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QUATERNIONIC GAMMA FUNCTIONS AND THEIR LOGARITHMIC DERIVATIVES AS SPECTRAL FUNCTIONS

Jean-François Burnol

We establish Connes's local trace formula (related to the explicit formulae of number theory) for the quaternions. This is done as an application of a study of the central operator $H = \log(|x|) + \log(|y|)$ in the context of invariant harmonic analysis. The multiplicative analysis of the additive Fourier transform gives a spectral interpretation to generalized "Tate Gamma functions" (closely akin to the Godement-Jacquet "$\gamma(s, \pi, \psi)$" functions.) The analysis of H leads furthermore to a spectral interpretation for the logarithmic derivatives of these Gamma functions (which are involved in "explicit formulae".)

1. Introduction

We establish for quaternions an analog of the trace formula obtained by Connes in [5] for a commutative local field K. This formula has the form $\mathbf{Tr}(\widetilde{P_\Lambda} P_\Lambda U_f) = 2\log(\Lambda) f(1) + W(f) + o(1)$ (for $\Lambda \to \infty$), where f is a test-function on $K^\times$, U_f is the operator of multiplicative convolution with f, P_Λ and $\widetilde{P_\Lambda}$ are cut-off projections (precise definitions will be given later), all acting on the Hilbert space of square-integrable functions on K. The contant term $W(f)$ was shown by Connes to be exactly the term arising in the "Weil's explicit formulae" [14] of number theory.

We have shown in this abelian local case (see [1], [2], and the related papers [3] and [4]) that the Weil term $W(f)$ can be written as $-H(f)(1)$ for a certain dilaton-invariant operator H. We study this operator in the non-commutative context of quaternions and then derive the (analog of) Connes's asymptotic formula. The proof would go through (with some simplifications of course) equally well in the abelian case.

We first give some elementary lemmas of independent interest about self-adjoint operators. We then study in multiplicative terms the additive Fourier Transform, and this immediately leads to the definition of certain "Quaternionic Tate Gamma" functions and to the analog of Tate's local functional equations ([13], [15]). This is of course very much related to the generalization to $GL(N)$ of Tate's Thesis in the work [6] of Godement-Jacquet (see [8], [9] for reviews and references to further works by other authors), where certain "$\gamma(s, \pi, \psi)$" functions, local L- and ϵ-factors and associated functional equations are studied. Also relevant is the classic monograph by Stein and Weiss [12] on harmonic

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analysis in euclidean spaces. In this paper we will follow a completely explicit and accordingly elementary approach.

We introduce the "conductor operator" $H = \log(|x|) + \log(|y|)$ and show how it gives an operator theoretic interpretation to the logarithmic derivatives of the Gamma functions (which are involved in explicit formulae.) It is then a simple matter to compute Connes's trace, and to obtain the asymptotic formula

$$\mathbf{Tr}(\widetilde{P_\Lambda} P_\Lambda U_f) = 2\log(\Lambda) f(1) - H(f)(1) + o(1)$$

in a form directly involving our operator H. Further work leads to a "Weil-like" formulation for the constant term $H(f)(1)$, if so desired.

2. Closed invariant operators

It is well known that any bounded operator on $L^2(\mathbb{R}, dx)$ which commutes with translations is diagonalized by the additive Fourier transform (see for example the Stein-Weiss monograph [12].) We need a generalization which applies to (possibly) unbounded operators on Hilbert spaces of the form $L^2(G, dx)$ where G is a topological group. Various powerful statements are easily found in the standard references on Hilbert spaces, usually in the language of spectral representations of abelian von Neumann algebras. For lack of a reference precisely suited to the exact formulation we will need, we provide here some simple lemmas with their proofs.

Definition 2.1. *Let L be a Hilbert space. A (possibly unbounded) operator M on L with domain D is said to commute with the bounded operator A if*

$$\forall v \in L : v \in D \Rightarrow \Big(\ A(v) \in D \text{ and } M(A(v)) = A(M(v))\ \Big).$$

Theorem 2.2. *Let L be a Hilbert space and G a (not necessarily abelian) group of unitary operators on L. Let $\mathcal{A}$ be the von Neumann algebra of bounded operators commuting with G. Let M be a (possibly unbounded) operator on L, with dense domain D. If the three following conditions are satisfied*

(1) *$\mathcal{A}$ is abelian,*

(2) *(M, D) is symmetric, and*

(3) *(M, D) commutes with the elements of G,*

then (M, D) has a unique self-adjoint extension. This extension commutes with the operators in $\mathcal{A}$.

Proof. We first replace (M, D) by its double-adjoint so that we can assume that (M, D) is closed (it is easy to check that conditions (2) and (3) remain valid). The problem is to show that it is self-adjoint. Let K be the range of the operator $M + i$. It is a closed subspace of L (as $\|(M+i)(\varphi)\|^2 = \|M(\varphi)\|^2 + \|\varphi\|^2$, and M is closed). Let R be the bounded operator onto D which is orthogonal projection onto K followed with the inverse of $M + i$. One checks easily that R belongs to $\mathcal{A}$, hence commutes with its adjoint R^* which will also belong to $\mathcal{A}$. Any vector ψ in the kernel of R is then in the kernel of R^* (as $< R^*\psi | R^*\psi > = < \psi | R R^*\psi > = 0$). So ψ belongs to the orthogonal complement to the range of

R, that is $\psi = 0$ as the range of R is D. So $K = L$ and in the same manner $(M - i)(D) = L$. By the basic criterion for self-adjointness (see [10]), M is self-adjoint. Let $A \in \mathcal{A}$. It commutes with the resolvent R hence leaves stable its range D. On D one has $RA(M + i) = AR(M + i) = A = R(M + i)A$ hence $A(M + i) = (M + i)A$ so A commutes with M. □

For the remainder of this section we let G be a locally compact, Hausdorff, topological *abelian* group and $\widehat{G}$ its dual group. We refer to [11] for the basics of harmonic analysis on G. In particular we have a Haar measure dx (unique up to a multiplicative constant) and a Hilbert space $L = L^2(G, dx)$. We also have a dual Haar measure dy on $\widehat{G}$ such that the Fourier transform $F(\varphi)(y) = \int \varphi(x)\overline{y(x)}dx$ is an isometry of L onto $L^2(\widehat{G}, dy)$. We sometimes identify the two Hilbert spaces without making explicit the reference to F: so when we write $f(y) \in L$ we really refer to $F^{-1}(f) \in L$ with $f \in L^2(\widehat{G}, dy)$. No confusion should arise. We will assume that dy is a σ-finite measure so that there exists $\psi \in L$ with the property $\psi(y) \neq 0$ *a.e.*.

Let $a(y)$ be a measurable function on $\widehat{G}$, not necessarily bounded. Let $D_a \subset L$ be the domain of square-integrable (equivalence classes of measurable) functions $\varphi(x)$ on G such that $a(y)F(\varphi)(y)$ belongs to $L^2(\widehat{G}, dy)$. And let M_a be the operator with domain D_a acting according to $\varphi \mapsto M_a(\varphi) = F^{-1}(a \cdot F(\varphi))$. We write $a = b$ if the two functions $a(y)$ and $b(y)$ are equal almost everywhere on $\widehat{G}$.

Lemma 2.3. *The operator (M_a, D_a) on $L^2(G, dx)$ commutes with G. Furthermore D_a is dense and (M_a, D_a) is a closed operator. If (M_b, D_b) extends (M_a, D_a), then in fact $a = b$ and $(M_b, D_b) = (M_a, D_a)$. The adjoint of (M_a, D_a) is $(M_{\overline{a}}, D_{\overline{a}})$ (of course $D_{\overline{a}} = D_a$.)*

Proof. We give the proof for completeness. The commutation with G-translations is clear. Then D_a contains (the inverse Fourier transform of) $\frac{\psi(y)}{\sqrt{1+|a(y)|^2}}$ and all its translates. Hence if f is orthogonal to D_a then the function $\frac{\overline{f(y)}\psi(y)}{\sqrt{1+|a(y)|^2}}$ on $\widehat{G}$ belongs to $L^1(\widehat{G}, dy)$ and has a vanishing "inverse Fourier transform", hence $f = 0$ (almost everywhere). It is also clear using $\frac{\psi(y)}{\sqrt{1+|a(y)|^2}}$ that if (M_b, D_b) extends (M_a, D_a), then $a = b$. Let us assume that the sequence φ_j is such that $\varphi = \lim \varphi_j$ and $\theta = \lim M_a(\varphi_j)$ both exist. Let us pick a pointwise on $\widehat{G}$ almost everywhere convergent subsequence $\varphi_{j_k}(y)$. Using Fatou's lemma we deduce that φ belongs to D_a. Using Fatou's lemma again we get the vanishing of $\int_G |\theta(y) - a(y)\varphi(y)|^2\, dy$, and this shows that (M_a, D_a) is a closed operator. Finally let $f(y)$ be in the domain of the adjoint of (M_a, D_a). There exists then an element θ of L such that for any $\varphi \in D_a$ the equality

$$\int f(y)\overline{a(y)\varphi(y)}\, dy = \int \theta(y)\overline{\varphi(y)}\, dy$$

holds. This implies that the two following functions of $L^1(\widehat{G}, dy)$:

$$\frac{f(y)\overline{a(y)\psi(y)}}{\sqrt{1+|a(y)|^2}} \quad \text{and} \quad \frac{\theta(y)\overline{\psi(y)}}{\sqrt{1+|a(y)|^2}}$$

have the same Fourier transform on G, hence are equal almost everywhere. So $f \in D_{\overline{a}}$ and $(M_a)^*(f) = (M_{\overline{a}})(f)$. □

Theorem 2.4. *Let* (M, D) *be a* closed operator *on* $L^2(G, dx)$ *commuting with* G*-translations. Then* $(M, D) = (M_a, D_a)$ *for a (unique) multiplicator* a*.*

Note 2.5. For a bounded M and $G = \mathbb{R}$, this is proven in the classical monograph by Stein and Weiss [12], as a special case of a more general statement applying in L^p spaces.

Proof. Let us first assume that M is bounded. We use the (inverse Fourier transform of the) function $\psi(y)$ and define $a(y)$ to be $\dfrac{M(\psi)(y)}{\psi(y)}$. Let us consider the domain D consisting of all finite linear combinations of translates of ψ. It is dense by the argument using unicity of Fourier transform in L^1 we have used previously. Then $(M, D) \subset (M_a, D_a)$, hence (M_a, D_a) is also an extension of the closure of (M, D). As M is assumed to be bounded this is (M, L). But this means that $D_a = L$ and that $M = M_a$ (we then note that necessarily a is essentially bounded).

The next case is when M is assumed to be self-adjoint. Its resolvents $R_1 = (M - i)^{-1}$ and $R_2 = (M + i)^{-1}$ are bounded and commute with G. Hence they correspond to multiplicators $r_1(y)$ and $r_2(y)$. The kernel of R_1 is orthogonal to the range of $R_2 = R_1^*$ which is all of D, so in fact it is reduced to $\{0\}$. Hence $r_1(y)$ is almost everywhere non-vanishing. Let $f \in D$ and $g = M(f)$. As $R_1(M(f) - i\,f) = f$ we get $g(y) = \dfrac{1 + i\,r_1(y)}{r_1(y)} \cdot f(y)$ and defining $a(y)$ to be $\dfrac{1 + i\,r_1(y)}{r_1(y)}$ we see that (M_a, D_a) is an extension of (M, D). Taking the adjoints we deduce that (M, D) is an extension of $(M_{\overline{a}}, D_{\overline{a}})$. So all three are equal (and a is real-valued).

For the general case we use the theorem of polar decomposition (see for example [10]). There exists a non-negative self-adjoint operator $|M|$ with the same domain as M and a partial isometry U such that $M = U|M|$. Further conditions are satisfied which make $|M|$ and U unique: so they also commute with G. It follows from what was proven previously that $(M, D) \subset (M_a, D_a)$ for an appropriate a (the product of the multiplicators associated to the self-adjoint $|M|$ and the bounded U). The adjoint (M^*, D^*) also has a dense domain and commutes with G, so in the same manner $(M^*, D^*) \subset (M_b, D_b)$ for an appropriate b. The inclusion $(M_{\overline{a}}, D_{\overline{a}}) \subset (M^*, D^*) \subset (M_b, D_b)$ implies $b = \overline{a}$ and $(M_a, D_a) = (M, D)^{**}$. But the double-adjoint coincides with the closed operator (M, D). □

Let us mention an immediate corollary:

Corollary 2.6. *A closed symmetric operator on $L^2(G, dx)$ commuting with G is self-adjoint, and a symmetric operator which has a dense domain and commutes with G is essentially self-adjoint.*

3. Tate's functional equations

Our first concern will be to introduce numerous notations. Let $\mathbb{H}$ be the space of quaternions with $\mathbb{R}$-basis $\{1, i, j, k\}$ and table of multiplication $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. A typical quaternion will be denoted $x = x_0 + x_1 i + x_2 j + x_3 k$, its conjugate $\overline{x} = x_0 - x_1 i - x_2 j - x_3 k$, its real part $\mathrm{Re}(x) = x_0$, its (reduced) norm $n(x) = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2$.

$\mathbb{H}$ can also be considered as a left $\mathbb{C}$-vector space with basis $\{1, j\}$. We then write $a = x_0 + x_1 i$ and $b = x_2 + x_3 i$. Then $jaj^{-1} = \overline{a}$, $x = a + bj$, and $n(x) = a\overline{a} + b\overline{b}$. The action of $\mathbb{H}$ on itself by right-multiplication sends $x = a + bj$ to the 2×2 complex matrix

$$R_x = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}.$$

We write V for the complex vector space of complex-linear forms $\alpha : \mathbb{H} \to \mathbb{C}$. The forms $A : x \mapsto a$ and $B : x \mapsto b$ are a basis of V. We have a left action of $\mathbb{H}$ on V with $x \in \mathbb{H}$ acting as $\alpha(y) \mapsto \alpha(yx)$. This left action represents the quaternion x by the matrix

$$L_x = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

Also let $V_N = \mathrm{SYM}^N(V)$, for $N = 0, 1, \dots$ be the $N+1$-dimensional complex vector space with basis the monomials $A^j B^{N-j}, 0 \leq j \leq N$.

Let $G = \mathbb{H}^\times$ be the multiplicative group (with typical element g) and $G_0 = \{g \in G |\ n(g) = 1\}$ its maximal compact subgroup. Through the assignment $g \mapsto L_g$ an isomorphism $G_0 \sim SU(2)$ is obtained, and the V_N's give the complete list of (isomorphism classes of) irreducible representations of G_0.

The additive Fourier Transform $\mathcal{F}$ is taken with respect to the additive character $x \mapsto \lambda(x) = e^{-2\pi\, i(x+\overline{x})}$. We note that $\lambda(xy) = \lambda(yx)$. The choice we make for the normalization of $\mathcal{F}$ is:

$$\mathcal{F}(\varphi)(y) = \widetilde{\varphi}(y) = \int \varphi(x)\lambda(-xy)\, dx,$$

where $dx = 4dx_0 dx_1 dx_2 dx_3$ is the unique self-dual Haar measure for λ. With these choices the function $\omega(x) = e^{-2\pi x\overline{x}}$ is its own Fourier transform.

Definition 3.1. *The module $|g|$ of $g \in \mathbb{H}^\times$ is defined by the equality of additive Haar measures on $\mathbb{H}$: $d(gx) = d(xg) = |g|dx$. It is expressed in terms of the reduced norm by $|g| = n(g)^2$.*

Note 3.2. The multiplicative (left- and right-) Haar measures on G are the multiples of $\frac{dg}{|g|}$.

One has a direct product $G = (0,\infty) \times G_0$, $g = rg_0$, $r = \sqrt{n(g)} = |g|^{1/4}$. We write $d\sigma$ for the Euclidean surface element on G_0 (for the coordinates x_i), so that $dx = 4r^3\,dr d\sigma$. The rule for integrating functions of r is $\int g(r)dx = 8\pi^2 \int_0^\infty g(r)r^3 dr$ as is checked with $\omega(x)$. So $d\sigma = 2\pi^2\, d^*g_0$ where d^*g_0 is the Haar measure on G_0 with total mass 1.

Definition 3.3. *The normalized Haar measure on G is defined to be $d^*g = \frac{1}{2\pi^2}\frac{dg}{|g|} = 4\frac{dr}{r}\,d^*g_0$. It is chosen so that its push-forward under the module map $g \mapsto u = |g| \in \mathbb{R}^{\times +}$ is $\frac{du}{u} = 4\frac{dr}{r}$.*

The multiplicative group G acts in various unitary ways on $L^2 := L^2(\mathbb{H}, dx)$:

$$L_1(g) : \varphi(x) \mapsto |g|^{1/2}\varphi(xg) \qquad R_1(g) : \varphi(x) \mapsto |g|^{1/2}\varphi(gx)$$

and also $L_2(g) = R_1(g^{-1})$ and $R_2(g) = L_1(g^{-1})$.

Definition 3.4. *The* Inversion I *is the unitary operator on $L^2(\mathbb{H}, dx)$ acting as $\varphi(x) \mapsto \frac{1}{|x|}\varphi(\frac{1}{x})$. The* Gamma operator *is the composite $\Gamma = \mathcal{F}I$.*

Theorem 3.5. *The Gamma operator commutes with both left actions L_1 and L_2 and with both right actions R_1 and R_2 of G on L^2.*

Proof. One just checks that $\mathcal{F}$ intertwines L_1 with L_2, and also R_1 with R_2 and that the inversion I also intertwines L_1 with L_2, and R_1 with R_2. ☐

Definition 3.6. *The* basic isometry *is the map $\phi(x) \mapsto f(g) = \sqrt{2\pi^2\,|g|}\;\phi(g)$ between $L^2(\mathbb{H}, dx)$ and $L^2(G, d^*g)$.*

Note 3.7. It is convenient to avoid using any notation at all for the basic isometry. So we still denote by $\mathcal{F}$ the additive Fourier transform transported to the multiplicative setting. The inversion I becomes $f(g) \mapsto f(g^{-1})$. The Gamma operator is still denoted Γ when viewed as acting on $L^2(G, d^*g)$.

The spectral decomposition of $L^2((0,\infty), \frac{du}{u})$ is standard Fourier (or Mellin) theory (alternatively we can apply Theorem **2.4** here): any bounded operator M commuting with multiplicative translations is given by a measurable bounded multiplier $a(\tau)$ in dual space $L^2(\mathbb{R}, \frac{d\tau}{2\pi})$:

$$G_1(u) = \lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda}\psi(\tau)u^{-i\tau}\frac{d\tau}{2\pi} \Longrightarrow M(G_1)(u) = \lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda}a(\tau)\psi(\tau)u^{-i\tau}\frac{d\tau}{2\pi}.$$

On the other hand the spectral decomposition of $L^2(G_0,\, d^*g_0)$ is part of the Peter-Weyl theory: it tells us that $L^2(G_0,\, d^*g_0)$ decomposes under the $L_1 \times R_1$ action by $G_0 \times G_0$ into a countable direct sum $\oplus_{N\geq 0} W_N$ of finite dimensional irreducible, non-isomorphic, modules. This is also the isotypical decomposition under either L_1 alone or R_1 alone (for which W_N then contains $N+1$ copies of V_N.) Using the standard theory of tensor products of separable Hilbert spaces (see for example [10]) we have:

Lemma 3.8. *The isotypical decomposition of $L^2(G, d^*g)$ under the compact group $G_0 \times G_0$ acting through $L_1 \times R_1$ is*

$$L^2(G, d^*g) = L^2((0,\infty), \frac{du}{u}) \otimes L^2(G_0, d^*g_0) = \oplus_N L^2((0,\infty), \frac{du}{u}) \otimes W_N.$$

Lemma 3.9. *Let M be a bounded operator on L^2 which commutes with both the L_1 and R_1 actions of G. Then to each integer $N \geq 0$ is associated an (essentially bounded) multiplicator $a_N(\tau)$ on $\mathbb{R}$, unique up to equality almost everywhere, such that*

$$\begin{aligned} &\psi \in L^2(\mathbb{R}, \frac{d\tau}{2\pi}),\ G_1(u) = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \psi(\tau) u^{-i\tau} \frac{d\tau}{2\pi} \\ &\Rightarrow \forall F \in W_N \quad M(FG_1) = FG_2,\ \textit{with} \\ &G_2(u) = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} a_N(\tau)\psi(\tau) u^{-i\tau} \frac{d\tau}{2\pi} \end{aligned}$$

and where FG_1 is the function $g \mapsto F(\frac{g}{|g|^{1/4}})G_1(|g|)$ and FG_2 the function $g \mapsto F(\frac{g}{|g|^{1/4}})G_2(|g|)$.

Proof. Let us take $f \in L^2((0,\infty), \frac{du}{u})$ and consider the linear operator M^f on $L^2(G_0, d^*g_0)$:

$$F(g_0) \mapsto \left(g_0 \mapsto \int_0^\infty \overline{f(u)} M(f \otimes F)(g_0\, u^{1/4}) \frac{du}{u} \right).$$

It commutes with the action of $G_0 \times G_0$ hence stabilizes each W_N and is a multiple a_N^f of the identity there. On the other hand, if we choose F_1 and F_2 in W_N and consider

$$f \mapsto \left(u \mapsto \int_{G_0} \overline{F_2(g_0)} M(f \otimes F_1)(g_0\, u^{1/4})\, d^*g_0 \right)$$

we obtain a bounded operator $M(F_1, F_2)$ on $L^2((0,\infty), \frac{du}{u})$ commuting with dilations and such that

$$< f|M(F_1,F_2)(f) >=< F_2|M^f(F_1) >= a_N^f < F_2|F_1 >,$$

where the left-hand bracket is computed in $L^2((0,\infty), \frac{du}{u})$ while the next two are in $L^2(G_0, d^*g_0)$. So $M(F_1, F_2)$ depends on (F_1, F_2) only through $< F_2|F_1 >$. We then let $a_N(\tau)$ be the spectral multiplier associated to $M(F, F)$ for an arbitrary F satisfying $< F|F >= 1$. □

Corollary 3.10. *The von Neumann algebra $\mathcal{A}$ of bounded operators commuting simultaneously with the left and right actions of the multiplicative quaternions on $L^2(\mathbb{H}, dx)$ is abelian.*

Lemma 3.11. *A self-adjoint operator M commuting with both left and right actions of G commutes with any operator of the von Neumann algebra $\mathcal{A}$. In particular it commutes with Γ.*

Proof. One applies Theorem **2.2**. □

Definition 3.12. *The* quaternionic Tate Gamma functions *are the multiplicators $\gamma_N(\tau)$ ($N \geq 0$) associated to the unitary operator* Γ.

Note 3.13. This generalizes the Gamma functions of Tate for $K = \mathbb{R}$ and $K = \mathbb{C}$ ([13]). In all cases they are indexed by the characters of the maximal compact subgroup of the multiplicative group $K^\times$.

Lemma 3.14. *There is a smooth function in the equivalence class of $\gamma_N(\tau)$.*

Proof. If the function $G_1(u)$ on $(0,\infty)$ is chosen smooth with compact support (so that $\psi(\tau)$ is entire) then, for any $F \in W_N$ the function FG_1, viewed in the additive picture, is smooth on $\mathbb{H}$, has compact support, and vanishes identically in a neighborhood of the origin. So its image under the inversion also belongs to the Schwartz class in the additive picture on $\mathbb{H}$. Hence $\Gamma(FG_1)$ can be written as $|g|^{1/2}\phi(g)$ for some Schwartz function $\phi(x)$ of the additive variable x. One checks that this then implies that $G_2(u)$ is a Schwartz function of the variable $\log(u)$ (we assume that F does not identically vanish of course), hence that $\gamma_N(\tau)\psi(\tau)$ is a Schwartz function of τ. The various allowable ψ's have no common zeros so the conclusion follows. $\square$

Note 3.15. From now on γ_N refers to this unique smooth representative. It is everywhere of modulus 1 as Γ is a unitary operator.

Note 3.16. Any function $F \in W_N$ will now be considered as a function on all of $G = \mathbb{H}^\times$ after extending it to be constant along each radial line. It is not defined at $x = 0$ of course.

Let $F \in W_N$. For $\mathrm{Re}(s) > 0$, $F(x)|x|^{s-1}$ is a tempered distribution on $\mathbb{H}$, hence has a distribution-theoretic Fourier Transform. At first we only consider $s = \frac{1}{2} + i\tau$:

Lemma 3.17. *As distributions on* $\mathbb{H}$

$$\mathcal{F}(F(\frac{1}{x})\,|x|^{-\frac{1}{2}+i\tau}) = \gamma_N(\tau)F(x)\,|x|^{-\frac{1}{2}-i\tau}.$$

Proof. We have to check the identity:

$$\int F(\frac{1}{y})\,|y|^{-\frac{1}{2}+i\tau}\widetilde{\varphi}(y)\,dy = \gamma_N(\tau)\cdot\int F(x)\,|x|^{-\frac{1}{2}-i\tau}\varphi(x)\,dx$$

for all Schwartz functions $\varphi(x)$ with Fourier Transform $\widetilde{\varphi}(y)$. Both integrals are analytic in $\tau \in \mathbb{R}$, hence both sides are smooth (bounded) functions of τ. It will be enough to prove the identity after integrating against $\psi(\tau)\,\frac{d\tau}{2\pi}$ with an arbitrary Schwartz function $\psi(\tau)$. With the notations of Lemma **3.9**, we have to check

$$\int F(\frac{1}{y})G_1(\frac{1}{y})|y|^{-\frac{1}{2}}\widetilde{\varphi}(y)\,dy = \int F(x)\,G_2(x)|x|^{-\frac{1}{2}}\varphi(x)\,dx.$$

But, by Lemma **3.9**, and by Definition **3.12**, $F(x)\,G_2(x)|x|^{-\frac{1}{2}}$ is just the Fourier Transform in $L^2(\mathbb{H}, dx)$ of $F(\frac{1}{y})G_1(\frac{1}{y})|y|^{-\frac{1}{2}}$, so this reduces to the L^2-identity

$$\int \psi(y)\widetilde{\varphi}(y)\,dy = \int \widetilde{\psi}(x)\varphi(x)\,dx.$$

□

Theorem 3.18. *Let $F \in W_N$. There exists an analytic function $\Gamma_N(s)$ in $0 < \mathrm{Re}(s) < 1$ depending only on $N \in \mathbb{N}$ and such that the following identity of tempered distributions on $\mathbb{H}$ holds for each s in the critical strip* $(0 < \mathrm{Re}(s) < 1)$:

$$\mathcal{F}(F(\frac{1}{x})\,|x|^{s-1}) = \Gamma_N(s)F(x)\,|x|^{-s}.$$

Proof. We have to check an identity:

$$\int F(\frac{1}{y})\,|y|^{s-1}\widetilde{\varphi}(y)\,dy = \Gamma_N(s)\cdot\int F(x)\,|x|^{-s}\varphi(x)\,dx$$

for all Schwartz functions $\varphi(x)$ with Fourier Transform $\widetilde{\varphi}(y)$. Both integrals are analytic in the strip $0 < \mathrm{Re}(s) < 1$, their ratio is thus a meromorphic function, which depends neither on F nor on φ as it equals $\gamma_N(\tau)$ on the critical line. Furthermore for any given s we can choose $\varphi(x) = \overline{F(x)}\alpha(|x|)$, with α having very small support around $|x| = 1$ to see that this ratio is in fact analytic. □

Note 3.19. This is the analog for quaternions of Tate's "local functional equation" [13], in the distribution theoretic flavor advocated by Weil [15]. We followed a different approach than Tate, as his proof does not go through that easily in the non-commutative case.

Let $\Gamma(s)$ be Euler's Gamma function $(\int_0^\infty e^{-u}u^s\,\frac{du}{u})$.

Theorem 3.20. *We have for each $N \in \mathbb{N}$:*

$$\Gamma_N(s) = i^N(2\pi)^{2-4s}\frac{\Gamma(2s+\frac{N}{2})}{\Gamma(2(1-s)+\frac{N}{2})}.$$

Proof. Let $0 \le j \le N$ and $\omega_j(x) = \overline{A(x)}^{N-j}\overline{B(x)}^j e^{-2\pi x\overline{x}} = \overline{a}^{N-j}\,\overline{b}^j\,\omega(x)$. One checks that $\widetilde{\omega_j}(y) = (-1)^j\,i^N\,\alpha^{N-j}\,\overline{\beta}^j\,\omega(y)$ $(y = \alpha + \beta j)$. We choose as homogeneous function $F_j(x) = a^{N-j}\,b^j\,|x|^{-N/4}$. For these choices the identity of Theorem **3.18** becomes

$$i^N\int(\alpha\overline{\alpha})^{N-j}(\beta\overline{\beta})^j e^{-2\pi y\overline{y}}|y|^{s-1-N/4}dy =$$
$$\Gamma_N(s)\int(a\overline{a})^{N-j}(b\overline{b})^j e^{-2\pi x\overline{x}}|x|^{-s-N/4}dx.$$

Adding a suitable linear combinations of these identities for $0 \le j \le N$ gives

$$i^N\int(y\overline{y})^N\,e^{-2\pi y\overline{y}}|y|^{s-1-N/4}dy = \Gamma_N(s)\int(x\overline{x})^N\,e^{-2\pi x\overline{x}}|x|^{-s-N/4}dx,$$

hence the result after evaluating the integrals in terms of $\Gamma(s)$. □

4. The central operator $H = \log(|x|) + \log(|y|)$

Definition 4.1. *We let $\Delta \subset \mathcal{C}^\infty(G)$ be the vector space of finite linear combinations of functions $f(g) = F(g_0)K(\log(|g|))$ with F in one of the W_N's (hence smooth) and K a Schwartz function on $\mathbb{R}$. It is a dense sub-domain of L^2.*

Theorem 4.2. *Δ is stable under $\mathcal{F}$.*

Proof. We have to show that $\gamma_N(\tau)$ is a multiplier of the Schwartz class. Let $h_N(\tau) = -i\frac{\gamma_N'(\tau)}{\gamma_N(\tau)}$. Using Theorem **3.20** and the partial fraction expansion of the logarithmic derivative of $\Gamma(s)$ (as in [1] for the real and complex Tate Gamma functions), or Stirling's formula, or any other means, one finds $h_N(\tau) = O(\log(1+|\tau|))$, $h_N^{(k)}(\tau) = O(1)$, so that $\gamma_N^{(k)}(\tau) = O(\log(1+|\tau|)^k)$. □

Let A be the operator on $L^2(\mathbb{H}, dx)$ of multiplication with $\log(|x|)$. As it is unbounded, we need a domain and we choose it to be Δ. Of course (A, Δ) is essentially self-adjoint. It is unitarily equivalent to the operator (B, Δ), $B = \mathcal{F}A\mathcal{F}^{-1}$. Clearly:

Lemma 4.3. *The domain Δ is stable under A and B.*

Definition 4.4. *The* conductor operator *is the operator $H = A + B$:*

$$H = \log(|x|) + \log(|y|).$$

This is an unbounded operator defined initially on the domain Δ.

Lemma 4.5. *The conductor operator (H, Δ) commutes with the left and with the right actions of G and is symmetric.*

This is clear. Applying now Theorem **2.2** we deduce:

Theorem 4.6. *The conductor operator (H, Δ) has a unique self-adjoint extension.*

We will simply denote by H and call "conductor operator" this self-adjoint extension.

Theorem 4.7. *The conductor operator H commutes with the inversion I.*

Proof. Indeed, it commutes with Γ by Theorem **2.2** and it commutes with $\mathcal{F}$ by construction. □

We now want to give a concrete description of its spectral functions.

Definition 4.8. *Let for each $N \in \mathbb{N}$ and $\tau \in \mathbb{R}$:*

$$h_N(\tau) = -i\frac{\gamma_N'(\tau)}{\gamma_N(\tau)},$$
$$k_N(\tau) = -h_N'(\tau)$$

Explicit computations prove that the functions h_N are left-bounded ($\exists C\,\forall\tau\,\forall N\quad h_N(\tau)\ \geq\ -C$) and that the functions k_N are bounded ($\exists C\,\forall\tau\,\forall N\ |k_N(\tau)|\leq C$.) We need not reproduce these computations here, which use only the partial fraction expansion of Euler's gamma function, as similar results are provided in [1] in the real and complex cases.

Let $f(g)=F(g_0)\phi(|g|)$ be an element of Δ, $F\in W_N\subset L^2(G_0,d^*g_0)$, $\phi\in L^2((0,\infty),\frac{du}{u})$, ϕ being a Schwartz function of $\log(u)$ ($u=|g|$). We can also consider f to be given as a pair $\{F,\psi\}$ with $\psi(\tau)=\int_0^\infty\phi(u)u^{i\tau}\frac{du}{u}$ being a Schwartz function of τ. Then $A(f)$ is given by the pair $\{F,D(\psi)\}$ where D is the differential operator $\frac{1}{i}\frac{d}{d\tau}$. This implies that $\Gamma A\Gamma^{-1}(f)$ corresponds to the pair $\{F,D(\psi)-h_N\cdot\psi\}$. On the other hand $\Gamma A\Gamma^{-1}=-B$ so $H(f)$ corresponds to the pair $\{F,h_N\cdot\psi\}$. The commutation with the inversion I translates into $h_N(-\tau)=h_N(\tau)$. Also: $K=i[B,A]=-i[A,H]$ sends the pair $\{F,\psi\}$ to $\{F,k_N\cdot\psi\}$, hence is bounded and anti-commutes with the Inversion. We have proved:

Theorem 4.9. *The operator* $\log(|x|)+\log(|y|)$ *is self-adjoint, left-bounded, commutes with the left- and right- dilations, commutes with the Inversion, and its spectral functions are the functions* $h_N(\tau)$. *The operator* $i\,[\log(|y|),\log(|x|)]$ *is bounded, self-adjoint, commutes with the left- and right- dilations, anti-commutes with the Inversion and its spectral functions are the functions* $k_N(\tau)$.

We now conclude this chapter with a study of some elementary distribution-theoretic properties of H. For this we need the analytic functions of s ($0<\mathrm{Re}(s)<1$) indexed by $N\in\mathbb{N}$:

$$H_N(s)=\frac{d}{ds}\log(\Gamma_N(s))$$

(so that $h_N(\tau)=H_N(\frac{1}{2}+i\tau)$).

Lemma 4.10. *Let* $\varphi(x)$ *be a Schwartz function on* $\mathbb{H}$. *Then* $H(\varphi)$ *is continuous on* $\mathbb{H}\backslash\{0\}$, *is* $O(\log(|x|))$ *for* $x\to 0$, *and is* $O(1/|x|)$ *for* $|x|\to\infty$. *Furthermore, for any* $F\in W_N$ *(constant along radial lines), the following identity holds for* $0<\mathrm{Re}(s)<1$:

$$\int H(\varphi)(x)F(x)|x|^{-s}dx=H_N(s)\int\varphi(x)F(x)|x|^{-s}dx.$$

Proof. Assuming the validity of the estimates we see that both sides of the identity are analytic functions of s, so it is enough to prove the identity on the critical line:

$$\int H(\varphi)(x)F(x)|x|^{-\frac{1}{2}-i\tau}dx=h_N(\tau)\int\varphi(x)F(x)|x|^{-\frac{1}{2}-i\tau}dx.$$

As in the proof of Lemma **3.17**, it is enough to prove it after integrating against an arbitrary Schwartz function $\psi(\tau)$. With $G(u)=\int_{-\infty}^{\infty}\psi(\tau)u^{-i\tau}\frac{d\tau}{2\pi}$, and using Theorem **4.9** this becomes

$$\int H(\varphi)(x)F(x)G(|x|)|x|^{-\frac{1}{2}}dx=\int\varphi(x)H(FG)(x)|x|^{-\frac{1}{2}}dx$$

(on the right-hand-side $H(FG)$ is computed in the multiplicative picture, on the left-hand-side $H(\varphi)$ is evaluated in the additive picture). The self-adjointness of H reduces this to $\overline{H(\overline{\varphi})} = H(\varphi)$, which is a valid identity.

For the proof of the estimates we observe that $B(\varphi)$ is the Fourier transform of an L^1-function hence is continuous, so that we only need to show that it is $O(1/|x|)$ for $|x| \to \infty$. For this we use that $B(\varphi)$ is additive convolution of $-\varphi$ with the distribution $G = \mathcal{F}(-\log(|y|))$. The estimate then follows from the formula for G given in the next lemma. $\square$

Lemma 4.11. *The distribution* $G(x) = \mathcal{F}(-\log(|y|))$ *is given as:*

$$G(\varphi) = \int_{|x|\le 1} (\varphi(x)-\varphi(0))\,\frac{dx}{2\pi^2\,|x|} + \int_{|x|>1} \varphi(x)\,\frac{dx}{2\pi^2\,|x|} + (4\log(2\pi)+4\gamma_e-2)\varphi(0).$$

Proof. We have used the notation $\gamma_e = -\Gamma'(1)$ for the Euler-Mascheroni's constant ($= 0.577\dots$). Let Δ_s for $\mathrm{Re}(s) > 0$ be the homogeneous distribution $|x|^{s-1}$ on $\mathbb{H}$. It is a tempered distribution. The formula

$$\Delta_s(\varphi) = \int_{|x|\le 1} (\varphi(x) - \varphi(0))|x|^{s-1}\,dx + \int_{|x|>1} \varphi(x)|x|^{s-1}\,dx + \frac{2\pi^2}{s}\varphi(0)$$

defines its analytic continuation to $\mathrm{Re}(s) > -\frac{1}{4}$, with a simple pole at $s = 0$. Using

$$\mathcal{F}(\Delta_s) = \Gamma_0(s)\Delta_{1-s}$$

for $s = 1 - \varepsilon$, $\varepsilon \to 0$, and expanding in ε gives

$$\begin{aligned}&\varphi(0) + \varepsilon G(\varphi) + O(\varepsilon^2) = \\ &\Gamma_0(1-\varepsilon)\cdot\left\{\frac{2\pi^2}{\varepsilon}\varphi(0) + \int_{|x|\le 1}(\varphi(x)-\varphi(0))\,\frac{dx}{|x|} + \int_{|x|>1}\varphi(x)\,\frac{dx}{|x|} + O(\varepsilon)\right\}.\end{aligned}$$

As $\Gamma_0(1-\varepsilon) = \frac{1}{2\pi^2}(\varepsilon + (4\log(2\pi) + 4\gamma_e - 2)\varepsilon^2 + O(\varepsilon^3))$ the result follows. $\square$

5. The trace of Connes for quaternions

Let $f(g)$ be a smooth function with compact support on $\mathbb{H}^\times$. Let U_f be the bounded operator $\int f(g)L_2(g)\,d^*g$ on $L^2(\mathbb{H}, dx)$ of left multiplicative convolution. So

$$U_f : \varphi(x) \mapsto \int_G f(g)\frac{1}{\sqrt{|g|}}\varphi(g^{-1}x)\,d^*g.$$

The composition $U_f\,\mathcal{F}$ of U_f with the Fourier Transform $\mathcal{F}$ acts as

$$\begin{aligned}
\varphi(x) \quad &\mapsto \quad \int_G \int_{\mathbb{H}} f(g)\frac{1}{\sqrt{|g|}}\lambda(-g^{-1}xy)\varphi(y)\,dy\,d^*g \\
&= \quad \int_{\mathbb{H}} \int_G f(\frac{1}{g})\sqrt{|g|}\lambda(-gxy)\,d^*g\,\varphi(y)\,dy \\
&= \quad \frac{1}{\sqrt{2\pi^2}}\int_{y\in\mathbb{H}} \left(\int_{Y\in\mathbb{H}} f(\frac{1}{Y})\frac{1}{\sqrt{2\pi^2|Y|}}\lambda(-Yxy)\,dY\right)\varphi(y)\,dy \\
&= \quad \frac{1}{\sqrt{2\pi^2}}\int_{\mathbb{H}} \mathcal{F}(I(f)_a)(xy)\varphi(y)\,dy.
\end{aligned}$$

In this last equation $I(f)_a$ is the additive representative $\frac{1}{\sqrt{2\pi^2|Y|}}f(\frac{1}{Y})$ of $I(f)$. Finally denoting similarly with $\Gamma(f)_a$ the additive representative of $\Gamma(f)$ we obtain

$$(U_f\mathcal{F})(\varphi)(x) = \frac{1}{\sqrt{2\pi^2}}\int_{\mathbb{H}} \Gamma(f)_a(xy)\varphi(y)\,dy.$$

As f has compact support on $\mathbb{H}^\times$ we note that $I(f)_a$ is smooth with compact support on $\mathbb{H}$ and that $\Gamma(f)_a$ belongs to the Schwartz class. Following Connes ([5], for $\mathbb{R}$ or $\mathbb{C}$ instead of $\mathbb{H}$), our goal is to compute the trace of the operator $\widetilde{P_\Lambda}P_\Lambda\,U_f$, where $\widetilde{P_\Lambda} = \mathcal{F}P_\Lambda\mathcal{F}^{-1}$ and P_Λ is the cut-off projection to functions with support in $|x| \leq \Lambda$. Our reference for trace-class operators will be [7]. We recall that if A is trace-class then for any bounded B, AB and BA are trace-class and have the same trace. Also if K_1 and K_2 are two Hilbert-Schmidt operators given for example as L^2-kernels $k_1(x,y)$ and $k_2(x,y)$ on a measure space (X,dx) then $A = K_1^*\,K_2$ is trace-class and its trace is the Hilbert-Schmidt scalar product of K_1 and K_2:

$$\mathbf{Tr}(K_1^*\,K_2) = \int\int \overline{k_1(x,y)}\,k_2(x,y)\;dx\,dy.$$

The operator $P_\Lambda\mathcal{F}^{-1}P_\Lambda$ is an operator with kernel a smooth function restricted to a finite box (precisely it is $\lambda(xy),\ |x|,|y| \leq \Lambda$). Such an operator is trace class, as is well-known (one classical line of reasoning is as follows: taking a smooth function $\rho(x)$ with compact support, identically 1 on $|x| \leq \Lambda$, and Q_ρ the multiplication operator with ρ, one has $P_\Lambda\mathcal{F}^{-1}P_\Lambda = P_\Lambda Q_\rho\mathcal{F}^{-1}Q_\rho P_\Lambda$, so that it is enough to prove that $Q_\rho\mathcal{F}^{-1}Q_\rho$ is trace-class. This operator has a smooth kernel with compact support, so we can put the system in a box, and reduce to an operator K with smooth kernel on a torus. Then $K = (1+\Delta)^{-n}(1+\Delta)^nK$ with Δ the positive Laplacian. For n large enough, $(1+\Delta)^{-n}$ is trace-class, while $(1+\Delta)^nK$ is at any rate bounded.)

So Connes's operator $\widetilde{P_\Lambda}P_\Lambda\,U_f = \mathcal{F}\cdot P_\Lambda\mathcal{F}^{-1}P_\Lambda\cdot U_f$ is indeed trace class and

$$\mathbf{Tr}(\widetilde{P_\Lambda}P_\Lambda\,U_f) = \mathbf{Tr}(P_\Lambda\mathcal{F}^{-1}P_\Lambda\cdot U_f\mathcal{F}) = \mathbf{Tr}(P_\Lambda\mathcal{F}^{-1}P_\Lambda\cdot P_\Lambda U_f\mathcal{F}P_\Lambda)$$

can be computed as a Hilbert-Schmidt scalar product:

$$\mathbf{Tr}(\widetilde{P_\Lambda} P_\Lambda\, U_f) = \frac{1}{\sqrt{2\pi^2}} \int\int_{|x|,|y|\leq\Lambda} \lambda(xy)\Gamma(f)_a(xy)\,dx\,dy,$$

using the change of variable $(x,y)\mapsto(Y=xy,y)$

$$\mathbf{Tr}(\widetilde{P_\Lambda} P_\Lambda\, U_f) = \sqrt{2\pi^2}\int_{|Y|\leq\Lambda^2}\lambda(Y)\Gamma(f)_a(Y)\left(\int_{\frac{|Y|}{\Lambda}\leq|y|\leq\Lambda}\frac{dy}{2\pi^2|y|}\right)dY$$

$$(\mathbf{C})\qquad \mathbf{Tr}(\widetilde{P_\Lambda} P_\Lambda\, U_f) = \sqrt{2\pi^2}\int_{|Y|\leq\Lambda^2}\Big(2\log(\Lambda)-\log(|Y|)\Big)\lambda(Y)\Gamma(f)_a(Y)\,dY$$

This integral is an inverse (additive) Fourier transform evaluated at 1. As $\Gamma = \mathcal{F}I$ itself involves a Fourier transform the final result is just $\sqrt{2\pi^2}M_\Lambda(I(f)_a)(1)$ where M_Λ is the self-adjoint operator $(2\log(\Lambda)-B)_+ = \max(2\log(\Lambda)-B,0)$. If we recall that $\sqrt{2\pi^2}$ is involved in the basic isometry from the additive to the multiplicative picture, we can finally express everything back in the multiplicative picture:

Theorem 5.1. *The Connes operator* $\widetilde{P_\Lambda} P_\Lambda\, U_f$ *is a trace-class operator and satisfies*

$$\begin{aligned}\mathbf{Tr}(\widetilde{P_\Lambda} P_\Lambda\, U_f) &= (2\log(\Lambda)-B)_+(I(f))(1)\\ \mathbf{Tr}(\widetilde{P_\Lambda} P_\Lambda\, U_f) &= 2\log(\Lambda)f(1)-H(f)(1)+o(1)\end{aligned}$$

For the last line we used that $B(I(f))(1) = H(I(f))(1) = H(f)(1)$ as $H = \log(|x|)+\log(|y|)$ commutes with the Inversion I. The error is $o(1)$ for $\Lambda\to\infty$ as it is bounded above in absolute value (assuming $\Lambda > 1$) by

$$\sqrt{2\pi^2}\int_{|Y|\geq\Lambda^2}\log(|Y|)\ |\Gamma(f)_a(Y)|\ dY$$

and $\Gamma(f)_a$ is a Schwartz function of $Y\in\mathbb{H}$. We note that if needed the Lemma **4.11** gives to the term $H(f)(1)$ a form more closely akin to the Weil's explicit formulae of number theory. We note that Connes's computation in [5] also goes through an intermediate stage essentially identical with **(C)** and that the identification of the constant term with Weil's expression for the explicit formula of number theory then requires a further discussion. The main result of [1] and of this paper is thus the direct connection between H and the logarithmic derivatives of the Tate Gamma functions involved in the explicit formulae.

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Université de Nice- Sophia Antipolis, Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice Cédex 02, France.

E-mail address: `burnol@math.unice.fr`