Mathematical Research Letters **8**, 713–728 (2001)

ON A CONJECTURE OF KASHIWARA

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To Yu.I.Manin on his 65th birthday

1. Introduction

1.1. There is a number of deep results on irreducible perverse sheaves of geometric origin on varieties over $\mathbb{C}$ proved by reduction modulo p and a purity argument (see, e.g., §6.2 of[BBD]). We give a general method to deduce them *without the geometric origin assumption* from a plausible conjecture by A.J. de Jong on local systems modulo l on varieties over a finite field of characteristic $p \neq l$. To demonstrate the method we deduce from de Jong's conjecture the following surprising conjecture formulated by Kashiwara [K].

1.2. Conjecture $\mathbf{Kash}_{\mathrm{top}}(\mathbb{C})$. Suppose that "algebraic variety" means "algebraic variety over $\mathbb{C}$" and "perverse sheaf on X" means "perverse sheaf of F-vector spaces on $X(\mathbb{C})$ equipped with the usual topology, which is constructible in the Zariski sense", where F is a field of characteristic 0.

1. Let $\pi : X \to Y$ be a proper morphism of algebraic varieties and M a semisimple perverse sheaf on X. Then the complex $\pi_*(M)$ is "semisimple", i.e., isomorphic to a direct sum of complexes of the form $N_k[k]$ where the N_k's are semisimple perverse sheaves.

2. In the above situation the hard Lefschetz theorem holds, i.e., if $u \in H^2(X, \mathbb{Z})$ is the class of a relatively ample line bundle on X then multiplication by u^k, $k > 0$, induces an isomorphism $H^{-k}\pi_*(M) \to H^k\pi_*(M)$.

3. Let M be a semisimple perverse sheaf on an algebraic variety X. Let f be a regular function on X. Denote by Ψ_f the corresponding nearby cycle functor. Let W denote the monodromy filtration of $\Psi_f(M)$ (i.e., the unique exhaustive increasing filtration with the following property: if $T_u \in \operatorname{Aut}\Psi_f(M)$ is the unipotent automorphism from the Jordan decomposition of the monodromy $T \in \operatorname{Aut}\Psi_f(M)$ and $N := T_u - \mathrm{id}$ then $NW_k\Psi_f(M) \subset W_{k-2}\Psi_f(M)$ and the morphisms $N^k : \mathrm{gr}_k^W(\Psi_f(M)) \to \mathrm{gr}_{-k}^W(\Psi_f(M))$ are isomorphisms). Then the perverse sheaf $\mathrm{gr}^W(\Psi_f(M))$ is semisimple.

Remark. Statement 3 implies the semisimplicity of $\mathrm{gr}^W(\Phi_f(M))$, where Φ_f is the vanishing cycle functor. Indeed, if M is irreducible and $\operatorname{Supp} M \not\subset$

Received August 21, 2001.
Partially supported by NSF grant DMS-0100108.

$\{x \in X | f(x) = 0\}$ then $(\Phi_f(M), N) \simeq \mathrm{Im}(T - \mathrm{id} : \Psi_f(M) \to \Psi_f(M))$, so $\mathrm{gr}^W(\Phi_f(M))$ considered just as a perverse sheaf (with forgotten grading) is isomorphic to a direct summand of $\mathrm{gr}^W(\Psi_f(M))$.

1.3. Beilinson, Bernstein, Deligne, and Gabber proved that the above conjecture holds if M "has geometric origin" (see [BBD], §§6.2.4 – 6.2.10 for the proof of statements 1 and 2 in this case). We will show that it holds if the rank of each irreducible component of M on its smoothness locus is ≤ 2, and in the general case it would follow from a plausible conjecture formulated by de Jong [dJ]. We need the following particular case of de Jong's conjecture.

Conjecture dJ(n). Let X be an absolutely irreducible normal scheme over a finite filed $\mathbb{F}_q$. Let $\mathbb{F}$ be a finite field such that $\mathrm{char}\,\mathbb{F} \neq \mathrm{char}\,\mathbb{F}_q$. Suppose that $\rho_0 : \pi_1(X \otimes \bar{\mathbb{F}}_q) \to GL(n, \mathbb{F})$ is an absolutely irreducible representation and $\rho_t : \pi_1(X \otimes \bar{\mathbb{F}}_q) \to GL(n, \mathbb{F}[[t]])$ is a deformation of ρ_0 (i.e., a continuous morphism such that the composition of ρ_t and the evaluation morphism $GL(n, \mathbb{F}[[t]]) \to GL(n, \mathbb{F})$ equals ρ_0). If ρ_t extends to a morphism $\pi_1(X) \to GL(n, \mathbb{F}[[t]])$ then the deformation ρ_t is trivial (i.e., $\rho_t(\gamma) = g_t \rho(\gamma) g_t^{-1}$, $g_t \in \mathrm{Ker}(GL(n, \mathbb{F}[[t]]) \to GL(n, \mathbb{F}))$, $g_0 = 1$).

1.4. Main Theorem. *Conjecture dJ(n) implies Kashiwara's conjecture for irreducible perverse sheaves whose rank over the smoothness locus equals n.*

1.5. De Jong [dJ] has proved dJ(n) (and, in fact, a stronger conjecture) for $n \leq 2$. His method is to reduce the statement to the case that X is a smooth projective curve and then to prove a version of the Langlands conjecture (more precisely, he proves that given a continuous representation $\sigma : \pi_1(X) \to GL(n, \mathbb{F}((t)))$, $n \leq 2$, whose restriction to $\pi_1(X \otimes \bar{\mathbb{F}}_q)$ is absolutely irreducible there exists a nonzero $\mathbb{F}((t))$-valued unramified cusp form on $GL(n)$ over the adeles of X which is an eigenfunction of the Hecke operators with eigenvalues related in the usual way with the eigenvalues of $\sigma(Fr_v)$, $v \in X$). Then he uses the fact that the space of cusp forms with given central character is finite-dimensional and therefore the eigenvalues of Hecke operators belong to $\bar{\mathbb{F}}$ if the central character is defined over $\bar{\mathbb{F}}$.

1.6. Remark. The above Conjecture $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ implies a similar statement for holonomic $\mathcal{D}$-modules with regular singularities. Kashiwara [K] conjectured that the regular singularity assumption is, in fact, unnecessary (in this case statement 3 is more complicated, see [K]). I cannot prove this.

1.7. For an algebraically closed field k denote by $\mathrm{Kash}_l(k)$ the analog of Conjecture $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ with $\mathbb{C}$ replaced by k, the usual topology replaced by the etale topology and F assumed to be a finite extension of $\mathbb{Q}_l$, $l \neq \mathrm{char}\,k$. One has $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C}) \Rightarrow \mathrm{Kash}_l(\mathbb{C})$; on the other hand, if $\mathrm{Kash}_l(\mathbb{C})$ is true for infinitely many primes l then $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ holds (the easy Lemma 2.5 below shows that the field F from $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ can be assumed to be a number field without loss of generality; then use arguments from [BBD], §6.1.1). One also has

$\mathrm{Kash}_l(\mathbb{C}) \Rightarrow \mathrm{Kash}_l(k)$ for every k of characteristic 0 (indeed, a perverse $\mathbb{Q}_l$-sheaf on a variety over k is defined over a countable subfield $k_0 \subset k$, which can be embedded into $\mathbb{C}$). In fact, a specialization argument in the spirit of [BBD], §6.1.6 shows that $\mathrm{Kash}_l(k) \Leftrightarrow \mathrm{Kash}_l(\bar{\mathbb{Q}})$ for any k of characteristic 0.

Remark. A similar specialization argument shows that $\mathrm{Kash}_l(k)$ is equivalent to $\mathrm{Kash}_l(\bar{\mathbb{F}}_p)$ for any algebraically closed field k of characteristic $p > 0$. But I cannot prove $\mathrm{Kash}_l(\bar{\mathbb{F}}_p)$.

1.8. To prove that de Jong's conjecture implies $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ we use the fact that $\mathrm{Kash}_l(\bar{\mathbb{F}}_p)$ is true if M comes from a perverse sheaf M_0 on a variety X_0 over $\mathbb{F}_{p^m}$ with $X_0 \otimes_{\mathbb{F}_{p^m}} \bar{\mathbb{F}}_p = X$. This is well known if M is pure, but according to Lafforgue ([L], Corollary VII.8) every absolutely irreducible l-adic perverse sheaf on X_0 becomes pure after tensoring it by a rank 1 sheaf on $\operatorname{Spec} \mathbb{F}_{p^m}$ (this follows from the Langlands conjecture for $GL(n)$ over a functional field proved in [L]). We also use the moduli of local systems on a complex variety to get rid of certain singularities (see 2.6). This is why I cannot deduce $\mathrm{Kash}_l(\bar{\mathbb{F}}_p)$ from de Jong's conjecture even though we use characteristic p arguments to deduce $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$.

Even if it is possible to prove de Jong's conjecture, it would be great if somebody finds a direct proof of $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ or its $\mathcal{D}$-module version. This has already been done in some particular cases. In the case that Y is a point, X is smooth, and M is a local system statement 2 was proved by C. Simpson [Si]. Using $\mathcal{D}$-modules and Simpson's idea of mixed twistor structures C. Sabbah [Ss] has recently proved statements 2,3 of Kashiwara's conjecture under the assumption that X is smooth and M is a local system extendable to a local system on a compactification on X; he has also proved statement 1 under the additional assumption that Y is projective.

2. Outline of the proof

In this section we prove Theorem 1.4 modulo some lemmas. Their proofs (which are standard) can be found in §3 and §6.

2.1. We can assume that the perverse sheaf M from Conjecture $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ is irreducible and is not supported on a closed subvariety of X different from X. Then there is a smooth open $j : U \hookrightarrow X$, $U \neq \emptyset$, such that $M = j_{!*}M_U$ for some irreducible lisse perverse sheaf M_U on U. Denote by n the rank of M_U.

A lisse perverse sheaf on $U(\mathbb{C})$ is the same as a local system on U or a representation of $\pi_1(U, u)$, $u \in U(\mathbb{C})$. So irreducible lisse perverse sheaves on $U(\mathbb{C})$ form an algebraic stack of finite type over $\mathbb{Z}$. The corresponding coarse moduli scheme Irr_n^U is of finite type over $\mathbb{Z}$. The functor $A \mapsto \mathrm{Irr}_n^U(A)$ on the category of commutative rings is the sheaf (for the etale topology) associated to the presheaf $\underline{\mathrm{Irr}}_n^U$ where $\underline{\mathrm{Irr}}_n^U(A)$ is the set of isomorphism classes of rank n locally free sheaves of A-modules N on $U(\mathbb{C})$ such that $N \otimes_A k$ is irreducible for every field k equipped with a homomorphism $A \to k$. It is easy to see that if

$\operatorname{Pic} A = 0$ and every Azumaya algebra over A is isomorphic to $\operatorname{End} A^n$ then the map $\underline{\mathrm{Irr}}_n^U \to \mathrm{Irr}_n^U$ is bijective. In particular, this is true if A is a complete local ring with finite residue field.

Notice that locally constant sheaves with finite fibers on $U(\mathbb{C})$ and U_{et} are the same. Moreover, if U has a model U_E over a subfield $E \subset \mathbb{C}$ (i.e., $U = U_E \otimes_E \mathbb{C}$) then they are the same as locally constant sheaves with finite fibers on $(U_E \otimes_E \bar{E})_{\mathrm{et}}$, where $\bar{E}$ is the algebraic closure of E in $\mathbb{C}$, so that $\mathrm{Gal}(\bar{E}/E)$ acts on the disjoint union of the completions $(\widehat{\mathrm{Irr}_n^U})_z$ of Irr_n^U at all possible closed points $z \in \mathrm{Irr}_n^U$.

2.2. Fix a closed point $z \in \mathrm{Irr}_n^U$ and a model U_E of U over a finitely generated subfield $E \subset \mathbb{C}$ so that z is $\mathrm{Gal}(\bar{E}/E)$-invariant. Then $\mathrm{Gal}(\bar{E}/E)$ acts on $(\widehat{\mathrm{Irr}_n^U})_z$. In §6 we will prove the following lemma.

2.3. Lemma. *(a) The action of* $\mathrm{Gal}(\bar{E}/E)$ *on* $(\widehat{\mathrm{Irr}_n^U})_z$ *is unramified almost everywhere, i.e., it factors through* $\pi_1(\operatorname{Spec} R, \operatorname{Spec} \bar{E})$ *for some finitely generated ring R with field of fractions E. For such R we have a well-defined Frobenius automorphism* Fr_v *of* $(\widehat{\mathrm{Irr}_n^U})_z$ *corresponding to a closed point $v \in \operatorname{Spec} R$ and an embedding of the henselization R_v into $\bar{E}$: namely* $\mathrm{Fr}_v \in \pi_1(\operatorname{Spec} R, \operatorname{Spec} \bar{E})$ *is the image of the Frobenius element of* $\pi_1(\operatorname{Spec} R, \bar{v})$ *under the isomorphism* $\pi_1(\operatorname{Spec} R, \bar{v}) \xrightarrow{\sim} \pi_1(\operatorname{Spec} R, \operatorname{Spec} \bar{E})$ *corresponding to the embedding* $R_v \hookrightarrow \bar{E}$.

(b) Assume that Conjecture dJ(n) holds. Then there exists R as in (a) such that for every closed point $v \in \operatorname{Spec} R$ and $k \in \mathbb{N}$ the fixed point scheme $\mathrm{Fix}(\mathrm{Fr}_v^k, (\widehat{\mathrm{Irr}_n^U})_z)$ *is finite over $\mathbb{Z}_l$, where l is the characteristic of the residue field of z.*

2.4. Now let us prove Theorem 1.4. If M is as in Conjecture $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ and F' is a field containing F then each of the 3 statements of the conjectures holds for $M \otimes_F F'$ if and only if it holds for M. So for $i = 1, 2, 3$ there is a subset $\mathrm{Bad}_i^{\mathbb{Q}} \subset \mathrm{Irr}_n^U \otimes \mathbb{Q}$ such that for every field $F \supset \mathbb{Q}$ and every rank n absolutely irreducible lisse perverse sheaf of F-vector spaces M_U on U the i-th statement of Conjecture $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ does not hold for $M = j_{!*}M_U$ if and only if $M_U \in \mathrm{Bad}_i^{\mathbb{Q}}(F)$. In §3 we will prove the following statement.

2.5. Lemma. $\mathrm{Bad}_i^{\mathbb{Q}} \subset \mathrm{Irr}_n^U \otimes \mathbb{Q}$ *is constructible for $i = 1, 2, 3$.*

2.6. Denote by Bad_i the closure of $\mathrm{Bad}_i^{\mathbb{Q}}$ in Irr_n^U. We have to prove that $\mathrm{Bad}_i^{\mathbb{Q}} = \emptyset$. If $\mathrm{Bad}_i^{\mathbb{Q}} \neq \emptyset$ then there is an open $V \subset \mathrm{Irr}_n^U$ such that $\mathrm{Bad}_i^{\mathbb{Q}} \cap (V \otimes \mathbb{Q})$ is closed in $V \otimes \mathbb{Q}$ and $V \cap \mathrm{Bad}_i \subset V$ is non-empty and smooth over $\mathbb{Z}$ (if equipped with the reduced scheme structure). Take a closed point $z \in V \cap \mathrm{Bad}_i$ and denote by $(\widehat{\mathrm{Bad}}_i)_z$ the preimage of Bad_i in $(\widehat{\mathrm{Irr}_n^U})_z$. Fix E and U_E as in 2.2.

2.7. Lemma. $(\widehat{\mathrm{Bad}}_i)_z$ *is* $\mathrm{Gal}(\bar{E}/E)$*-stable if* E *is big enough in the following sense. We assume that* $j : U \hookrightarrow X$ *comes from a morphism* $j_E : U_E \hookrightarrow X_E$ *of schemes over* E*. We also assume that in the cases* $i = 1, 2$ $\pi : X \to Y$ *comes from a morphism* $\pi_E : X_E \to Y_E$ *of schemes over* E *and in the case* $i = 3$ *the function* f *is defined over* E*. Besides, in the case* $i = 2$ *we assume that our relatively ample line bundle on* X *comes from a bundle* $\mathcal{L}_E$ *on* X_E*.*

Proof. Let us consider $(\widehat{\mathrm{Irr}}_n^U)_z$ as a scheme rather than a formal scheme. It suffices to show that if F is a finite extension of $\mathbb{Q}_l$ and $O \subset F$ is the ring of integers then the subset $T \subset \mathrm{Mor}(\mathrm{Spec}\, O, (\widehat{\mathrm{Irr}}_n^U)_z)$ consisting of morphisms $f : \mathrm{Spec}\, O \to (\widehat{\mathrm{Irr}}_n^U)_z$ such that $f(\mathrm{Spec}\, F) \subset (\widehat{\mathrm{Bad}}_i)_z$ is $\mathrm{Gal}(\bar{E}/E)$-invariant. A morphism $f : \mathrm{Spec}\, O \to (\widehat{\mathrm{Irr}}_n^U)_z$ defines an isomorphism class of lisse perverse sheaves of O-modules M_U on $U(\mathbb{C})$, and $f(\mathrm{Spec}\, F) \subset (\widehat{\mathrm{Bad}}_i)_z$ if and only if the i-th statement of Conjecture $\mathrm{Kash}_{\mathrm{top}}(\mathbb{C})$ does not hold for the irreducible perverse sheaf $M = j_{!*}(M_U \otimes_O F)$. We can consider M_U (resp. M) as a perverse O-sheaf (resp. a perverse F-sheaf) on the scheme $U_E \otimes_E \bar{E}$ (resp. $X_E \otimes_E \bar{E}$) rather than on the topological space $U(\mathbb{C})$ (resp. on $X(\mathbb{C})$). So T is $\mathrm{Gal}(\bar{E}/E)$-invariant. □

2.8. Lemma. *Let* E *be as in 2.7 and* R*,* l *as in 2.3(b). Let* v *be a closed point of* $\mathrm{Spec}\, R$ *and* $i \in \{1, 2, 3\}$*. Then the fixed point scheme* $\mathrm{Fix}_k := \mathrm{Fix}(\mathrm{Fr}_v^k, (\widehat{\mathrm{Bad}}_i)_z)$ *is finite and flat over* $\mathbb{Z}_l$ *for every* $k \in \mathbb{N}$*. For some* $k \in \mathbb{N}$ *it is not empty.*

Proof. We chose R so that finiteness is clear. We chose z so that $(\widehat{\mathrm{Bad}}_i)_z$ is smooth over $\mathbb{Z}_l$, therefore $\mathrm{Fix}_k \subset (\widehat{\mathrm{Bad}}_i)_z$ is defined by d equations, where d is the dimension of $(\widehat{\mathrm{Bad}}_i)_z$ over $\mathbb{Z}_l$. So finiteness implies that Fix_k is a complete intersection over $\mathbb{Z}_l$ and therefore flat over $\mathbb{Z}_l$ (this argument was used, e.g., in §3.14 of [dJ]). For some $k \in \mathbb{N}$ the automorphism Fr_v^k acts identically on the residue field of z. Then Fix_k is not empty. □

The above lemma contradicts the following statement, which will be proved in §6.

2.9. Lemma. *Let* E *be as in 2.7 and* R *as in 2.3(a). For each* $i \in \{1, 2, 3\}$ *there is a non-empty open* $W \subset \mathrm{Spec}\, R$ *such that for every closed point* $v \in W$ *and every* $k \in \mathbb{N}$ *the fixed point scheme* $\mathrm{Fix}(\mathrm{Fr}_v^k, (\widehat{\mathrm{Bad}}_i)_z)$ *has no* $\bar{\mathbb{Z}}_l$*-points, where* $\bar{\mathbb{Z}}_l$ *is the ring of integers of* $\bar{\mathbb{Q}}_l$*.*

3. Constructibility

In this section all rings are assumed to be commutative and Noetherian. Lemma 2.5 can be reformulated as follows.

3.1. Lemma. *Let N be a locally free sheaf of modules on $U(\mathbb{C})$ over an integral $\mathbb{Q}$-algebra A such that $N \otimes_A k$ is irreducible for all fields k equipped with a homomorphism $A \to k$. Let $M_U := N[d]$, $d := \dim U$, be the corresponding perverse sheaf. Denote by K the field of fractions of A. Let $i \in \{1,2,3\}$.*

(a) If the i-th statement of Conjecture $Kash_{\mathrm{top}}(\mathbb{C})$ holds for the perverse sheaf $M = j_{!*}M_U \otimes_A K$ then there is an $f \in A \setminus \{0\}$ such that it holds for $j_{!*}(M_U \otimes_A k)$ for every field k equipped with a homomorphism $A_f \to k$.

(b) If the i-th statement of Conjecture $Kash_{\mathrm{top}}(\mathbb{C})$ does not hold for $M = j_{!*}M_U \otimes_A K$ then there is an $f \in A \setminus \{0\}$ such that for every field k equipped with a homomorphism $A_f \to k$ it does not hold for $j_{!*}(M_U \otimes_A k)$.

The proof will be given in 3.11 after some preparatory lemmas.

3.2. Let X be an algebraic variety over $\mathbb{C}$. As usual, $D(X(\mathbb{C}), A)$ denotes the derived category of sheaves of A-modules on $X(\mathbb{C})$ and $D^b_c(X(\mathbb{C}), A) \subset D(X(\mathbb{C}), A)$ is the full subcategory of complexes with constructible cohomology sheaves (the notion of constructible set being understood in the sense of algebraic geometry). Recall that the perverse t-structure on $D^b_c(X(\mathbb{C}), A)$ is defined as follows: ${}^pD^{\geq 0}_c(X(\mathbb{C}), A)$ (resp. ${}^pD^{\leq 0}_c(X(\mathbb{C}), A)$) is the full subcategory of $D^b_c(X(\mathbb{C}), A)$ consisting of complexes C such that every irreducible subvariety $Y \overset{\nu}{\hookrightarrow} X$ has a non-empty open subset $Y' \subset Y$ such that $H^i\nu^!C|_{Y'} = 0$ (resp.$H^i\nu^*C|_{Y'} = 0$) for $i < 2\dim S$ (resp. for $i > -2\dim S$). We will write ${}^pD^{\geq 0}$, ${}^pD^{\leq 0}$ instead of ${}^pD^{\geq 0}_c$, ${}^pD^{\leq 0}_c$. Let ${}^pD_{\geq 0}(X(\mathbb{C}), A) \subset D^b_c(X(\mathbb{C}), A)$ denote the full subcategory of complexes C such that $C \overset{L}{\otimes}_A N$ belongs to ${}^pD^{\geq 0}(X(\mathbb{C}), A)$ for all finitely generated A-modules N.

3.3. Lemma. *Suppose we have a stratification $X = \bigcup_\nu X_\nu$. Denote by i_ν the embedding $X_\nu \hookrightarrow X$.*

(i) $C \in {}^pD^{\geq 0}(X(\mathbb{C}), A)$ if and only if $i^!_\nu C \in {}^pD^{\geq 0}(X_\nu(\mathbb{C}), A)$ for all ν.

(ii) $C \in {}^pD_{\geq 0}(X(\mathbb{C}), A)$ if and only if $i^!_\nu C \in {}^pD_{\geq 0}(X_\nu(\mathbb{C}), A)$ for all ν.

Proof. (i) is obvious. (ii) follows from (i) because the functor $\overset{L}{\otimes}_A N$ commutes with $i^!_\nu$ □

3.4. Lemma. *If $C \in {}^pD_{\geq 0}(X(\mathbb{C}), A)$ then $A' \overset{L}{\otimes}_A C \in {}^pD_{\geq 0}(X(\mathbb{C}), A)$ for every A-algebra A' (not necessarily finite over A).*

Proof. Using 3.3(ii) we reduce the proof to the case where X is smooth and C is lisse. This case is equivalent to that of $X = \operatorname{Spec}\mathbb{C}$, which is well known and easy. □

3.5. Let M be a perverse sheaf of A-modules. If N is a finitely generated A-module the perverse sheaf $N \otimes_A M$ is defined by $N \otimes_A M := \tau_{\geq 0} N \overset{L}{\otimes}_A M$, where $\tau_{\geq 0}$ denotes perverse truncation. Quite similarly one defines $A' \otimes_A M$ for an A-algebra A'.

A perverse sheaf of A-modules M is said to be *flat* if $M \in {}^pD_{\geq 0}(X(\mathbb{C}), A)$; this is equivalent to exactness of the functor $N \mapsto N \otimes_A M$ on the category of finitely generated A-modules N.

3.6. Lemma. *Let A be an integral ring.*

(i) If M is a perverse sheaf of A-modules then there is an $f \in A \setminus \{0\}$ such that $A_f \otimes_A M$ is flat over A_f.

(ii) If $C \in {}^pD^{\geq 0}(X(\mathbb{C}), A)$ then there is an $f \in A \setminus \{0\}$ such that $A_f \otimes_A C \in {}^pD_{\geq 0}(X(\mathbb{C}), A)$.

Proof. Clearly (ii)⇒(i). The proof of (ii) is similar to that of 3.4. □

3.7. Lemma. *Let A be an integral ring and K its field of fractions. Let M be a perverse sheaf of A-modules on $X(\mathbb{C})$. Let $j : U \hookrightarrow X$ be an open embedding. Suppose that $M \otimes_A K = j_{!*}j^*(M \otimes_A K)$. Then there is an $f \in A \setminus \{0\}$ such that for every field k equipped with a homomorphism $A_f \to k$ one has $M \otimes_A k = j_{!*}j^*(M \otimes_A k)$.*

Proof. Consider the embedding $i : X \setminus U \to X$. The equality $M \otimes_A K = j_{!*}j^*(M \otimes_A K)$ means that $i^*M \otimes_A K \in {}^pD^{<0}(X(\mathbb{C}), K)$ and $i^!M \otimes_A K \in {}^pD^{>0}(X(\mathbb{C}), K)$. Localizing A one can assume that $i^*M \in {}^pD^{<0}(X(\mathbb{C}), A)$ and $i^!M \in {}^pD^{>0}(X(\mathbb{C}), A)$. Moreover, by 3.6 we can assume that M is flat and $i^!M \in {}^pD_{>0}(X(\mathbb{C}), A)$. Then for every field k equipped with a homomorphism $A \to k$ one has $M \otimes_A k = M \overset{L}{\otimes}_A k$, so $i^*(M \otimes_A k) = (i^*M) \overset{L}{\otimes}_A k \in {}^pD^{<0}(X(\mathbb{C}), k)$ and $i^!(M \otimes_A k) = i^!(M \overset{L}{\otimes}_A k) = (i^!M) \overset{L}{\otimes}_A k$ is in ${}^pD^{>0}(X(\mathbb{C}), k)$ by 3.4. Therefore $M \otimes_A k = j_{!*}j^*(M \otimes_A k)$. □

3.8. Lemma. *Let A, K, M, X be as in 3.7. Suppose that* $\operatorname{char} K = 0$.

(i) If the perverse sheaf $M \otimes_A K$ is semisimple then there is an $f \in A \setminus \{0\}$ such that $M \otimes_A k$ is semisimple for every morphism from A_f to a field k.

(ii) If $M \otimes_A K$ is not semisimple then there is an $f \in A \setminus \{0\}$ such that $M \otimes_A k$ is not semisimple for every morphism from A_f to a field k.

Proof. (i) We can assume that $M \otimes_A K$ is irreducible. Localizing A we can assume that the support of M equals the support of $M \otimes_A K$. So we can assume that this support equals X. Then there is a smooth open subset $j : U \hookrightarrow X$ such that $M \otimes_A K = j_{!*}j^*(M \otimes_A K)$ and $j^*M \otimes_A K$ is lisse and irreducible. Then $j^*M \otimes_A \bar{K}$ is semisimple (here we use that $\operatorname{char} K = 0$), so localizing A we can assume that $j^*M \otimes_A k$ is semisimple for every morphism from A to a field k. It remains to apply 3.7.

(ii) Let $0 \to N_K \to M_K \to P_K \to 0$ be a nontrivial extension of perverse sheaves with $M_K = M \otimes_A K$. Localizing A we can assume that it comes from an exact sequence $0 \to N \to M \to P \to 0$ of flat perverse sheaves of A-modules on $X(\mathbb{C})$. It defines an element $u \in H^1(C)$, $C := R\operatorname{Hom}(P, N)$, such that the image of u in $H^1(C) \otimes_A K$ is nonzero. For every morphism from A to a field k the sequence of perverse sheaves $0 \to N \otimes_A k \to M \otimes_A k \to P \otimes_A k \to 0$ is still exact and its class in $\operatorname{Ext}^1(P \otimes_A k, N \otimes_A k) = H^1(C \overset{L}{\otimes}_A k)$ is the image of u. As C is quasi-isomorphic to a finite complex of finitely generated A-modules, there exists $f \in A \setminus 0$ such that for every morphism from A_f to a field k the image of u in $H^1(C \overset{L}{\otimes}_A k)$ is nonzero. □

3.9. Lemma. *Let A, K be as in 3.8 and $\mathcal{C} \in D_c^b(X(\mathbb{C}), A)$.*

(i) Suppose that $\mathcal{C} \otimes_A K$ is "semisimple", i.e., isomorphic to a direct sum of complexes of the form $N_j[j]$, where the N_j's are semisimple perverse sheaves. Then there is an $f \in A \setminus \{0\}$ such that $\mathcal{C} \overset{L}{\otimes}_A k$ is "semisimple" for every morphism from A_f to a field k.

(ii) If $\mathcal{C} \otimes_A K$ is not "semisimple" then there is an $f \in A \setminus \{0\}$ such that $\mathcal{C} \overset{L}{\otimes}_A k$ is not "semisimple" for every morphism from A_f to a field k.

Proof. (i) follows from 3.8(i). Let us prove (ii). By 3.6(i) we can assume that the perverse cohomology sheaves $H^i\mathcal{C}$ are flat over A. Then $H^j(\mathcal{C} \overset{L}{\otimes}_A k) = H^j\mathcal{C} \otimes_A k$, so if $H^j\mathcal{C} \otimes_A K$ is not semisimple for some j then by 3.8 there is an $f \in A \setminus \{0\}$ such that $H^j(\mathcal{C} \overset{L}{\otimes}_A k)$ is not semisimple for every morphism from A_f to a field k. If $H^j\mathcal{C} \otimes_A K$ is semisimple for all j then there exists j such that the canonical element $u \in \operatorname{Hom}(\tau_{>j}\mathcal{C}, (\tau_{\le j}\mathcal{C})[1])$ has nonzero image in $\operatorname{Hom}(\tau_{>j}\mathcal{C}, (\tau_{\le j}\mathcal{C})[1]) \otimes_A K$. Just as in the proof of 3.8(ii) this implies that there is an $f \in A \setminus \{0\}$ such that the image of u in $\operatorname{Hom}(\tau_{>j}\mathcal{C} \overset{L}{\otimes}_A k, (\tau_{\le j}\mathcal{C} \overset{L}{\otimes}_A k)[1])$ is nonzero for every morphism from A_f to a field k. □

3.10. Lemma. *Let A, K be as in 3.7 and $\varphi : \mathcal{C}_1 \to \mathcal{C}_2$ a morphism in $D_c^b(X(\mathbb{C}), A)$. If the morphism $\varphi_K : \mathcal{C}_1 \otimes_A K \to \mathcal{C}_2 \otimes_A K$ induced by φ is an isomorphism then there is an $f \in A \setminus \{0\}$ such that $\varphi_k : \mathcal{C}_1 \overset{L}{\otimes}_A k \to \mathcal{C}_2 \overset{L}{\otimes}_A k$ is an isomorphism for every morphism from A_f to a field k. If φ_K is not an isomorphism then there is an $f \in A \setminus \{0\}$ such that φ_k is not an isomorphism for every morphism from A_f to a field k.* □

3.11. Proof of Lemma 3.1. By 3.6(i) and 3.7, localizing A we can assume that $j_{!*}M_U$ is A-flat and $(j_{!*}M_U) \otimes_A k = j_{!*}(M_U \otimes_A k)$ for every morphism from A to a field k.

To prove Lemma 3.1 for $i = 1$ apply 3.9 to $\mathcal{C} = \pi_* j_{!*} M_U$ and notice that $\pi_* j_{!*}(M_U \otimes_A k) = (\pi_* j_{!*} M_U) \overset{L}{\otimes}_A k$. To prove the lemma for $i = 2$ notice that by 3.6(i) one can assume that the perverse sheaves $H^r \pi_* j_{!*} M_U$, $r \in \mathbb{Z}$, are A-flat and so $H^r \pi_* j_{!*}(M_U \otimes_A k) = (H^r \pi_* j_{!*} M_U) \otimes_A k$. Then apply 3.10.

Now let us prove the lemma for $i = 3$. As Ψ_f is exact, $L := \Psi_f j_{!*} M_U$ is an A-flat perverse sheaf and $\Psi_f j_{!*}(M_U \otimes_A k) = L \otimes_A k$. As $\dim \operatorname{End}(L \otimes_A K) < \infty$ and $\operatorname{char} K = 0$, the monodromy $T \in \operatorname{Aut}(L \otimes_A K)$ satisfies $p(T)^m = 0$ for some $m \in \mathbb{N}$ and some monic polynomial p with nonzero discriminant D and constant term c. Localizing A we can assume that D, c are invertible in A and the equality $p(T)^m = 0$ holds in $\operatorname{End} L$. There is a unique $q \in A[t]/(p(t))$ such that $q \equiv 1 \mod p(t)$ and $p(t/q) = 0$. Define $T_u \in \operatorname{Aut} L$ by $T_u := q(T)$, then T_u becomes equal to the unipotent part of T after any base change $A \to k$, where k is a field. There is a unique exhaustive increasing filtration W on L such that $(T-1)W_k L \subset W_{k-2} L$ and the morphisms $(T-1)^k : \operatorname{gr}^W_k L \to \operatorname{gr}^W_{-k} L$ are isomorphisms (this holds for a unipotent automorphism in any abelian category). By 3.6(i), localizing A we can assume that $\operatorname{gr}^W L$ is A-flat. Then for every morphism from A to a field k the filtration on $L \otimes_A k$ induced by W is the weight filtration and $\operatorname{gr}(L \otimes_A k) = (\operatorname{gr}^W L) \otimes_A k$. It remains to apply 3.8. □

4. Very good models

In this section we remind the results of [BBD] used in §6.

4.1. Let E be a field finitely generated over a prime field, X a scheme of finite type over E and $M_1, \dots, M_k \in D^b_c(X, \mathbb{F})$, where $\mathbb{F}$ is a finite field whose characteristic l is different from that of E. Let $R \subset E$ be a subring of finite type over $\mathbb{Z}$ whose field of fractions equals E.

A *model* of X over R is a scheme X_R of finite type over R with an isomorphism $X_R \otimes_R E \xrightarrow{\sim} X$. A model of $(X, M_1, \dots, M_k)$ over R is a collection $(X_R, (M_1)_R, \dots, (M_k)_R)$ where X_R is a model of X over R and $(M_i)_R$ is an object of $D^b_c(X_R, \mathbb{F})$ whose pull-back to X is identified with M_i.

For every R and E as above every collection $(X, M_1, \dots, M_k)$ has a model over R. Given a model $(X_R, (M_1)_R, \dots, (M_k)_R)$ we denote by X_u the fiber of X_R over a geometric point u of $\operatorname{Spec} R$ and by $(M_i)_u$ the $*$-restriction of M_i to X_u. We write $X_{\bar{E}}$, $(M_i)_{\bar{E}}$ instead of $X_{\operatorname{Spec} \bar{E}}$, $(M_i)_{\operatorname{Spec} \bar{E}}$ and denote by p_R the morphism $X_R \to \operatorname{Spec} R$. We need the following ad hoc definition.

4.2. Definition. A model $(X_R, (M_1)_R, \dots, (M_k)_R)$ is *good* if

1) l is invertible in R;

2) for every geometric point u of $\operatorname{Spec} R$ and every i, j the morphism

$$((p_R)_* \mathrm{R}\mathbf{Hom}((M_i)_R, (M_j)_R))_u \to R\operatorname{Hom}((M_i)_u, (M_j)_u) \tag{4.1}$$

is an isomorphism and the morphism

$$((p_R)_* \mathrm{R}\mathbf{Hom}((M_i)_R, (M_j)_R))_u \to R\operatorname{Hom}((M_i)_{\bar{E}}, (M_j)_{\bar{E}}) \tag{4.2}$$

induced by an embedding of the strict henselization R_u into $\bar{E}$ is also an isomorphism.

4.3. Remark. In the above definition we understand R**Hom** as $\mathrm{R}\mathbf{Hom}_{\mathbb{F}}$ and $R\,\mathrm{Hom} := R\,\mathrm{Hom}_{\mathbb{F}}$. But if O is a local ring with residue field $\mathbb{F}$ then the analogs of (4.1) and (4.2) with R**Hom** replaced by $\mathrm{R}\mathbf{Hom}_O$ and $R\,\mathrm{Hom}$ replaced by $R\,\mathrm{Hom}_O$ are still isomorphisms (because $\mathrm{R}\mathbf{Hom}_O((M_i)_R,(M_j)_R)$ equals $\mathrm{R}\mathbf{Hom}((M_i)_R,(M_j)_R)\otimes R\,\mathrm{Hom}_O(\mathbb{F},\mathbb{F})$, etc.).

4.4. According to §6.1 of [BBD], *every model $(X_R,(M_1)_R,\dots,(M_k)_R)$ becomes good after a base change of the form $R\to R_f$, $f\in R\setminus\{0\}$.* More precisely, after replacing R by some R_f the following properties (which are stronger than the above property 2) hold:

a) for every geometric point u of $\operatorname{Spec} R$ and every i,j the morphism $(\mathrm{R}\mathbf{Hom}((M_i)_R,(M_j)_R))_u\to \mathrm{R}\mathbf{Hom}((M_i)_u,(M_j)_u)$ is an isomorphism;

b) the complexes $(p_R)_*K_{ij}$, $K_{ij}:=\mathrm{R}\mathbf{Hom}((M_i)_R,(M_j)_R)$, are lisse and for every geometric point u of $\operatorname{Spec} R$ and every i,j the morphism $((p_R)_*K_{ij})_u\to \mathrm{R}\Gamma(X_u,(K_{ij})_u)$ is an isomorphism.

This is shown in §6.1 of [BBD] by reducing to the case where each $(M_i)_R$ is the extension by zero of a local system on a locally closed subscheme of X_R and using Theorem 1.9 from [De3].

4.5. Suppose we have a model $(X_R,(M_1)_R,\dots,(M_k)_R)$. Let R_u denote the strict henselization of R at a geometric point u of $\operatorname{Spec} R$. Choose an algebraic closure $\bar E\supset E$ and an embedding $R_u\hookrightarrow \bar E$. Let O be a local Artinian ring whose residue field $O/\mathfrak{m}$ is a finite extension of $\mathbb{F}$. Let $D^{\{M_i\}}(X_u,O)\subset D^b_c(X_u,O)$ be the thick triangulated subcategory generated by the complexes $(M_i)_u\otimes_{\mathbb{F}} O/\mathfrak{m}$ (according to [Ve], a triangulated subcategory $\mathcal{B}$ of a triangulated category $\mathcal{A}$ is *thick* if every object of $\mathcal{A}$ which is a direct summand of an object of $\mathcal{B}$ belongs to $\mathcal{B}$). Let $D_{\{M_i\}}(X_u,O)\subset D^b_c(X_u,O)$ denote the full subcategory of complexes C such that $C\overset{L}{\otimes}_O O/\mathfrak{m}\in D^{\{M_i\}}(X_u,O/\mathfrak{m})$. Notice that $D_{\{M_i\}}(X_u,O)\subset D^{\{M_i\}}(X_u,O)\cap D_{\mathrm{prf}}(X_u,O)$, where $D_{\mathrm{prf}}(X_u,O)\subset D^b_c(X_u,O)$ is the full subcategory of complexes of finite Tor-dimension. (In fact, one can prove [1] that $D_{\{M_i\}}(X_u,O)=D^{\{M_i\}}(X_u,O)\cap D_{\mathrm{prf}}(X_u,O)$, but we do not need this fact). One also has the similar categories $D^{\{M_i\}}(X_R\otimes_R R_u,O)$, $D_{\{M_i\}}(X_R\otimes_R R_u,O)$, $D^{\{M_i\}}(X_{\bar E},O)$, and $D_{\{M_i\}}(X_{\bar E},O)$. The following lemma and its proof is a version of §6.1.9 of [BBD].

4.6. Lemma. *If a model $(X_R,(M_1)_R,\dots,(M_k)_R)$ is good then the functors*

$$D_{\{M_i\}}(X_R\otimes_R R_u,O)\to D_{\{M_i\}}(X_{\bar E},O), \tag{4.3}$$

---

[1]If $C\in D^{\{M_i\}}(X_u,O)$ then for every $N\in\mathbb{N}$ one has an exact triangle $K\to C\overset{L}{\otimes}_O O/\mathfrak{m}\to K'$ such that $K\in D^{\{M_i\}}(X_u,O/\mathfrak{m})$ and $H^jK'=0$ for $j>-N$ (put $K:=C\overset{L}{\otimes}_O P$, where P is a perfect complex of O-modules such that $H^0P=O/\mathfrak{m}$ and $H^jP=0$ if $j\neq 0$ and $j>-N'$). If one also has $C\in D_{\mathrm{prf}}(X_u,O)$ then for N big enough the morphism $C\overset{L}{\otimes}_O O/\mathfrak{m}\to K'$ is zero, so $C\overset{L}{\otimes}_O O/\mathfrak{m}$ is a direct summand of $K\in D^{\{M_i\}}(X_u,O/\mathfrak{m})$ and therefore $C\overset{L}{\otimes}_O O/\mathfrak{m}\in D^{\{M_i\}}(X_u,O/\mathfrak{m})$.

$$D_{\{M_i\}}(X_R \otimes_R R_u, O) \to D_{\{M_i\}}(X_u, O) \tag{4.4}$$

and their analogs for $D^{\{M_i\}}$ *are equivalences, so one gets equivalences of triangulated categories*

$$D_{\{M_i\}}(X_{\bar{E}}, O) \xrightarrow{\sim} D_{\{M_i\}}(X_u, O) \tag{4.5}$$

$$D^{\{M_i\}}(X_{\bar{E}}, O) \xrightarrow{\sim} D^{\{M_i\}}(X_u, O) \tag{4.6}$$

Proof. It suffices to prove the lemma for $D^{\{M_i\}}$. To this end, use Remark 4.3 and the fact that every idempotent endomorphism of an object of the derived category of sheaves comes from a direct sum decomposition (see [Ne], ch. 1, Proposition 1.6.8.) ☐

4.7. Remark. Consider the point $v \in \operatorname{Spec} R$ corresponding to the geometric point u. The henselization R_v of R at v is embedded into $\bar{E}$; denote by E_v its field of fractions. Then $\operatorname{Gal}(\bar{E}/E_v)$ acts on R_u, $\bar{E}$, and u, so it acts on the categories $D_{\{M_i\}}(X_R \otimes_R R_u, O)$, $D_{\{M_i\}}(X_{\bar{E}}, O)$, and $D_{\{M_i\}}(X_u, O)$ (an action of a monoid Γ on a category C is a monoidal functor from Γ to the monoidal category of functors $C \to C$). The equivalence (4.5) is $\operatorname{Gal}(\bar{E}/E_v)$-equivariant because it is a composition of two $\operatorname{Gal}(\bar{E}/E_v)$-equivariant equivalences.

4.8. Now suppose that the M_i's are perverse sheaves whose pull-backs to $X_{\bar{E}}$ are semisimple. A good model $(X_R, (M_1)_R, \dots, (M_k)_R)$ is said to be *very good* if $(M_i)_u$ is a semisimple perverse sheaf for every i and every geometric point u of $\operatorname{Spec} R$. In this situation if $(M_i)_{\bar{E}}$ is absolutely irreducible (i.e., $(M_i)_{\bar{E}}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ is irreducible) then $(M_i)_u$ is absolutely irreducible for every geometric point u of $\operatorname{Spec} R$ (because (4.5) induces an isomorphism $\operatorname{End}(M_i)_{\bar{E}} \simeq \operatorname{End}(M_i)_u$).

In the case of a very good model $D^{\{M_i\}}(X_u, O)$ consists of all objects of $D^b_c(X_u, O)$ such that all irreducible components of the reduction modulo $\mathfrak{m}$ of their perverse cohomology sheaves occur in $((M_1)_u \oplus \dots \oplus (M_k)_u) \otimes_{\mathbb{F}} O/\mathfrak{m}$. There is a similar description of $D^{\{M_i\}}(X_{\bar{E}}, O)$. *So the equivalences (4.6) and (4.5) corresponding to a very good model send perverse sheaves to perverse sheaves and the same is true for the equivalences inverse to (4.5) and (4.5).*

Using the principles explained in §6.1.7 of [BBD] one shows that *every model* $(X_R, (M_1)_R, \dots, (M_k)_R)$ *becomes very good after a base change of the form* $R \to R_f$.

4.9. Now let O be a complete discrete valuation ring whose residue field $O/\mathfrak{m}$ is a finite extension of $\mathbb{F}$. According to Deligne's definition of the l-adic derived category (see §2.2.14 of [BBD] or §1.1.2 of [De4]), $D^b_c(X_u, O)$ is the inverse limit of $D_{\mathrm{prf}}(X_u, O/\mathfrak{m}^r)$, $r \in \mathbb{N}$. We define $D_{\{M_i\}}(X_u, O)$ to be the inverse limit of $D_{\{M_i\}}(X_u, O/\mathfrak{m}^r)$. Clearly $D_{\{M_i\}}(X_u, O) \subset D^b_c(X_u, O)$ is the full subcategory of complexes $C \in D^b_c(X_u, O)$ such that $C \overset{L}{\otimes}_O O/\mathfrak{m} \in D^{\{M_i\}}(X_u, O/\mathfrak{m})$. Same is true for $X_{\bar{E}}$ and $X_R \otimes_R R_u$. By 4.6, in the case of a good model we have the equivalences (4.3) – (4.5). It easily follows from 4.8 that in the case of a very good model the equivalence (4.5) sends perverse sheaves to perverse sheaves and the same is true for the equivalence inverse to (4.5).

5. A lemma on nearby cycles

5.1. We keep the notation from 4.1, but now we suppose that $\operatorname{char} E = 0$. Let $f \in H^0(X, \mathcal{O}_X)$ and $M \in D^b_c(X, \mathbb{F})$. Let (X_R, M_R, f_R) be a model for (X, M, f) (i.e., (X_R, M_R) is a model for (X, M) and $f_R \in H^0(X_R, \mathcal{O}_{X_R})$ extends f). For each geometric point u of $\operatorname{Spec} R$ we have the nearby cycle complex $\Psi_{f_u}(M_u) \in D^b_c(Y_u, \mathbb{F})$, where $f_u = f_R|_{X_u}$ and Y_u is the fiber of the subscheme $Y_R \subset X_R$ defined by $f_R = 0$. Lemma 5.4 below essentially says that after localizing R the complexes $\Psi_{f_u}(M_u)$ come from a single object of $D^b_c(X_R, \mathbb{F})$. To formulate the lemma precisely we need some notation.

Define $X_R[f_R^{1/m}] \subset X_R \times \mathbb{A}^1$ by the equation $f_R(x) = t^m$, $x \in X_R$, $t \in \mathbb{A}^1$. The direct image of the constant sheaf on $X_R[f_R^{1/m}]$ with fiber $\mathbb{F}$ with respect to the projection $X_R[f_R^{1/m}] \to X_R$ is denoted by $\mathcal{E}_m$. If $m|m'$ then $\mathcal{E}_m$ is embedded into $\mathcal{E}_{m'}$. Denote by $\mathcal{E}^{(m)}$ the direct limit of $\mathcal{E}_{ml^n}$, $n \in \mathbb{N}$, where $l := \operatorname{char} \mathbb{F}$. Finally, put $\Psi^{(m)}_{f_R}(M_R) := (\nu_R)_* \nu_R^*(M_R \otimes \mathcal{E}^{(m)})|_{Y_R}$, where $\nu_R : X_R \setminus Y_R \to X_R$ is the embedding. Clearly $\Psi^{(ml)}_{f_R}(M_R) = \Psi^{(m)}_{f_R}(M_R)$. For every geometric point u of $\operatorname{Spec} R$ one has the base change morphism

$$(\Psi^{(m)}_{f_R}(M_R))_u \to \Psi^{(m)}_{f_u}(M_u) \tag{5.1}$$

and the obvious morphism

$$\Psi^{(m)}_{f_u}(M_u) \to \Psi_{f_u}(M_u)\ . \tag{5.2}$$

Let κ_u be the residue field of u and $I := \operatorname{Gal}(\overline{\kappa_u((t))}/\kappa_u((t)))$. If m, l are invertible on u and coprime then $\Psi^{(m)}_{f_u}(M_u) = \Psi_{f_u}(M_u)^{I_m}$, where I_m is the unique normal subgroup of I such that $I/I_m \simeq \mathbb{Z}_l \times \mathbb{Z}/m\mathbb{Z}$. The following ad hoc definition will be used in 6.2.4.

5.2. Definition. A model (X_R, M_R, f_R) of (X, M, f) is Ψ_f*-good* if there exists $m \in \mathbb{N}$ such that

a) $m^{-1}, l^{-1} \in R$,

b) the morphisms (5.1) and (5.2) are isomorphisms for all u,

c) for every $\overline{m} \in m\mathbb{N}$ the morphism $\Psi^{(m)}_{f_R}(M_R) \to \Psi^{(\overline{m})}_{f_R}(M_R)$ is an isomorphism over $Y_R \otimes_{\mathbb{Z}} \mathbb{Z}[\overline{m}^{-1}]$,

d) the cohomology sheaves of $\Psi^{(m)}_{f_R}(M_R)$ are constructible.

5.3. Remarks. (i) Maybe d) holds automatically.

(ii) If properties a) – d) hold for (X_R, M_R, f_R) and some m then for every $m' \in m\mathbb{N}$ they hold for $(X_{R'}, M_{R'}, f_{R'})$ and m', where $R' := R[1/m']$, $X_{R'} := X_R \otimes_R R'$, $f_{R'} := f_R|_{X_{R'}}$.

5.4. Lemma. *Every model (X_R, M_R, f_R) of (X, M, f) becomes Ψ_f-good after a base change of the form $R \to R_g$, $g \in R \setminus \{0\}$.*

Proof. Using resolution of singularities in characteristic 0 reduce the proof to the case that $X_R = \operatorname{Spec} R[t_1, \dots t_n]$, $f_R = t_1^{k_1} \dots t_n^{k_n}$, and M_R is the constant sheaf (cf. the proof of Theorem 2.3.1 of [De2]). In this case take m to be the l.c.m. of $k_1, \dots, k_n$ and perform the base change $R \to R[l^{-1}m^{-1}]$ (cf. the proof of Theorem 3.3 of [De1]). □

6. Proof of Lemmas 2.3 and 2.9

We fix a model U_E of U over a finitely generated subfield $E \subset \mathbb{C}$. Let $\mathbb{F}$ denote the residue field of z (which is finite), and $l := \operatorname{char} \mathbb{F}$. While proving Lemmas 2.3 and 2.9 we can replace E by its finite extension, so we can assume that z is the isomorphism class of a lisse perverse $\mathbb{F}$-sheaf M_{U_E} on U_E. The pull-back $M_{U_{\bar{E}}}$ of M_{U_E} to $U_{\bar{E}} := U_E \otimes_E \bar{E}$ is absolutely irreducible. We will use the notions of "good" and "very good" from 4.2 and 4.8.

Given a finitely generated ring R with field of fractions E and a closed point $v \in \operatorname{Spec} R$ we choose an embedding of the henselization R_v into $\bar{E}$. We will use this embedding in the situation of 2.3(a). The field of fractions of R_v is denoted by E_v. The embedding $R_v \to \bar{E}$ defines a geometric point $\bar{v}$ of $\operatorname{Spec} R$ and an embedding of the strict henselization $R_{\bar{v}}$ into $\bar{E}$ (if $E_{\bar{v}} \subset \bar{E}$ is the maximal extension of E_v unramified at v and L is the integral closure of R_v in $E_{\bar{v}}$ then $\bar{v}$ is defined to be the closed point of L; then $R_{\bar{v}} = L$).

6.1. Proof of Lemma 2.3. Choose a very good model (U_R, M_{U_R}) of (U_E, M_{U_E}) over a finitely generated ring R with field of fractions E such that U_R is smooth over R and $M_{U_R}[-d]$ is a local system, $d := \dim U$. Then R has the properties required in Lemma 2.3. Indeed, let $v \in \operatorname{Spec} R$ be a closed point. Our $(\widehat{\operatorname{Irr}}_n^U)_z$ is the base of the universal deformation of the local system $M_{U_{\bar{E}}}[-d]$. Let $M_{U_{\bar{v}}}$ be the pull-back of M_{U_R} to $U_{\bar{v}} := U_R \otimes_R \bar{v}$; as explained in 4.8, $M_{U_{\bar{v}}}$ is absolutely irreducible. By 4.8, for every Artinian local ring O with residue field $\mathbb{F}$ the equivalence (4.6) (with X replaced by U) induces an equivalence between the O-linear category of perverse sheaves of O-modules on $U_{\bar{E}}$ and that on $U_{\bar{v}}$. So we get a canonical isomorphism between $(\widehat{\operatorname{Irr}}_n^U)_z$ and the base of the universal deformation of $M_{U_{\bar{v}}}[-d]$. It is $\operatorname{Gal}(\bar{E}/E_v)$-equivariant by 4.7. So the action of $\operatorname{Gal}(\bar{E}/E_v)$ on $(\widehat{\operatorname{Irr}}_n^U)_z$ is unramified, and Conjecture dJ(n) implies that for every $k \in \mathbb{N}$ the fixed point scheme of $\operatorname{Fr}_v^k : (\widehat{\operatorname{Irr}}_n^U)_z \to (\widehat{\operatorname{Irr}}_n^U)_z$ is finite.

6.2. Proof of Lemma 2.9. Recall that in the lemma i denotes the number of a statement in Kashiwara's conjecture (see 2.4). We assume that E is big enough in the sense of 2.7. We fix $j_E : U_E \hookrightarrow X_E$, in the cases $i = 1, 2$ we fix $\pi_E : X_E \to Y_E$, and in the case $i = 2$ we also fix $\mathcal{L}_E$ (see 2.7).

6.2.1. Let $M_1, \dots, M_k$ be the irreducible components of the perverse cohomology sheaf $H^0(j_E)_! M_U$. The pull-backs of $M_1, \dots, M_k$ to $X_{\bar{E}} := X_E \otimes_E \bar{E}$ are semisimple.

Notice that if O is a complete discrete valuation ring whose residue field $O/\mathfrak{m}$ contains $\mathbb{F}$ and $\tilde{M}$ is a perverse sheaf of O-modules on $U_{\bar{E}}$ such that $\tilde{M}/\mathfrak{m}\tilde{M} \simeq M_{U_{\bar{E}}} \otimes_{\mathbb{F}} (O/\mathfrak{m})$ then the perverse sheaf $(j_{\bar{E}})_{!*}\tilde{M}/\mathfrak{m}(j_{\bar{E}})_{!*}\tilde{M}$ is a quotient of $H^0(j_{\bar{E}})_! M_U \otimes_{\mathbb{F}} (O/\mathfrak{m})$ and so each of its irreducible components occurs in $((M_1)_{\bar{E}} \oplus \dots (M_k)_{\bar{E}}) \otimes_{\mathbb{F}} (O/\mathfrak{m})$.

6.2.2. Let R be as in 2.3(a). Localizing R (i.e., replacing R by R_f for some nonzero $f \in R$) we can choose a very good model $(X_R, (M_1)_R, \dots, (M_k)_R)$ of $(X_E, M_1, \dots, M_k)$.

6.2.3. *Proof for $i = 1, 2$.* Let $N_1, \dots, N_m$ be the irreducible components of all perverse cohomology sheaves of the complexes $(\pi_E)_* M_1, \dots, (\pi_E)_* M_k$. Localizing R we can choose a very good model $(Y_R, (N_1)_R, \dots, (N_m)_R)$ of $(Y_E, N_1, \dots, N_m)$. Further localizing R we can extend $\pi_E : X_E \to Y_E$ to a proper morphism $\pi_R : X_R \to Y_R$. In the case $i = 2$ after another localization we can extend $\mathcal{L}_E$ to a relatively ample line bundle on X_R. Further localizing R we can assume that $(\pi_R)_*(M_1)_R, \dots, (\pi_R)_*(M_k)_R$ belong to the triangulated subcategory of $D_c^b(Y_R, \mathbb{F})$ generated by $(N_1)_R, \dots, (N_m)_R$. We claim that after these localizations the fixed point schemes $\mathrm{Fix}(\mathrm{Fr}_v^k, (\widehat{\mathrm{Bad}}_i)_z)$, $i \in \{1, 2\}$, have no $\bar{\mathbb{Z}}_l$-points for every closed point $v \in \operatorname{Spec} R$ and every $k \in \mathbb{N}$. Let us prove this for $i = 1$ (the case $i = 2$ is quite similar).

Suppose there exists a Fr_v^k-invariant $\bar{\mathbb{Z}}_l$-point of $(\widehat{\mathrm{Bad}}_1)_z$. It comes from a Fr_v^k-invariant O-point ξ of $(\widehat{\mathrm{Bad}}_1)_z$, where O is the ring of integers in some finite extension $F \supset \mathbb{Q}_l$. We have $[O/\mathfrak{m} : \mathbb{F}] < \infty$, where $\mathfrak{m}$ is the maximal ideal of O and $\mathbb{F}$ is the residue field of z. Our ξ corresponds to an O-flat lisse perverse sheaf $\tilde{M}$ on $U_{\bar{E}}$ such that $\tilde{M}/\mathfrak{m}\tilde{M} \simeq M_{U_{\bar{E}}} \otimes_{\mathbb{F}} (O/\mathfrak{m})$ and the complex $(\pi_{\bar{E}})_*(j_{\bar{E}})_{!*}\tilde{M} \otimes_O F$ is not semisimple (here $M_{U_{\bar{E}}}$ is the pull-back of M_{U_E} to $U_{\bar{E}}$ and the morphisms $j_{\bar{E}} : U_{\bar{E}} \hookrightarrow X_{\bar{E}}$, $\pi_{\bar{E}} : X_{\bar{E}} \to Y_{\bar{E}}$ are induced by $j_E : U_E \hookrightarrow X_E$, $\pi_E : X_E \to Y_E$). Besides, $(\sigma^k)^*\tilde{M} \simeq \tilde{M}$, where $\sigma \in \mathrm{Gal}(\bar{E}/E_v)$ is a preimage of the Frobenius element of $\pi_1(\operatorname{Spec} R, \bar{v})$.

The perverse sheaf $(j_{\bar{E}})_{!*}\tilde{M} \otimes_O F$ is absolutely irreducible, the complex $(\pi_{\bar{E}})_*(j_{\bar{E}})_{!*}\tilde{M} \otimes_O F$ is not semisimple, and $(\sigma^k)^*(j_{\bar{E}})_{!*}\tilde{M} \simeq (j_{\bar{E}})_{!*}\tilde{M}$. By 6.2.1 $(j_{\bar{E}})_{!*}\tilde{M} \in D_{\{M_i\}}(X_{\bar{E}}, O)$, so $(\pi_{\bar{E}})_*(j_{\bar{E}})_{!*}\tilde{M} \in D_{\{N_i\}}(Y_{\bar{E}}, O)$. We have a commutative diagram of $\mathrm{Gal}(\bar{E}/E_v)$-equivariant functors

$$\begin{array}{ccc} D_{\{M_i\}}(X_{\bar{E}}, O) & \xrightarrow{\sim} & D_{\{M_i\}}(X_{\bar{v}}, O) \\ \downarrow & & \downarrow \\ D_{\{N_i\}}(Y_{\bar{E}}, O) & \xrightarrow{\sim} & D_{\{N_i\}}(Y_{\bar{v}}, O) \end{array} \tag{6.1}$$

where the horizontal arrows are the equivalences (4.5) and the vertical ones are $(\pi_{\bar{E}})_*$ and $(\pi_{\bar{v}})_*$. Let $P \in D_{\{M_i\}}(X_u, O)$ be the image of $(j_{\bar{E}})_{!*}\tilde{M}$. Then P is a perverse sheaf on $X_{\bar{v}}$ such that $P \otimes_O F$ is absolutely irreducible but the complex

$(\pi_{\bar{v}})_* P \otimes_O F$ is not semisimple. Besides, the isomorphism class of P is invariant with respect to $\pi_1(v_k, \bar{v})$, $v_k := \operatorname{Spec} \kappa_k$, where κ_k is the extension of order k of the residue field of v inside the residue field of $\bar{v}$. So P is the pull-back of a perverse sheaf P_0 on $X_{v_k} := X_R \otimes_R v_k$. According to Corollary VII.8 of [L] and Corollary 5.3.2 of [BBD], $P_0 \otimes_O F$ becomes pure after tensoring it by a rank 1 sheaf on v_k. So by §5.1.14 and §5.4.6 of [BBD] $(\pi_{\bar{v}})_* P \otimes_O F$ is semisimple, and we get a contradiction.

6.2.4. *Proof for* $i = 3$. Let $Y_E \subset X_E$ denote the subscheme $f = 0$. Let $N_1, \ldots, N_r$ be the irreducible components of the perverse sheaves $\Psi_f(M_1), \ldots, \Psi_f(M_k)$. Localizing R we can assume that f extends to a regular function f_R on X_R. Further localizing R we can choose a very good model $(Y_R, (N_1)_R, \ldots, (N_r)_R)$ of $(Y_E, N_1, \ldots, N_r)$ such that Y_R is the closed subscheme of X_R defined by $f_R = 0$. By 5.4, after another localization all the models (Y_E, N_j, f), $1 \le j \le r$, become Ψ_f-good in the sense of 5.2. By 5.3(ii) we can assume that the number m from the definition of "Ψ_f-good" does not depend on j. Further localizing R we can assume that $\Psi_{f_R}^{(m)}(M_1)_R, \ldots, \Psi_{f_R}^{(m)}(M_k)_R$ are in the triangulated subcategory of $D_c^b(Y_R, \mathbb{F})$ generated by $(N_1)_R, \ldots, (N_r)_R$. We claim that after these localizations $\operatorname{Fix}(\operatorname{Fr}_v^k, (\widehat{\operatorname{Bad}_3})_z)$ has no $\bar{\mathbb{Z}}_l$-points for every closed point $v \in \operatorname{Spec} R$ and every $k \in \mathbb{N}$.

We will explain only the parts of the proof that are not quite the same as in 6.2.3. First, instead of the semisimplicity theorem from [BBD] we use Gabber's theorem (§5.1.2 of [BB]). Second, the role of (6.1) is played by the diagram

$$\begin{array}{ccc} D_{\{M_i\}}(X_{\bar{E}}, O) & \xrightarrow{\sim} & D_{\{M_i\}}(X_{\bar{v}}, O) \\ \downarrow & & \downarrow \\ D_{\{N_i\}}(Y_{\bar{E}}, O[[\Gamma]]) & \xrightarrow{\sim} & D_{\{N_i\}}(Y_{\bar{v}}, O[[\Gamma]]) \end{array} \tag{6.2}$$

Here $O[[\Gamma]]$ is the completed group algebra of $\Gamma := \mathbb{Z}_l(1)$, i.e., the projective limit of $O[\Gamma/l^n\Gamma]$, $n \in \mathbb{N}$. We define the "O-constructible derived category" $D^b_{O\text{-}c}(Y_{\bar{E}}, O[[\Gamma]])$ as $\varprojlim \varinjlim D_c^b(Y_{\bar{E}}, (O/\mathfrak{m}^s)[\Gamma/l^n\Gamma])$ ($\varprojlim$ with respect to s and $\varinjlim$ with respect to n). We denote by $D_{\{N_i\}}(Y_{\bar{E}}, O[[\Gamma]])$ the full subcategory of objects $C \in D^b_{O\text{-}c}(Y_{\bar{E}}, O[[\Gamma]])$ such that the image of C in $D_c^b(Y_{\bar{E}}, O)$ belongs to $D_{\{N_i\}}(Y_{\bar{E}}, O)$. The upper horizontal arrow in (6.2) is the equivalence (4.5) and the lower one is its analog for $O[[\Gamma]]$. The vertical arrows in (6.2) are Ψ_f and $\Psi_{f_{\bar{v}}}$. The $O[[\Gamma]]$-structure on the nearby cycles comes from the identification of Γ with the Sylow l-subgroup of the tame quotient of $\operatorname{Gal}(\overline{K((t))}/K((t)))$, where K is an algebraically closed field, $\operatorname{char} K \neq l$. Just as in 6.2.3, $\operatorname{Gal}(\bar{E}/E_v)$ acts on the four categories from (6.2), and the arrows are $\operatorname{Gal}(\bar{E}/E_v)$-equivariant functors. To construct a canonical isomorphism between the two functors $D_{\{M_i\}}(X_{\bar{E}}, O) \to D_{\{N_i\}}(Y_{\bar{v}}, O[[\Gamma]])$ from (6.2) insert the functor $\Psi^{(m)}_{f_{R_{\bar{v}}}} : D_{\{M_i\}}(X_R \otimes_R R_{\bar{v}}, O) \to D_{\{N_i\}}(Y_R \otimes_R R_{\bar{v}}, O[[\Gamma]])$ as a vertical arrow between the two vertical arrows of (6.2). Here $\Psi^{(m)}_{f_{R_{\bar{v}}}}$ is the analog of $\Psi^{(m)}_{f_R}$ with R and X_R replaced by $R_{\bar{v}}$ and $X_R \otimes_R R_{\bar{v}}$.

Acknowledgements

I thank A. Beilinson for drawing my attention to Kashiwara's conjectures and numerous discussions. I also thank A.J. de Jong, who had informed me about his work [dJ] long before it was published.

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