Mathematical Research Letters **8**, 347–359 (2001)

UNSTABLE COHEN–MACAULAY ALGEBRAS

Mara D. Neusel

Abstract. We characterize Cohen–Macaulay algebras in the category $\mathcal{K}_{fg}$ of unstable Noetherian algebras over the Steenrod algebra via the depth of the $\mathcal{P}^*$-invariant ideals. This allows us to transfer the Cohen–Macaulay property to suitable subalgebras. We apply this to rings of invariants of finite groups and to the $\mathcal{P}^*$-inseparable closure.

For over 50 years the Steenrod algebra $\mathcal{P}^*$ and algebras over it have proven to be decisive tools in algebraic topology. Applications occur in cohomology theory, invariant theory of finite groups, and most recently in algebraic geometry. This makes it desirable to have a completely algebraic (and *not* topological) approach to the subject.

In this paper we follow the path begun in [12], [13], [14] and [16] of building a "complete" $\mathcal{P}^*$-invariant commutative algebra. We find a $\mathcal{P}^*$-invariant version of the classical characterization of Cohen–Macaulay algebras. This allows us to show that Cohen–Macaulayness is inherited by subalgebras provided that the $\mathcal{P}^*$-invariant prime ideal spectra are in bijective correspondence and the extension is integral. This applies in particular to an unstable algebra and its $\mathcal{P}^*$-inseparable closure. It also applies to rings of polynomial invariants of finite groups.

For an algebraic introduction to the Steenrod algebra and algebras (and modules) over it we refer to the introduction of [13], Chapters 10 and 11 in [19], or Chapters 8 – 10 in [17].

1. Cohen–Macaulay algebras

In this section we characterize unstable Cohen–Macaulay algebras. We start with notation and a review of some terminology.

Let $\mathbb{F}$ be a Galois field of characteristic p with q elements. We denote by H^* a non–negatively graded, connected, commutative Noetherian $\mathbb{F}$-algebra. Since all objects are graded, all homomorphisms are degree preserving (unless otherwise stated) and all ideals are homogeneous. Note also that the augmentation ideal is the unique homogeneous maximal ideal in H^*. Recall that a graded algebra

Received October 2, 2000.

Keywords: Steenrod Algebra, Cohen–Macaulay Algebras, Unstable Algebras, $\mathcal{P}^*$-Invariant Prime Ideal Spectrum, $\mathcal{P}^*$-Inseparable Closure, Polynomial Invariants of Finite Groups

is *connected* if and only if it is equal to $\mathbb{F}$ in degree zero. Throughout the whole paper the Krull dimension of H^* is

$$\dim(H^*) = n.$$

Let H^* have the additional structure of an unstable algebra over the Steenrod algebra $\mathcal{P}^*$. Namely, H^* is a left $\mathcal{P}^*$-module satisfying the Cartan formulae

$$\mathcal{P}^i(hh') = \sum_{j+k=i} \mathcal{P}^j(h)\mathcal{P}^k(h') \quad \forall h,\ h' \in H^*,\ \forall i \geq 0,$$

and the unstability condition

$$\mathcal{P}^i(h) = \begin{cases} h^q & \text{if } i = \deg(h) \\ 0 & \text{if } i > \deg(h) \end{cases} \quad \forall h \in H^*.$$

Of particular interest are unstable Noetherian algebras over $\mathcal{P}^*$ that arise as rings of invariants in the following way. Let

$$\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$$

be a faithful representation of a finite group G over the field $\mathbb{F}$. The group G acts via ρ on the n-dimensional vector space $V = \mathbb{F}^n$. This induces an action of G on the ring $\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n]$ of polynomial functions in n variables via

$$gf(v) := f\left(\rho(g)^{-1}(v)\right) \quad \forall f \in \mathbb{F}[V],\ g \in G,\ v \in V.$$

The subring $\mathbb{F}[V]^G$ of polynomials invariant under this action is an unstable Noetherian algebra over the Steenrod algebra. See [17] or [19] for an introduction to invariant theory and the use of Steenrod technology there. Since the ground field is finite, the general linear group $\mathrm{GL}(n,\ \mathbb{F})$ is finite. Its ring of invariants (in the tautological representation) was determined by L.E. Dickson and is therefore called the *Dickson algebra*

$$\mathcal{D}^*(n) = \mathbb{F}[V]^{\mathrm{GL}(n,\ \mathbb{F})} = \mathbb{F}[\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,n-1}].$$

It is a polynomial algebra generated by the *Dickson classes* $\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,n-1}$ (see Section 8.1 in [19], or the original [7]). The Dickson algebra plays a central role in the invariant theory of finite groups, where it provides us with a universal set of invariants present in every ring of invariants.

Even more crucial is its role in the entire category of unstable Noetherian algebras over $\mathcal{P}^*$. By the imbedding theorem, Theorem 8.1.5 in [13], sufficiently high powers of the Dickson classes provide us with a universal system of parameters for every unstable Noetherian algebra over $\mathcal{P}^*$. We will use this fact frequently in the following.

Recall that the *height of an ideal* I is defined as

$$\mathrm{ht}\,(I) = \min\{\mathrm{ht}\,(\mathfrak{p}) : I \subseteq \mathfrak{p},\ \mathfrak{p} \subset H^*\text{prime}\}.$$

We denote by $\mathrm{Proj}\,(H^*)$ the spectrum of homogeneous prime ideals in H^*, and by $\mathrm{Proj}_{\mathcal{P}^*}(H^*)$ the spectrum of $\mathcal{P}^*$-invariant homogeneous prime ideals, where an ideal is called $\mathcal{P}^*$-*invariant* if it is stable under the action of the Steenrod algebra $\mathcal{P}^*$.

In classical terms we can characterize Cohen–Macaulay algebras by looking at the depth of the ideals in the algebra in question. A sequence $h_1, \dots, h_k \in H^*$ is called a *regular sequence* if

$$h_1 \in H^*$$

and

$$\overline{h_i} \in H^*/(h_1, \dots, h_{i-1}) \quad \forall i = 2, \dots, k$$

are nonzero divisors, where

$$\overline{h_i} = \mathrm{pr}_i(h_i)$$

is the image under the canonical projection

$$\mathrm{pr}_i : H^* \longrightarrow H^*/(h_1, \dots, h_{i-1}).$$

The *depth of an ideal* I is the length of a maximal regular sequence with members in I. Recall that for every ideal I,

$$\mathrm{dp}\,(I) \le \mathrm{ht}\,(I).$$

We have equality for all ideals I if and only if the algebra is Cohen–Macaulay (see Theorem 3.3.2 in [2]).

Proposition 1.1. *Let H^* be an unstable Noetherian algebra over the Steenrod algebra $\mathcal{P}^*$. The following statements are equivalent:*

(1) *H^* is Cohen–Macaulay.*
(2) $\mathrm{dp}\,(\mathfrak{m}) = \mathrm{ht}\,(\mathfrak{m})$, *where $\mathfrak{m} \subset H^*$ is the augmentation ideal.*
(3) $\mathrm{dp}\,(\mathfrak{p}) = \mathrm{ht}\,(\mathfrak{p})$ *for every $\mathcal{P}^*$-invariant prime ideal $\mathfrak{p} \subset H^*$.*
(4) $\mathrm{dp}\,(I) = \mathrm{ht}\,(I)$ *for every $\mathcal{P}^*$-invariant ideal $I \subset H^*$.*
(5) *Every $\mathcal{P}^*$-invariant ideal $I \subseteq H^*$ of height k contains*

$$\mathbf{d}_{n,0}^{q^t}, \dots, \mathbf{d}_{n,k-1}^{q^t} \in I$$

for some suitably large $t \in \mathbb{N}$, and these elements form a regular sequence of maximal length in I.

Proof. Since the maximal ideal $\mathfrak{m} \subset H^*$ is $\mathcal{P}^*$-invariant, the implications

$$(4) \Rightarrow (3) \Rightarrow (2)$$

are obvious. The implication

$$(2) \Rightarrow (1)$$

follows from the corresponding classical result (cf. Theorem 3.3.2 in [2]) that a ring is Cohen–Macaulay if the height of every maximal ideal is equal to its depth. Since we are in the graded situation, we have precisely one maximal ideal, namely the augmentation ideal.

The implication

$$(5) \Rightarrow (4)$$

is clear. It remains only to show the implication (1) $\Rightarrow$ (5). To this end assume that H^* is Cohen–Macaulay. Since H^* is Noetherian, it contains a (q^s)th power of the Dickson algebra for some $s \geq 0$ such that

$$\mathcal{D}^*(n)^{q^s} = \mathbb{F}[\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,n-1}^{q^s}] \hookrightarrow H^*$$

is an integral extension. This follows from the imbedding theorem (Theorem 8.1.5 in [13]). By Macaulay's theorem (Theorem 3.3.5 (vi) in [2]) the powers of the Dickson classes

$$\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,n-1}^{q^s} \in H^*$$

form a homogeneous system of parameters and hence a regular sequence of maximal length in H^*. By Theorem 8.3.4 in [13] every $\mathcal{P}^*$-invariant prime ideal $\mathfrak{p} \subset H^*$ of height k contains a $\mathcal{P}^*$-invariant ideal of height k generated by k elements. By the imbedding theorem (Theorem 8.1.5 in [13]) we can choose these k elements to be the fractals of the top k Dickson classes

$$\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,k-1}^{q^s}.$$

Recall that a $\mathcal{P}^*$-invariant ideal $I \subset H^*$ has a primary decomposition

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_l,$$

where every primary ideal $\mathfrak{q}_i$, $i = 1, \ldots, l$, as well as its radical $\sqrt{\mathfrak{q}_i}$ is $\mathcal{P}^*$-invariant (see Theorem 3.5 in [16]). By what we have seen above, we know that

$$\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,k-1}^{q^s} \in \sqrt{I}.$$

Therefore

$$\mathbf{d}_{n,0}^{q^t}, \ldots, \mathbf{d}_{n,k-1}^{q^t} \in I \subset H^*$$

for some $t \geq s$, and this is a regular sequence because $\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,k-1}^{q^s}$ is a regular sequence (see Exercise 12 of Section 3.1 in [2]). □

Remark. Note carefully that we can reformulate conditions (2) and (3) as follows:

(2') The localization $H^*_{\mathfrak{m}}$ at the maximal ideal is a Cohen–Macaulay ring.
(3') The localization $H^*_{\mathfrak{p}}$ at a $\mathcal{P}^*$-invariant prime ideal is a Cohen–Macaulay ring for every $\mathfrak{p} \in \mathrm{Proj}_{\mathcal{P}^*}(H^*)$.

These localizations inherit naturally an action of the Steenrod algebra, but they are no longer unstable (cf. [20], [14]).

2. Integral extensions

Let

$$\phi : H^* \hookrightarrow \mathrm{K}^*$$

be an integral extension of unstable Noetherian algebras over the Steenrod algebra $\mathcal{P}^*$. This map induces a map between the homogeneous prime ideal spectra

$$\phi^* : \mathrm{Proj}\,(\mathrm{K}^*) \longrightarrow \mathrm{Proj}\,(H^*),\ \mathfrak{p} \mapsto \mathfrak{p} \cap H^*.$$

If we restrict this map to the subset of $\mathcal{P}^*$-invariant homogeneous prime ideals, we get

$$\phi^* \mid_{\mathcal{P}^*} : \mathrm{Proj}_{\mathcal{P}^*}(\mathrm{K}^*) \longrightarrow \mathrm{Proj}_{\mathcal{P}^*}(H^*)$$

because the contraction of a $\mathcal{P}^*$-invariant prime ideal is $\mathcal{P}^*$-invariant (see Lemma 2.1 in [12]). Note that

$$(\phi^*)^{-1}\,(\mathrm{Proj}_{\mathcal{P}^*}(H^*)) = \mathrm{Proj}_{\mathcal{P}^*}(\mathrm{K}^*)$$

is always true by Theorem 2.3 in [12]. In particular this means that $\phi^* \mid_{\mathcal{P}^*}$ is always surjective.

Theorem 2.1. *Keeping the above notation we assume that the induced map $\phi^* \mid_{\mathcal{P}^*}$ is bijective. If K^* is Cohen–Macaulay then so is H^*.*

Proof. By the imbedding theorem (Theorem 8.1.5 in [13]) we find a fractal of the Dickson algebra inside H^* such that

$$\mathcal{D}^*(n)^{q^s} = \mathbb{F}[\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,n-1}^{q^s}] \hookrightarrow H^*$$

is an integral extension. By the characterization of unstable Cohen–Macaulay algebras given in Proposition 1.1 (5) it is enough to show that for every $\mathcal{P}^*$-invariant prime ideal $\mathfrak{q} \subset H^*$ of height $\mathrm{ht}\,(\mathfrak{q}) = k$, we have a regular sequence

$$\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,k-1}^{q^s} \in \mathfrak{q}.$$

We proceed by induction on $\mathrm{ht}(\mathfrak{q})$.

If $\mathrm{ht}(\mathfrak{q}) = 0$ then

$$0 \leq \mathrm{dp}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{q}) = 0,$$

and there is nothing to show.

However, we need to start our induction with $\mathrm{ht}(\mathfrak{q}) = 1$. We have

$$\begin{array}{ccccc} \mathcal{D}^*(n)^{q^s} & \hookrightarrow & H^* & \hookrightarrow & \mathrm{K}^* \\ \cup & & \cup & & \cup \\ \mathfrak{q} \cap \mathcal{D}^*(n)^{q^s} & \leftrightarrow & \mathfrak{q} & \leftrightarrow & \mathfrak{q}^e, \end{array}$$

where $(-)^e$ denotes the extended ideal. The ideal $\mathfrak{q} \cap \mathcal{D}^*(n)^{q^s} \subseteq \mathcal{D}^*(n)^{q^s}$ is by construction $\mathcal{P}^*$-invariant and prime of height 1. Therefore

$$\mathfrak{q} \cap \mathcal{D}^*(n)^{q^s} = \left(\mathbf{d}_{n,0}^{q^s}\right)$$

by Theorem 9.2.5 in [17]. In other words,

$$\mathbf{d}_{n,0}^{q^s} \in \mathfrak{q} \subseteq \mathfrak{q}^e.$$

Since K^* is Cohen–Macaulay, the element $\mathbf{d}_{n,0}^{q^s}$ is a nonzero divisor in K^*, so it is not a zero divisor in H^*.

Assume that $\mathrm{ht}(\mathfrak{q}) = k > 1$. First we note that there exists precisely one prime ideal $\mathfrak{p} \subset \mathrm{K}^*$ such that

$$\mathfrak{q} = \mathfrak{p} \cap H^*.$$

Moreover, $\mathfrak{p}$ is also $\mathcal{P}^*$-invariant of height k, and

$$\mathfrak{q}^e \subseteq \sqrt{\mathfrak{q}^e} = \mathfrak{p}$$

by Theorem 2.3 in [12]. By Proposition 8.3.3 in [13] there exists a $\mathcal{P}^*$-invariant prime ideal $\mathfrak{q}' \subset \mathfrak{q}$ of height $k-1$. Hence, there is precisely one prime ideal $\mathfrak{p}' \subset \mathrm{K}^*$ of height $k-1$ such that

$$\mathfrak{q}' = \mathfrak{p}' \cap H^*.$$

This ideal $\mathfrak{p}'$ is also $\mathcal{P}^*$-invariant and we have

$$\mathfrak{q}'^e \subseteq \sqrt{\mathfrak{q}'^e} = \mathfrak{p}'.$$

By induction, we know that $\mathfrak{q}'$ has depth $k-1$, and we may choose the fractals

$$\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,k-2}^{q^s} \in \mathfrak{q}'$$

of the top $k-1$ Dickson classes as a regular sequence of maximal length in $\mathfrak{q}'$. Note that

$$\mathrm{ht}(\mathbf{d}_{n,0}^{q^s}, \ldots, \mathbf{d}_{n,k-2}^{q^s}) = \mathrm{ht}(\mathfrak{q}') = k-1.$$

Therefore also all ideals in the chain[1]

$$\left(\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s}\right)^e = \left(\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s}\right)_{\mathrm{K}^*} \subseteq \mathfrak{q}'^e \subseteq \sqrt{\mathfrak{q}'^e} = \mathfrak{p}'$$

have height $k-1$. Since K^* is Cohen–Macaulay, they also have depth $k-1$. In particular, the sequence

$$\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s} \in \mathfrak{p}'$$

is regular of maximal length for the prime ideal $\mathfrak{p}'$. We want to extend the regular sequence $\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s} \in \mathfrak{q}$ by another element h_k. For that we assume to the contrary that every element in $\mathfrak{q} \setminus (\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})$ is a zero divisor modulo $(\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})$. Then $\mathfrak{q}$ is contained in an associated prime ideal of $(\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})$. In other words, there exists an element $h \in H^*$ such that

$$\mathfrak{q} \subseteq \left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s}) : h\right).$$

Moreover,

$$\mathfrak{q}^e \subseteq \left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s}) : h\right)^e \subseteq \left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e : h\right).$$

So, if h is a zero divisor in $\mathrm{K}^* \big/ (\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e$ then

$$\mathrm{ht}\left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e : h\right) = k-1$$

because $(\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e = (\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})_{\mathrm{K}^*} \subset \mathrm{K}^*$ is generated by a regular sequence and is therefore height unmixed by Macaulay's unmixedness theorem (see Definition 3.3.1 in [2]). However, the formula

$$k = \mathrm{ht}\,(\mathfrak{q}^e) \leq \mathrm{ht}\left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e : h\right) = k-1$$

shows that this is nonsense. On the other hand, if h is not a zero divisor in $\mathrm{K}^* \big/ (\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e$, then

$$\left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e : h\right) = \left(\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s}\right)_{\mathrm{K}^*}.$$

Again, we get a contradiction by comparing heights

$$k = \mathrm{ht}\,(\mathfrak{q}^e) \leq \mathrm{ht}\left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e : h\right) = \mathrm{ht}\left((\mathbf{d}_{n,0}^{q^s}, \dots, \mathbf{d}_{n,k-2}^{q^s})^e\right) = k-1.$$

[1]The notation $(-)_{\mathrm{K}^*}$ emphasizes that we are looking at an ideal in K^*.

Hence
$$\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-2}^{q^s} \in \mathfrak{q}'$$
is a regular sequence that can be extended to a regular sequence
$$\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-2}^{q^s}, h_k \in \mathfrak{q}$$
of length k in $\mathfrak{q}$. Since this is possible for every prime ideal $\mathfrak{q}$ of height k, this means that no prime ideal of height k is contained in a prime ideal associated to
$$(\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-2}^{q^s})_{H^*}.$$
I.e., the ideal $(\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-2}^{q^s})_{H^*}$ is height unmixed and every associated prime ideal has height $k-1$. Finally we need to show that we can choose
$$h_k = \mathbf{d}_{n,k-1}^{q^s}.$$
To this end observe that the ideal
$$(\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-1}^{q^s}) \subseteq H^*$$
has height k because $\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-1}^{q^s}$ is part of a homogeneous system of parameters for H^* by the imbedding theorem (Theorem 8.1.5 in [13]). If $\mathbf{d}_{n,k-1}^{q^s}$ were a zero divisor modulo $(\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-2}^{q^s})$ in H^*, then this element would be contained in an associated prime ideal. So there is an associated prime ideal of $(\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-2}^{q^s})_{H^*}$ of height at least
$$\operatorname{ht}(\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-1}^{q^s}) = k-1.$$
However, this contradicts that the ideal $(\mathbf{d}_{n,0}^{q^s}, \dots , \mathbf{d}_{n,k-2}^{q^s})_{H^*}$ is height unmixed. Hence we can choose $h_k = \mathbf{d}_{n,k-1}^{q^s}$. This completes the induction step. □

Remark. The converse of the preceding theorem is false. For example, choose a polynomial ring $\mathbb{F}[x]$ in one linear variable over a finite field $\mathbb{F}$. This is a unstable Cohen–Macaulay algebra. By Proposition 11.4.1 in [19] there are exactly two $\mathcal{P}^*$-invariant prime ideals
$$(0) \subsetneq (x) \subsetneq \mathbb{F}[x].$$
Let $\mathbb{F}[x,\ y]$ be the polynomial algebra in two linear variables, and consider its quotient by the ideal $(xy,\ y^q)$. Since this ideal is $\mathcal{P}^*$-invariant we get an integral extension
$$\phi : \mathbb{F}[x] \hookrightarrow \mathbb{F}[x,\ y]/(xy,\ y^q)$$
in the category of unstable algebras. Note that the longest chain of $\mathcal{P}^*$-invariant prime ideals in the bigger algebra is
$$(y) \subsetneq (y,\ x) \subsetneq \mathbb{F}[x,\ y]/(xy,\ y^q).$$

Hence $\phi^* \mid_{\mathcal{P}^*}$ is bijective. However, the polynomial algebra $\mathbb{F}[x]$ is Cohen–Macaulay, but the bigger algebra is not. In fact, it has depth zero since every element is a zero divisor.

Recall that the Steenrod algebra contains an infinite family of derivations inductively defined by

$$\mathcal{P}^{\Delta_1} = \mathcal{P}^1 \quad \text{and} \quad \mathcal{P}^{\Delta_i} = [\mathcal{P}^{\Delta_{i-1}},\ \mathcal{P}^{q^{i-1}}] \quad \forall i \geq 2.$$

Set $\mathcal{P}^{\Delta_0}(h) = \deg(h)h$. Then an unstable algebra H^* is called $\mathcal{P}^*$-*inseparably closed* if there exists a pth root of h in H^* whenever

$$\mathcal{P}^{\Delta_i}(h) = 0 \quad \forall i \geq 0.$$

Recall that a pth root of h is an element $h' \in H^*$ such that

$$h'^p = h.$$

The $\mathcal{P}^*$-*inseparable closure* $\sqrt[\mathcal{P}^*]{\mathrm{H}^*}$ of an unstable algebra H^* is then the smallest unstable algebra containing H^* that is $\mathcal{P}^*$-inseparably closed. See Chapter 4 of [13] for an introduction to this notion and its basic properties.

Corollary 2.2. *Let H^* be reduced and Noetherian.*[2] *If $\sqrt[\mathcal{P}^*]{\mathrm{H}^*}$ is Cohen–Macaulay, then so is H^*.*

Proof. Let $\sqrt[\mathcal{P}^*]{\mathrm{H}^*}$ be Cohen–Macaulay. By Proposition 4.2.1 (4) in [13]

$$\phi : H^* \hookrightarrow \sqrt[\mathcal{P}^*]{\mathrm{H}^*} H^*$$

is an integral extension. By Theorem 4.3.1 in [13] it induces a bijection of the respective prime ideal spectra. By Theorem 2.3 in [12] $\mathcal{P}^*$-invariant prime ideals correspond to $\mathcal{P}^*$-invariant prime ideals, so we apply the preceding result and conclude that H^* is Cohen–Macaulay. □

We illustrate these results with examples from invariant theory.

Example 2.3. Let $\rho : G \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})$ be a faithful representation of a finite group G containing the special linear group $\mathrm{SL}(n,\ \mathbb{F})$. Then we have integral extensions

$$\phi : \mathcal{D}^*(n) = \mathbb{F}[V]^{\mathrm{GL}(n,\ \mathbb{F})} \hookrightarrow \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V]^{\mathrm{SL}(n,\ \mathbb{F})}.$$

By Corollary 9.2.4 in [17] the $\mathcal{P}^*$-invariant prime ideal spectrum of the Dickson algebra $\mathcal{D}^*(n)$ forms a single chain

$$(\star) \qquad (0) \subset (\mathbf{d}_{n,0}) \subset (\mathbf{d}_{n,0},\ \mathbf{d}_{n,1}) \subset \cdots \subset (\mathbf{d}_{n,0}, \ldots, \mathbf{d}_{n,n-1}).$$

[2]The $\mathcal{P}^*$-inseparable closure does not exist for algebras with nilpotent elements. The statement would not make sense without this assumption. In contrast, $\sqrt[\mathcal{P}^*]{\mathrm{H}^*}$ does make sense for non–Noetherian algebras. Indeed, $\sqrt[\mathcal{P}^*]{\mathrm{H}^*}$ is Noetherian if and only if H^* is Noetherian by Theorem 6.3.1 in [13].

The invariants of the special linear group are given by

$$\mathbb{F}[\mathbf{E},\ \mathbf{d}_{n,1}, \dots , \mathbf{d}_{n.n-1}],$$

where $\mathbf{E}^{q-1} = \mathbf{d}_{n,0}$ is the Euler class of the $\mathrm{SL}(n,\ \mathbb{F})$-action on the dual vector space V^* (see Section 7.2 in [15] or Theorem 8.1.8 in [18]). The $\mathcal{P}^*$-invariant prime ideal spectrum of $\mathbb{F}[V]^{\mathrm{SL}(n,\ \mathbb{F})}$ forms also just a single chain

$$(0) \subset (\mathbf{E}) \subset (\mathbf{E},\ \mathbf{d}_{n,1}) \subset \cdots \subset (\mathbf{E}, \mathbf{d}_{n,1}, \dots , \mathbf{d}_{n,n-1})$$

since these are precisely the prime ideals lying over those of the chain $(\star)$ (cf. Example 1 of Section 2 in [12]). Therefore, ϕ induces a bijection

$$\phi^* \mid_{\mathcal{P}^*} : \mathrm{Proj}_{\mathcal{P}^*}\left(\mathbb{F}[V]^{\mathrm{SL}(n,\ \mathbb{F})}\right) \longrightarrow \mathrm{Proj}_{\mathcal{P}^*}\left(\mathcal{D}^*(n)\right)$$

on the $\mathcal{P}^*$-invariant prime ideal spectra. Hence also

$$\phi^* \mid_{\mathcal{P}^*} : \mathrm{Proj}_{\mathcal{P}^*}\left(\mathbb{F}[V]^{\mathrm{SL}(n,\ \mathbb{F})}\right) \longrightarrow \mathrm{Proj}_{\mathcal{P}^*}\left(\mathbb{F}[V]^{G}\right)$$

is a bijection. Since the ring of invariants of the special linear group is Cohen–Macaulay, the preceding result tells us that $\mathbb{F}[V]^G$ is also Cohen–Macaulay. Now $\mathrm{Syl}_p(G) = \mathrm{Syl}_p(\mathrm{GL}\,(n, \mathbb{F}))$ has Cohen–Macaulay invariants by a theorem of M.–J. Bertin [4], so G also has Cohen–Macaulay invariants by Theorem 5.6.7 in [17]. As we explain next, this is just a special case of a more general phenomenon.

Let

$$\rho : G \hookrightarrow \mathrm{GL}\,(n, \mathbb{F})$$

be a faithful representation of a finite group G. Take a primitive $(q-1)$st root of unity η in $\mathbb{F}$, where $|\mathbb{F}| = q$. Then the diagonal matrix

$$\mathbf{D} = \mathrm{diag}\,(\eta, \dots , \eta)$$

generates a cyclic group $\mathbb{Z}/(q-1)$ of order $q-1$ in $\mathrm{GL}\,(n, \mathbb{F})$. We consider the invariants of the group $G \circ \mathbb{Z}/(q-1) \subset \mathrm{GL}\,(n, \mathbb{F})$ generated by G and $\mathbf{D}$. We have integral extensions

$$\mathbb{F}[V]^{G\circ\mathbb{Z}/(q-1)} \hookrightarrow \mathbb{F}[V]^G \hookrightarrow \mathbb{F}[V].$$

Let $\mathfrak{p} \in \mathrm{Proj}_{\mathcal{P}^*}(\mathbb{F}[V]^G)$. Then the group G acts transitively on the set of prime ideals $\mathfrak{p}_1, \dots , \mathfrak{p}_k$ in $\mathbb{F}[V]$ lying over $\mathfrak{p}$. These ideals are again $\mathcal{P}^*$-invariant [12]. By [18] the prime ideals $\mathfrak{p}_1, \dots , \mathfrak{p}_k$ are generated by linear forms, and moreover, every ideal generated by linear forms is $\mathcal{P}^*$-invariant and certainly prime. The affine variety associated to such a prime ideal is therefore just a linear subspace $U_i \subseteq V$ of dimension equal to $n - \mathrm{ht}\,(\mathfrak{p})$ for $i = 1, \dots , k$. This means that the orbit structure of the action of G on the Grassmann varieties of V characterizes the structure of $\operatorname{Proj}_{\mathcal{P}^*}(\mathbb{F}[V]^G)$ and vice versa (cf. Chapter 9 in [17]). By construction the orbit structure of the $G\circ\mathbb{Z}/(q-1)$-action on the Grassmannians of V is the same as the one of G. Therefore we find that

$$\operatorname{Proj}_{\mathcal{P}^*}(\mathbb{F}[V]^G) \cong \operatorname{Proj}_{\mathcal{P}^*}(\mathbb{F}[V]^{G\circ\mathbb{Z}/(q-1)}).$$

More generally, denote by $\Gamma(G)$ the maximal subgroup of $\mathrm{GL}\,(n,\mathbb{F})$ containing G such that the orbit structure on the Grassmannians of V is the same as for G.[3] Then with the preceding argumentation we find that

$$\operatorname{Proj}_{\mathcal{P}^*}(\mathbb{F}[V]^G) \cong \operatorname{Proj}_{\mathcal{P}^*}(\mathbb{F}[V]^{\Gamma(G)}).$$

Hence we have proven the following proposition.

Proposition 2.4. *If $\mathbb{F}[V]^G$ is Cohen–Macaulay, then so is $\mathbb{F}[V]^H$ for every group H with $G \subseteq H \subseteq \Gamma(G)$.* □

The above Example 2.3 follows from this result with the observation that $\Gamma(\mathrm{SL}(n,\mathbb{F})) = \mathrm{GL}\,(n,\mathbb{F})$. Another pair of groups that fall into this class are the symplectic group and the general symplectic group.

Example 2.5. Let $\mathrm{Sp}(2l,\mathbb{F})$ be the symplectic group with $\mathbb{F}$ a field of odd characteristic. Choose a symplectic basis $e_1,\dots,e_l,\ f_1,\dots,f_l$ such that

$$f(e_i,e_j) = 0 = f(f_i,f_j), \quad f(e_i,f_j) = \delta_{ij}$$

for a symplectic form f. Then the general symplectic group $\Gamma(\mathrm{Sp}(2l,\mathbb{F}))$ is generated by $\mathrm{Sp}(2l,\mathbb{F})$ and diagonal matrices

$$\operatorname{diag}(\underset{\longleftarrow l \longrightarrow}{\eta,\dots,\eta},\ \underset{\longleftarrow l \longrightarrow}{1,\dots,1}),$$

where $\eta \in \mathbb{F}^\times$. Note that while the symplectic group is the group of f-isometries, the general symplectic group is the group of f-similarities (cf. Section 2.4 in [10] or the classic [8] Chapter II).[4] In any case, both groups have the same orbit structure when acting on the Grassmannians of V. Since the invariants of $\mathrm{Sp}(2l,\mathbb{F})$ form a complete intersection ([6], Section 8.2 in [3], or [11]) they are Cohen–Macaulay. Hence so are the invariants of any group between $\mathrm{Sp}(2l,\mathbb{F})$ and $\Gamma(\mathrm{Sp}(2l,\mathbb{F}))$.

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[3] So $\Gamma(G)$ is in many cases a subgroup of the group generated by G and all diagonal matrices of $\mathrm{GL}\,(n,\mathbb{F})$. However, this rule of thumb can be very misleading. For example, take the field $\mathbb{F} = \mathbb{F}_3$ with 3 elements. If we restrict our attention to $n = 2$, then we find a quaternion group Q_8 in $\mathrm{SL}(2,\ \mathbb{F}_3)$, which acts transivitely on the hyperplanes. This is the only Grassmannian of interest (see Example 1 of Section 2 in [12]). In oher words, $\Gamma(Q_8) = \mathrm{GL}\,(2,\mathbb{F}_3)$.

[4] L.E. Dickson calls these groups the special abelian linear group $SA(2l,q)$ and the general abelian linear group $GA(2l,q)$.

Of course, as in Example 2.3 the index of the symplectic group $\mathrm{Sp}(2l,\mathbb{F})$ in $\Gamma(\mathrm{Sp}(2l,\mathbb{F}))$ is prime to the characteristic, so we could lift the Cohen–Macaulayness of the invariants of the smaller group to the invariants of the bigger group by observing that the relative transfer homomorphism gives us a splitting of the natural inclusion (Theorem 5.6.7 in [17]). In order to find an example such that we can't derive Cohen–Macaulayness from older methods (cf. Section 5.5 in [17]) we have to find a group $G \subseteq \mathrm{GL}\,(n,\mathbb{F})$ such that the index $|\,\Gamma(G) : G|$ is divisible by the characteristic and the dimension n is at least 4 (cf. Proposition 5.6.9 in [17]).

Example 2.6. We consider groups G that act transitively on all Grassmannians of V. This is the simplest possible orbit structure. Namely, $G \subseteq \mathrm{GL}\,(n,\mathbb{F})$ with $\Gamma(G) = \mathrm{GL}\,(n,\mathbb{F})$. In Proposition 8.4 of [5] we find a complete list of such groups consisting of three types:

(1) $\mathrm{SL}(n,\mathbb{F}) \subseteq G$. This is our Example 2.3.

(2) $\rho : G = A_7 \hookrightarrow \mathrm{GL}\,(4,\mathbb{F}_2)$. The explicit embedding of the alternating group A_7 into $\mathrm{GL}\,(4,\mathbb{F}_2)$ can be found in Theorem 268 of [8] or in Satz II 2.5 of [9]. The ring of invariants $\mathbb{F}[V]^{A_7}$ is Cohen–Macaulay by Theorem 3.13 in [1]. However, we can't derive anything new from this because the embedding ρ is just the natural embedding of A_7 into $A_8 \cong \mathrm{GL}\,(4,\mathbb{F}_2)$.

(3) $\rho : G = \Gamma(1,\mathbb{F}_{2^5}) \hookrightarrow \mathrm{GL}\,(5,\mathbb{F}_2)$. The general semilinear group $\Gamma(1,\mathbb{F}_{2^5})$ admits a natural embedding into $\mathrm{GL}\,(5,\mathbb{F}_2)$ (see Section 2.1 in [10]). Since $\Gamma(1,\mathbb{F}_{2^5})$ has order $31 \cdot 5$, which is not divisible by 2, the ring of invariants $\mathbb{F}[V]^{\Gamma(1,\mathbb{F}_{2^5})}$ is Cohen–Macaulay by Theorem 5.5.2 in [17]. By Proposition 8.4 in [5]

$$\Gamma\,(\Gamma(1,\mathbb{F}_{2^5})) = \mathrm{GL}\,(5,\mathbb{F}_2),$$

so all groups between them have Cohen–Macaulay invariants. Note that the index $|\,\mathrm{GL}\,(5,\mathbb{F}_2) : \Gamma\,(\Gamma(1,\mathbb{F}_{2^5}))\,|$ is divisible by the characteristic. Finally, we can also describe the image of $\Gamma(1,\mathbb{F}_{2^5})$ in $\mathrm{GL}\,(5,\mathbb{F}_2)$ as the normalizer subgroup of a Singer cycle in $\mathrm{GL}\,(5,\mathbb{F}_2)$ (cf. Satz II 7.3 in [9]).

3. Acknowledgments

I thank Dan Isaksen for proofreading and the referee for reference [1].

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Department of Mathematics, University of Notre Dame, 370 CCMB, Notre Dame, IN 46556-5683.
E-mail address: `Neusel.1@nd.edu`