of finite-type. Then some $L = L_i$ is a lattice and $V = L \otimes \mathbb{Q}_p$. But $L = \mathrm{Hom}_{\mathbf{MF}_R}(M, A_{\mathrm{crys}})$ for some free R-module M in $\mathbf{MF}^h_R$ and M is then a strongly divisible lattice in $\Delta := M \otimes K$ according to the terminology of [2] definition 7.7, and therefore, $M \otimes K$ is weakly admissible. Thus, by the main theorem of [2], $M \otimes K$ is $B_{\mathrm{crys}}-$admissible and $V = \mathrm{Hom}_{\mathbf{MF}_K}(M \otimes K, B_{\mathrm{crys}})$ is crystalline.

Thereby, we can define Crys_h, the maximal $h-$crystalline subrepresentation functor for $h \leq p-2$ compatibly on all $\mathbb{Z}_p[G]$ modules. For the purposes of this paper, we need the definition only for finite-type $\mathbb{Z}_p$-modules, $\mathbb{Q}_p$-vector spaces of finite dimension, and discrete torsion modules.

An easy consequence of the definitions is that if L is a finitely generated free $\mathbb{Z}_p$ representation, then $\mathrm{Crys}_h(L) = \varprojlim \mathrm{Crys}_h(L/p^nL)$. By the key property that

$$L_1 \subset L_2 \Rightarrow \mathrm{Crys}_h(L_1) = L_1 \cap \mathrm{Crys}_h(L_2)$$

we again have left exactness.

The definition of the derived functors is somewhat delicate, as in the case of continuous Galois cohomology. It is just defined as usual using injective resolutions in the case of discrete torsion G-modules. For a finitely generated $\mathbb{Z}_p$-module L with continuous G-action, we define it using Jannsen's technique of continuous étale cohomology [4]: that is, we send L to the inverse system

$L_f := (L/p^n)_{n\in\mathbf{N}}$ (the subscript f standing for 'formal') of finite length $\mathbb{Z}_p[G]$ modules, and take the derived functor of the functor which takes an inverse system $(M_i)_{i\in\mathbf{N}}$ of discrete $\mathbb{Z}_p$ representations to the group

$$\varprojlim \ \mathrm{Crys}(M_i).$$

(Note that by the paragraph above, when the inverse system comes from a finite-type $\mathbb{Z}_p$-module, this agrees with taking the crystalline part functor directly.)

Then for finite-dimensional $\mathbb{Q}_p$-vector spaces V, we define

$$R^i\ \mathrm{Crys}_h(V) := R^i\ \mathrm{Crys}_h(L) \otimes \mathbb{Q}_p$$

where $L \subset V$ is a G-invariant $\mathbb{Z}_p$-lattice. By considering the directed system of lattices in V, it is easy to see that this definition is independent of the lattice.

The important thing for us is to get long exact sequences in a reasonable class of situations. Exactly as in [4], theorem 5.14, we get the following result:

Proposition 1. *For a finitely-generated free $\mathbb{Z}_p$-module M with continuous G-action, there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow\ & Crys_h(M) \longrightarrow Crys_h(M\otimes\mathbb{Q}_p) \longrightarrow Crys_h(M\otimes\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \\ \longrightarrow\ & R^1\ Crys_h(M) \longrightarrow R^1\ Crys_h(M\otimes\mathbb{Q}_p) \longrightarrow R^1\ Crys_h(M\otimes\mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \dots \end{aligned}$$

Proof. We recall some notation from Jannsen's paper: For a $\mathbb{Z}_p$ module L, we denote by $T(L)$ the inverse system given by $L[p^n]$ with transition maps $p : L[p^{n+1}]\rightarrow L[p^n]$. We also have the inverse system (L,p), whose objects are L in each degree and whose transition maps are all p. If $T_pL = \varprojlim T(L)$ denotes the usual Tate module of L, then we have

$$T_pL\otimes\mathbb{Q}_p \simeq \varprojlim(L,p)$$

for any torsion $\mathbb{Z}_p$-module L ([4], Lemma 5.9). Now, starting with a finitely generated free $\mathbb{Z}_p$-module M, we can associate the torsion module $V := M\otimes(\mathbb{Q}_p/\mathbb{Z}_p)$ and we have $M_f \simeq T(V)$ from which we get an exact sequence of inverse systems ([4], p.231):

$$0\rightarrow M_f\rightarrow(V,p)\rightarrow V\rightarrow 0$$

where the last V refers to the inverse system that has V in each degree with identities as transition maps. From this, we get a long exact sequence for $R^i\ \mathrm{Crys}_h$. Thus, it suffices to show that

$$R^i\ \mathrm{Crys}(V,p) \simeq R^i\ \mathrm{Crys}(M_f)\otimes\mathbb{Q}_p.$$

Note that for an injective object I in the category of discrete torsion $\mathbb{Z}_p[G]$-modules, $\mathrm{Crys}(I)$ is p-divisible: Take any $v \in \mathrm{Crys}(I)$. v lies inside a finite-length crystalline submodule N of $\mathrm{Crys}(I)$. By injectivity of I, it suffices to show that N injects to a finite-length crystalline representation N' inside which v becomes a p-multiple (for then N' will map to I, hence to $\mathrm{Crys}(I)$, providing the desired divisibility). This follows, for example, from Wach's theorem [10], equating finite-length crystalline representations with sub-quotients of rational crystalline representations.

By the same argument as in [4], prop. 1.6, lemma 5.2, and cor. 5.3, it follows that for an injective I as above, $T(I)$ and (I,p) are both acyclic inverse systems for R^i Crys_h. Therefore, if $V \hookrightarrow I^\cdot$ is an injective resolution for V, then $T(V) \hookrightarrow T(I^\cdot)$ and $(V,p) \hookrightarrow (I^\cdot, p)$ are both acyclic resolutions. But

$$T_p(\mathrm{Crys}(I^\cdot)) \otimes \mathbb{Q}_p \simeq \varprojlim(\mathrm{Crys}(I^\cdot), p)$$

as complexes, so that

$$H^i(T_p(\mathrm{Crys}(I^\cdot))) \otimes \mathbb{Q}_p \simeq H^i((\mathrm{Crys}(I^\cdot), p))$$

By the above-mentioned isomorphism between $T(V)$ and M_f we get

$$R^i\ \mathrm{Crys}(M) \otimes \mathbb{Q}_p \simeq R^i\ \mathrm{Crys}((V,p))$$

A systematic study of these functors will be presented in the forthcoming Ph.D. thesis of the second author.

2. The p-complement to Grothendieck's formula

In this section, we will continue to assume that K is absolutely unramified, and furthermore, that $p > 2$.

We will be using one more functor FF which associates to a p-adic representation its maximal 'finite and flat' part. Of course, one needs to define finite flat p-adic representations in a general setting. For finite $\mathbb{Z}_p$ representations, finite flat means the usual thing: a finite representation is finite flat if it's isomorphic to the $\bar{K}$ points of a finite flat commutative group scheme over R. A finite-type $\mathbb{Z}_p$-representation is defined to be finite flat if it is the inverse limit of finite finite flat representations (the double adjective seems unfortunately unavoidable). Finally, an arbitrary $\mathbb{Z}_p$ representation for G is said to be finite flat if it is the direct limit of finite flat representations of finite type.

For finite representations, the property of being finite flat is closed under passing to sub-objects and quotient objects (using Zariski closure and construction of good quotient schemes), so the same is true for any $\mathbb{Z}_p$ representation. Thus it makes sense to speak of the maximal finite flat subrepresentation of any representation, and the associated functor FF is left exact. Thus, we can consider its derived functors. In fact, by Fontaine-Laffaille's description of finite flat group schemes ([2], section 9) FF is nothing but Crys_1. Notice, however, that FF is defined over an arbitrary local field, not necessarily absolutely unramified.

We will also need the trivial observation that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence in $\mathbf{MF}_R$, M_1 and M_3 are in $\mathbf{MF}_R^h$, and M_2 is in $\mathbf{MF}_R^{h'}$ for some h', then in fact, M_2 is in $\mathbf{MF}_R^h$. This follows by noting that the morphisms are strict so that any F^iM_2 (the i-th step of the filtration for M_2) for $i > h$ would have to be zero when intersected with M_1 and when mapped to M_3, and hence, must be zero.

Thus, we have an obvious corresponding statement for $h-$ and h'-crystalline representations.

We now return to the problem of expressing the p-part of Γ in an analogous manner to Grothendieck's formula for $l \neq p$

$$\Gamma(l) \cong H^1(I, T_l A)_{\text{tor}}. \tag{1}$$

To derive the above formula, Grothendieck shows ([3], p. 132) that

$$\Gamma[l^n] \cong A[l^n]^f/(A^0[l^n])^f \tag{2}$$

where $\Gamma[l^n]$ (resp. $A[l^n]$) denotes the kernel of multiplication by l^n on Γ (resp. $A(\overline{K})$), and the superscript f denotes the "finite part" (denoted the "fixed part" by Grothendieck in [3], section 2.2.3), i.e., the points that extend to a map from $\mathrm{Spec}(\overline{R})$ to $\mathbf{A}$, or equivalently, the $\overline{K}$ points of the maximal finite flat subgroup scheme of $\mathbf{A}[l^n]$. Similarly, $(A^0[l^n])^f$ denotes the $\overline{K}$ points of the maximal finite flat subgroup scheme of $\mathbf{A}^0[l^n]$ which can also be thought of as the points of $A[l^n]^f$ which reduce mod p to a point in A_0^0, the connected component of the identity in the special fiber. The key point then is that the finite part coincides with the inertia invariants of $A[l^n]$ (resp. $A^0[l^n]$) [3] (Proposition 2.2.5) and the formula (1) follows easily.

In the case of $l = p$, (2) still holds (provided one assumes semi-stability, since the formula only depends on the multiplication-by-p map being finite flat, and hence, surjective on A^0), but it is no longer the case that the fixed part and inertia invariants coincide. However, we will show below that (for $l \neq p$) the finite part coincides with the *maximal h-crystalline part* for any $1 \leq h \leq p-2$ (recall that $p > 2$). This will allow us to derive, in a completely analogous manner to Grothendieck, the following:

Theorem 1. *Let A be an abelian variety over the absolutely unramified local field K with semi-stable reduction and $1 \leq h \leq p-2$. Then*

$$\Gamma(p) \cong R^1 \ \mathit{Crys}_h(T_p A)_{tor}.$$

Proof. We will first show that $\mathrm{Crys}_h(A[p^n]) = (A[p^n])^f$. For this, we note that the fixed part of $A[p^n]$ is none other than $FF(A[p^n])$. That is, the fixed part is finite-flat by definition, giving us one inclusion

$$(A[p^n])^f \subset FF(A[p^n]).$$

Now let $\mathcal{V}$ denote the finite-flat group scheme extending $FF(A[p^n])$, so that if V is the generic fiber of $\mathcal{V}$, we have

$$V(\overline{K}) \cong FF(A[p^n])$$

as G-modules. From the inclusion $FF(A[p^n]) \subset A[p^n]$, we have a map

$$V \to A.$$

We need to show that this map extends to a map $\mathcal{V} \to \mathbf{A}$, thereby showing that the finite part is actually "finite inside $\mathbf{A}$." However, restricting to the connected component V^0 of V, we find that the image must actually land in the finite part of A. This follows because $A[p^n]/A[p^n]^f$ is unramified ([3], Proposition 5.6). By results of Raynaud [8], this extends to a map $\mathcal{V}^0 \to \mathbf{A}^f$. Hence by Lemma 5.9.2 of [3], we get a unique map $\mathcal{V} \to \mathbf{A}$ extending the two previous maps, and giving us the opposite inclusion. (This is essentially the same argument as in [9], Lemma 6.2.)

We saw above that $FF = \mathrm{Crys}_1$, as functors. We will now show that one can replace Crys_1 by any of the Crys_h's in our setting. In fact, we will see from the proof that if any general Crys functor were defined for finite representations, then that could be used as well.

We certainly have an inclusion

$$FF(A[p^n]) \hookrightarrow \mathrm{Crys}_h(A[p^n])$$

which induces an inclusion of $\mathrm{Crys}_h(A[p^n])/FF(A[p^n])$ into the unramified G-module $A[p^n]/FF(A[p^n])$. Thus, $\mathrm{Crys}_h(A[p^n])/FF(A[p^n])$ is unramified as well, and hence finite-flat as a representation. Therefore $\mathrm{Crys}_h(A[p^n])$ sits in the middle of a short exact sequence whose outer terms are both crystalline of height one (actually, the last is of height 0):

$$0 \to FF(A[p^n]) \to \mathrm{Crys}_h(A[p^n]) \to \mathrm{Crys}_h(A[p^n])/FF(A[p^n]) \to 0$$

By the observation made earlier, we see that $\mathrm{Crys}_h(A[p^n])$ is itself crystalline of height one, and thus equal to $\mathrm{Crys}_1(A[p^n]) = FF(A[p^n]) = (A[p^n])^f$. As $A^0[p^n]/FF(A^0[p^n])$ is contained in $A[p^n]/FF(A[p^n])$, it is also unramified and an entirely similar argument gives $\mathrm{Crys}_h(A^0[p^n]) = (A^0[p^n])^f$ as well.

The isomorphism (2) thus becomes

$$\Gamma[p^n] \cong \mathrm{Crys}_h(A[p^n])/\ \mathrm{Crys}_h(A^0[p^n])$$

However,

Claim. $\mathrm{Crys}_h(A[p^n])/\ \mathrm{Crys}_h(A^0[p^n]) \simeq$

$$\mathrm{Crys}_h(T_pA \otimes \mathbb{Z}_p/p^n\mathbb{Z}_p)/\ \mathrm{Crys}_h(T_pA) \otimes \mathbb{Z}_p/p^n\mathbb{Z}_p.$$

Proof. The equality between the 'numerators' is obvious, so we need to see that $\mathrm{Crys}_h(T_pA) \otimes \mathbb{Z}_p/p^n\mathbb{Z}_p$ is equal to $\mathrm{Crys}_h(A^0[p^n])$. But

$$\mathrm{Crys}_h(T_pA) \cong \varprojlim(\ \mathrm{Crys}_h(A[p^n]) = \varprojlim(A[p^n]^f)$$

Hence $\mathrm{Crys}_h(T_pA) \simeq (T_pA)^f$. Thus,

$$\mathrm{Crys}_h(T_pA) \otimes \mathbb{Z}_p/p^n\mathbb{Z}_p \cong (T_pA)^f \otimes \mathbb{Z}_p/p^n\mathbb{Z}_p \cong (A^0[p^n])^f \cong \mathrm{Crys}_h(A^0[p^n]).$$

(The key point is the second isomorphism, as explained in [3]. That is, if you take the finite part of the Tate module and then reduce mod p^n, then you end up in $(A^0[p^n])^f$ because the multiplication by p map is finite and surjective only on $\mathbf{A}^0$.)

Applying direct limits, we get the formula

$$\Gamma(p) \cong \mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p/\mathbb{Z}_p)/\ \mathrm{Crys}_h(T_pA) \otimes\ \mathbb{Q}_p/\mathbb{Z}_p.$$

This plays a role analogous to Grothendieck's formula [3] (Proposition 11.2).

Following Grothendieck, we next apply Crys_h to the short exact sequence

$$0 \longrightarrow T_pA \longrightarrow T_pA \otimes \mathbb{Q}_p \longrightarrow T_pA \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0,$$

and obtain the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Crys}_h(T_pA) &\longrightarrow \mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p) \longrightarrow \mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \\ \longrightarrow R^1\,\mathrm{Crys}_h(T_pA) &\longrightarrow R^1\,\mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p) \longrightarrow R^1\,\mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \dots \end{aligned}$$

Note that

$$\mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p) = (\,\mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p) \cap T_pA) \otimes \mathbb{Q}_p = \mathrm{Crys}_h(T_pA) \otimes \mathbb{Q}_p.$$

Thus the kernel of the map of

$$R^1\,\mathrm{Crys}_h(T_pA) \to R^1\,\mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p)$$

is $\mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p/\mathbb{Z}_p)/\,\mathrm{Crys}_h(T_pA) \otimes \mathbb{Q}_p/\mathbb{Z}_p$, i.e. $\Gamma(p)$. Since $\Gamma(p)$ is torsion and $R^1\,\mathrm{Crys}_h(T_pA \otimes \mathbb{Q}_p)$ torsion-free, we do indeed find that

$$\Gamma(p) \cong R^1\,\mathrm{Crys}_h(T_pA)_{\mathrm{tor}}.$$

Remark. From the proof, it is clear that one could have just used the functor FF for the theorem in which case one could extend the theorem to the case of $e \leq p-2$ by eliminating the Fontaine-Laffaille theory (the restriction on ramification is still necessary in order to use Raynaud's theory). However, this would have made the analogy to the l-case less natural, since a crystalline representation is clearly the correct general notion which sets the formula into a broad context. In particular, the definition of FF on $\mathbb{Q}_p$ representations is rather artificial compared to Crys. It would of course have been nicer to replace Crys_h by a general Crys even for the integral representations.

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