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COHEN–MACAULAY QUOTIENTS OF NORMAL SEMIGROUP RINGS VIA IRREDUCIBLE RESOLUTIONS

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Abstract. For a radical monomial ideal I in a normal semigroup ring $k[Q]$, there is a unique minimal *irreducible resolution* $0 \to k[Q]/I \to \overline{W}^0 \to \overline{W}^1 \to \cdots$ by modules $\overline{W}^i$ of the form $\bigoplus_j k[F_{ij}]$, where the F_{ij} are (not necessarily distinct) faces of Q. That is, $\overline{W}^i$ is a direct sum of quotients of $k[Q]$ by prime ideals. This paper characterizes Cohen–Macaulay quotients $k[Q]/I$ as those whose minimal irreducible resolutions are *linear*, meaning that $\overline{W}^i$ is pure of dimension $\dim(k[Q]/I) - i$ for $i \geq 0$. The proof exploits a graded ring-theoretic analogue of the Zeeman spectral sequence [Zee63], thereby also providing a combinatorial topological version involving no commutative algebra. The characterization via linear irreducible resolutions reduces to the Eagon–Reiner theorem [ER98] by Alexander duality when $Q = \mathbb{N}^d$.

1. Introduction

Let $Q \subseteq \mathbb{Z}^d$ be an affine semigroup, which we require to be saturated except in Section 2. We assume everywhere for simplicity that Q generates $\mathbb{Z}^d$ as a group, and that Q has trivial unit group. The real cone $\mathbb{R}_{\geq 0}Q$ is a polyhedral cell complex; let $\Delta \subseteq \mathbb{R}_{\geq 0}Q$ be a closed polyhedral subcomplex. Corresponding to Δ is the ideal I_Δ inside the semigroup ring $k[Q]$, generated (as a k-vector space) by all monomials in $k[Q]$ not lying on any face of Δ. Thus $k[Q]/I_\Delta$ is spanned by monomials lying in Δ.

This paper has three goals, the latter two only for saturated affine semigroups Q:

- Define the notion of *irreducible resolution* for Q-graded $k[Q]$-modules.
- Introduce the *Zeeman double complex* for Δ.
- Characterize Cohen–Macaulay quotients $k[Q]/I_\Delta$ in terms of the above items.

An irreducible resolution (Definition 2.1) of a $\mathbb{Z}^d$-graded $k[Q]$-module is an injective-like resolution, in which the summands are quotients of $k[Q]$ by irreducible monomial ideals rather than indecomposable injectives. Minimal irreducible resolutions exist uniquely up to isomorphism for all Q-graded modules M

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(Theorem 2.4). When $k[Q]$ is normal and $M = k[Q]/I_\Delta$, every summand is isomorphic to a semigroup ring $k[F]$, considered as a quotient module of $k[Q]$, for some face $F \in \Delta$ (Corollary 3.5).

The Zeeman double complex $D(\Delta)$ consists of $k[Q]$-modules that are direct sums of semigroup rings $k[F]$ for faces $F \in \Delta$ (Definition 3.1). Its naturally defined differentials come from the maps in the chain and cochain complexes of Δ.

Here is the idea behind the Cohen–Macaulay criterion, Theorem 4.2. Although the total complex of the Zeeman double complex $D(\Delta)$ is an example of an irreducible resolution (Theorem 3.4), its large number of summands keeps it far from being minimal. However, the cancellation afforded by the horizontal differential of $D(\Delta)$ sometimes causes the resulting vertical differential (on the horizontal cohomology) to be a minimal irreducible resolution. This fortuitous cancellation occurs precisely when Δ is Cohen–Macaulay over k, in which case the horizontal cohomology occurs in exactly one column of $D(\Delta)$.

Part 2 of Theorem 4.2, which characterizes the Cohen–Macaulay property by collapsing at E^1 of the ordinary Zeeman spectral sequence for Δ (Definition 3.6), may be of interest to algebraic or combinatorial topologists. Its statement as well as its proof are independent of the surrounding commutative algebra.

The methods involving Zeeman double complexes and irreducible resolutions should have applications beyond those investigated here; see Section 5 for possibilities.

Notational conventions. The saturation of Q is the intersection with $\mathbb{Z}^d$ of the positive half-spaces defined by primitive integer-valued functionals $\tau_1, \ldots, \tau_n$ on $\mathbb{Z}^d$. The i^{th} *facet* of Q (for $i = 1, \ldots, n$) is the subset $F_i \subseteq Q$ on which τ_i vanishes. More generally, an arbitrary *face* of Q is defined by the vanishing of a linear functional on $\mathbb{Z}^d$ that is nonnegative on Q (it is not required that Q be saturated for this to make sense). The (Laurent) monomial in $k[\mathbb{Z}^d]$ with exponent α is denoted by $\mathbf{x}^\alpha$, although sets of monomials in $k[\mathbb{Z}^d]$ are frequently identified with their exponent sets in $\mathbb{Z}^d$.

All cellular homology and cohomology groups are taken with coefficients in the field k unless otherwise stated. We work here always with nonreduced (co)homology of the usually unbounded polyhedral complex Δ, which corresponds to the reduced (co)homology of an always bounded transverse hyperplane section of Δ, homologically shifted by 1.

All modules in this paper, including injective modules, are $\mathbb{Z}^d$-graded unless otherwise stated. Elementary facts regarding the category of $\mathbb{Z}^d$-graded $k[Q]$-modules, especially $\mathbb{Z}^d$-graded injective modules, hulls, and resolutions, can be found in [GW78], although some crucial facts will be reviewed as necessary.

2. Irreducible resolutions

The definitions and results in this section hold for unsaturated as well as saturated affine semigroups Q, with the same proofs.

Recall that an ideal W inside of $k[Q]$ is *irreducible* if W can not be expressed as an intersection of two ideals properly containing it.

Definition 2.1. An *irreducible resolution* $\overline{W}^{\bullet}$ of a $k[Q]$-module M is an exact sequence

$$0 \to M \to \overline{W}^0 \to \overline{W}^1 \to \cdots \qquad \overline{W}^i = \bigoplus_{j=1}^{\mu_i} k[Q]/W^{ij}$$

in which each W^{ij} is an irreducible monomial ideal of $k[Q]$. The irreducible resolution is called *minimal* if all the numbers μ_i are simultaneously minimized (among irreducible resolutions of M), and *linear* if $\overline{W}^i$ is pure of Krull dimension $\dim(M) - i$ for all i. (By convention, modules of negative dimension are zero.)

The fundamental properties of quotients $\overline{W} := k[Q]/W$ by irreducible monomial ideals W are inherited from the corresponding properties of indecomposable injective modules. Recall that each such *indecomposable injective module* is a vector space $k\{\alpha + E_F\}$ spanned by the monomials in $\alpha + E_F$, where

$$E_F \quad = \quad \{f - a \mid f \in F \text{ and } a \in Q\} \tag{1}$$

is the "negative tangent cone" along the face F of Q. The vector space $k\{\alpha+E_F\}$ carries an obvious structure of $k[Q]$-module.

In what follows, the $\mathbb{Z}^d$-graded injective hull of a $\mathbb{Z}^d$-graded module M is denoted by $E(M)$, so that, in particular, $E(k[F]) = k\{E_F\}$. Elementary behavior of $\mathbb{Z}^d$-graded injective hulls can be found in [GW78], the main facts required here being:

1. M has a minimal injective resolution, unique up to noncanonical isomorphism.
2. Any injective resolution of M is (noncanonically) the direct sum of a minimal injective resolution and a split exact injective resolution of 0.
3. The minimal injective resolution of M has finitely many indecomposable summands in each cohomological degree, if M is finitely generated.

Define the *Q-graded part* of M to be the submodule $\bigoplus_{a\in Q} M_a$ generated by elements whose degrees lie in Q.

Lemma 2.2. *A monomial ideal W is irreducible if and only if $\overline{W} := k[Q]/W$ is the Q-graded part of some indecomposable injective module.*

Proof. ($\Leftarrow$) The module $k\{\alpha+E_F\}_Q$ is clearly isomorphic to $\overline{W}$ for some ideal W. Supposing that $W \neq k[Q]$, we may as well assume $\alpha \in Q$ by adding an element far inside F, so that $\mathbf{x}^\alpha \in \overline{W}$ generates an essential submodule $k\{\alpha + F\}$; that is, every nonzero submodule of $\overline{W}$ intersects $k\{\alpha + F\}$ nontrivially. Suppose $W = W_1 \cap W_2$. The copy of $k\{\alpha + F\}$ inside $\overline{W}$ must include into $\overline{W}_j$ for $j = 1$ or 2. Indeed, if both induced maps $k\{\alpha + F\} \to \overline{W}_j$ have nonzero kernels, then they intersect in a nonzero submodule of $k\{\alpha + F\}$ because $k[F]$ is a domain. The essentiality of $k\{\alpha + F\} \subseteq \overline{W}$ then forces $\overline{W} \to \overline{W}_j$ to be an inclusion for some j. We conclude that W contains this W_j, so $W = W_j$ is irreducible.

($\Rightarrow$) Let W be an irreducible ideal and $\overline{W} = k[Q]/W$. Considering the injective hull $E(\overline{W}) = J_1 \oplus \cdots \oplus J_r$, the composite $k[Q] \to \overline{W} \to E(\overline{W})$ has kernel $W = W_1 \cap \cdots \cap W_r$, where $\overline{W}_j = (J_j)_Q$. Since W is irreducible, we must have $W_j = W$ for some j. We conclude that $E(\overline{W}) = J_j$, and $W = W_j$. □

Lemma 2.3. *For any finitely generated module M, there exists $\beta \in \mathbb{Z}^d$ such that $M_\beta \neq 0$, and for all $\gamma \in \beta + Q$, the inclusion $M_\gamma \hookrightarrow E(M)_\gamma$ is an isomorphism.*

Proof. Suppose that $E(M) = \bigoplus_{\alpha,F} k\{\alpha + E_F\}^{\mu(\alpha,F)}$, where we assume that $\alpha + E_F \neq \alpha' + E_{F'}$ whenever $(\alpha, F) \neq (\alpha', F')$. Now fix a pair (α, F) so that $\alpha + E_F$ is maximal inside $\mathbb{Z}^d$ among all such subsets appearing in the direct sum. Clearly we may assume F is maximal among faces of Q appearing in the direct sum. Pick an element f that lies in the relative interior of F.

By (1) and the maximality of F, some choice of $r \in \mathbb{N}$ pushes the $\mathbb{Z}^d$-degree $\alpha + r \cdot f \in \alpha + E_F$ outside of $\alpha' + E_{F'}$ for all (α', F') satisfying $F' \neq F$. Moreover, the maximality of (α, F) implies that $\alpha + r \cdot f \notin \alpha' + E_F$ whenever $\alpha' \neq \alpha$.

The prime ideal P_F satisfying $k[Q]/P_F = k[F]$ is minimal over the annihilator of M. Therefore, if $M' = (0 :_M P_F)$ is the submodule of M annihilated by P_F, the composite injection $M' \hookrightarrow M \hookrightarrow E(M)$ becomes an isomorphism onto its image after homogeneous localization at P_F—that is, after inverting the monomial $\mathbf{x}^f$. It follows that choosing $r \in \mathbb{N}$ large enough forces isomorphisms

$$(2) \qquad M_{\alpha + r\cdot f} \;\stackrel{\approx}{\to}\; E(M)_{\alpha + r \cdot f} \;\stackrel{\approx}{\to}\; \big(k\{\alpha + E_F\}^{\mu(\alpha,F)}\big)_{\alpha + r\cdot f} \;\cong\; k^{\mu(\alpha,F)}.$$

Setting $\beta = \alpha + r \cdot f$, the multiplication map $\mathbf{x}^{\gamma - \beta} : E(M)_\beta \to E(M)_\gamma$ for $\gamma \in \beta + Q$ is either zero or an isomorphism, because $E(M)$ agrees with $k\{\alpha + E_F\}^{\mu(\alpha,F)}$ in degrees β and γ by construction. The previous sentence holds with M in place of $E(M)$ by (2), because M is a submodule of $E(M)$. □

Not every module has an irreducible resolution, because being Q-graded is a prerequisite. However, Q-gradedness is the only restriction, as the next theorem shows.

Theorem 2.4. *Let $M = M_Q$ be a finitely generated Q-graded module. Then:*

1. *M has a minimal irreducible resolution, unique up to noncanonical isomorphism.*
2. *Any irreducible resolution of M is (noncanonically) the direct sum of a minimal irreducible resolution and a split exact irreducible resolution of* 0.
3. *The minimal irreducible resolution of M has finitely many irreducible summands in each cohomological degree.*
4. *The minimal irreducible resolution of M has finite length; that is, it vanishes in all sufficiently high cohomological degrees.*
5. *The Q-graded part of any injective resolution of M is an irreducible resolution.*
6. *Every irreducible resolution of M is the Q-graded part of an injective resolution.*

Proof. Lemma 2.2 implies part 5. Parts 1–3 therefore follow from part 6 and the corresponding facts about $\mathbb{Z}^d$-graded injective resolutions before Lemma 2.2. Part 4, on the other hand, is false for injective resolutions whenever $k[Q]$ is not isomorphic to a polynomial ring, so we prove it separately at the end.

Focusing now on part 6, let $\overline{W}^{\bullet}$ be an irreducible resolution of M, and set $J^0 = E(\overline{W}^0)$. The inclusion $M \hookrightarrow J^0$ has Q-graded part $M \hookrightarrow \overline{W}^0$ by Lemma 2.2. Making use of the defining property of injective modules, extend the composite inclusion $\overline{W}^0/M \hookrightarrow \overline{W}^1 \hookrightarrow E(\overline{W}^1)$ to a map $J^0/M \to E(\overline{W}^1)$, and let K^0 be the kernel. Then K^0 has zero Q-graded part because $\overline{W}^0/M \hookrightarrow \overline{W}^1$ is a monomorphism. The injective hull $K^0 \hookrightarrow E(K^0)$ therefore has zero Q-graded part. Extending $K^0 \hookrightarrow E(K^0)$ to a map $J^0/M \to E(K^0)$ yields an injection $J^0/M \to J^1 := E(K^0) \oplus E(\overline{W}^1)$ whose Q-graded part is $\overline{W}^0/M \hookrightarrow \overline{W}^1$. Replacing M, 0 and 1 by image$(\overline{W}^{i-1} \to \overline{W}^i)$, i and $i+1$ in this discussion produces the desired injective resolution by induction.

Finally, for the length-finiteness in part 4, consider the set $V(M)$ of degrees $a \in Q$ such that M_b vanishes for all $b \in a + Q$. The vector space $k\{V(M)\}$ is naturally an ideal in $k[Q]$. Lemma 2.3 implies that $V(M) \subsetneq V(\overline{W}/M)$ whenever $\overline{W}$ is the Q-graded part of an injective hull of M and $M \neq 0$ (that is, $V(M) \neq Q$). The noetherianity of $k[Q]$ plus this strict containment force the sequence of ideals

$$k\{V(M)\} \;\subseteq\; k\{V(\overline{W}^0/M)\} \;\subseteq\; k\{V(\overline{W}^1/\mathrm{image}(\overline{W}^0))\} \;\subseteq\; \cdots$$

to stabilize at the unit ideal of $k[Q]$ after finitely many steps. □

Remark 2.5. From another perspective, Theorem 2.4 says that the category of Q-graded modules has enough injectives, with the resulting "minimal injective resolutions" being unique and finite. Lemma 2.2 says that the indecomposable injectives in the category of Q-graded modules have the form $k[Q]/W$ for irreducible ideals W.

Examples of irreducible resolutions include Theorem 3.4, below, as well as the proof of Lemma 3.3, which contains the irreducible resolution of the canonical module of $k[F]$ in (3). In general, any injective resolution of a $\mathbb{Z}^d$-graded module yields an irreducible resolution of its Q-graded part, although the indecomposable injective summands with zero Q-graded part get erased. In particular, the "cellular injective resolutions" of [Mil00] become what should be called "cellular irreducible resolutions" here.

3. Zeeman double complex

This section introduces the Zeeman double complex and its resulting spectral sequences for saturated affine semigroups Q. The total complex of the Zeeman double complex in Theorem 3.4 provides a natural but generally nonminimal irreducible resolution for $k[Q]/I_\Delta$.

Assume for this section that Q is saturated. For each face $G \in \Delta$, let $k[G]$ be the affine semigroup ring for G, considered as a quotient of $k[Q]$, and denote by e_G the canonical generator of $k[G]$ in $\mathbb{Z}^d$-graded degree $\mathbf{0}$. Also, for each face

$F \in \Delta$, let kF be a 1-dimensional k-vector space spanned by F in $\mathbb{Z}^d$-graded degree $\mathbf{0}$.

Definition 3.1. Consider the $k[Q]$-module $D(\Delta) = \bigoplus_{F \supseteq G} kF \otimes_k k[G]$ generated by

$$\{F \otimes_k e_G \mid F, G \in \Delta \text{ and } F \supseteq G\},$$

with $k[Q]$ acting only on the right hand factors $k[G]$. Doubly index the generators so that $D(\Delta)_{pq}$ is generated by

$$\{F \otimes e_G \mid p = \dim F \text{ and } -q = \dim G\},$$

and hence $\{\mathbf{0}\} \otimes e_{\{\mathbf{0}\}} \in D(\Delta)_{00}$, with the rest of the double complex in the fourth quadrant. Now define the *Zeeman double complex* of $\mathbb{Z}^d$-graded $k[Q]$-modules to be $D(\Delta)$, with vertical differential ∂ and horizontal differential δ as in the diagram:

$$\partial e_G \;=\; \sum_{\substack{G' \subset G \\ \text{is a facet}}} \varepsilon(G', G) e_{G'} \qquad \begin{array}{ccc} F \otimes \partial e_G & & \\ \partial \uparrow & & \\ F \otimes e_G & \xrightarrow{\delta} & \delta F \otimes e_G \end{array} \qquad (-1)^q \delta F \;=\; \sum_{\substack{F \subset F' \\ \text{is a facet}}} \varepsilon(F, F') F',$$

where the signs $\varepsilon(G', G)$ and $\varepsilon(F, F')$ come from a fixed incidence function on Δ (defined by relative orientations of the polyhedral faces, say).

For each fixed G, the elements $F \otimes e_G$ generate a summand of $D(\Delta)$ closed under the horizontal differential δ. Taking the sum over all G yields the horizontal complex

$$(D(\Delta), \delta) = \bigoplus_{G \in \Delta} C^{\bullet}(\Delta_G) \otimes k[G], \quad \text{where} \quad \Delta_G = \{F \in \Delta \mid F \supseteq G\}$$

is the *part of* Δ *above* G. It is straightforward to verify that $C^{\bullet}(\Delta_G)$ is isomorphic to the reduced cochain complex $\widetilde{C}^{\bullet}\mathrm{link}(G, \Delta)$ of the link of G in Δ (also known as the vertex figure of G in Δ), but with $\varnothing$ in homological degree $\dim G$ instead of -1:

$$C^{i+1+\dim G}(\Delta_G) \quad \cong \quad \widetilde{C}^{i}\mathrm{link}(G, \Delta).$$

The cohomology $H^i(C^{\bullet}(\Delta_G))$ is also called the *local cohomology* $H^i_G(\Delta)$ of Δ near G. Since the complex $C^{\bullet}(\Delta_G)$ is naturally a subcomplex of $C^{\bullet}(\Delta_{G'})$ whenever $G' \subseteq G$, the natural restriction maps $H^i_G(\Delta) \to H^i_{G'}(\Delta)$ make local cohomology into a sheaf on Δ. The following is immediate from the above discussion.

Lemma 3.2. *In column p, the vertical complex $(H_\delta D(\Delta), \partial)$ of $k[Q]$-modules has $\bigoplus_{\dim G = q} H^p_G(\Delta) \otimes_k k[G]$ in cohomological degree $-q$. The vertical differential ∂ is comprised of the natural maps $H^p_G(\Delta) \otimes e_G \to H^p_{G'}(\Delta) \otimes \varepsilon(G', G) e_{G'}$ for facets G' of G.*

We'll need to know the vertical cohomology $H_\partial D(\Delta)$ of $D(\Delta)$, too.

Lemma 3.3. *$H_\partial D(\Delta) = \bigoplus_{F\in\Delta}\omega_{k[F]}$, where $\omega_{k[F]}$ is the canonical module of $k[F]$, and each summand $\omega_{k[F]}$ sits along the diagonal in bidegree $(p,q) = (\dim F, -\dim F)$.*

Proof. Collecting the terms with fixed F yields the tensor product of kF with

$$0 \to k[F] \to \bigoplus_{\text{facets } F' \text{ of } F} k[F'] \to \cdots \to \bigoplus_{\text{rays } v\in F} k[v] \to k \to 0. \tag{3}$$

The $\mathbb{Z}^d$-graded degree a part of this complex is zero unless $a \in F$, in which case we get the local homology complex $C_\bullet(F_{F''})$ of F near the face F'' containing a in its relative interior. Such local homology is zero unless $F'' = F$. Therefore, the only homology of (3) is the canonical module $\omega_{k[F]}$, being the kernel of the first map. □

Theorem 3.4. *The total complex $\operatorname{tot} D(\Delta)$ of the Zeeman double complex is an irreducible resolution of $k[Q]/I_\Delta$.*

Proof. The spectral sequence obtained by first taking vertical cohomology of $D(\Delta)$ has $H_\partial D(\Delta) = E^1 = E^\infty$ by Lemma 3.3. The same lemma implies that the cohomology of $\operatorname{tot} D(\Delta)$ is zero except in degree $p+q=0$, and that the nonzero cohomology has a filtration whose associated graded module is $\bigoplus_{F\in\Delta}\omega_{k[F]}$. The Hilbert series of this cohomology module equals that of $k[Q]/I_\Delta$.

On the other hand, the map $\phi : k[Q] \to D(\Delta)$ sending $1 \mapsto \sum_{F\in\Delta}\epsilon_F F\otimes e_F$ has kernel I_Δ, for any choice of signs $\epsilon_F = \pm 1$. The image of ϕ is thus isomorphic to $k[Q]/I_\Delta$. Choosing the signs

$$\epsilon_F = (-1)^{\dim F(\dim F+1)/2} = \begin{cases} -1 & \text{if } \dim F \equiv 1,2 \pmod 4 \\ 1 & \text{if } \dim F \equiv 0,3 \pmod 4 \end{cases}$$

forces $(\delta+\partial)(\sum_{F\in\Delta}\epsilon_F F\otimes e_F) = 0$, thanks to the factor $(-1)^q$ in the definition of δ. By Hilbert series considerations, the image of ϕ equals the kernel of $\delta+\partial$. □

Corollary 3.5. *Every summand in the minimal irreducible resolution of the quotient $k[Q]/I_\Delta$ by a radical monomial ideal I_Δ is isomorphic to $k[F]$ for some face F of Q.*

Proof. Every summand in the total Zeeman complex of Theorem 3.4 has the desired form. Now apply Theorem 2.4.2. □

The spectral sequence in the proof of Theorem 3.4 always converges rather early, at E^1. The other spectral sequence, however, obtained by first taking the horizontal cohomology H_δ, may be highly nontrivial.

Definition 3.6. The *$\mathbb{Z}^d$-graded Zeeman spectral sequence* for the polyhedral complex Δ is the spectral sequence $\mathbb{Z}E^\bullet_{pq}(\Delta)$ on the double complex $D(\Delta)$ obtained by taking horizontal homology first, so $\mathbb{Z}E^2_{pq}(\Delta) = H_\partial H_\delta D(\Delta)$. The *ordinary Zeeman spectral sequence* for Δ is the $\mathbb{Z}^d$-graded degree $\mathbf{0}$ piece $ZE^\bullet_{pq}(\Delta) = \mathbb{Z}E^\bullet_{pq}(\Delta)_{\mathbf{0}}$.

4. Characterization of Cohen–Macaulay quotients

This section contains a characterization of Cohen–Macaulayness in terms of irreducible resolutions coming from the Zeeman double complex $D(\Delta)$. As in the previous section, assume that Q is saturated.

Definition 4.1. The polyhedral complex Δ is *Cohen–Macaulay over* k if the local cohomology over k of Δ near every face $G \in \Delta$ satisfies $H^i_G(\Delta) = 0$ for $i < \dim \Delta$.

Theorem 4.2. *Let $I = I_\Delta$ be a radical monomial ideal. The following are equivalent.*

1. *Δ is Cohen–Macaulay over k.*
2. *The only nonzero vector spaces $ZE^1_{pq}(\Delta)$ lie in column $p = \dim(\Delta)$.*
3. *The complex $\mathbb{Z}E^1(\Delta)$ is a minimal linear irreducible resolution of $k[Q]/I$.*
4. *$k[Q]/I$ has a linear irreducible resolution.*
5. *$k[Q]/I$ is a Cohen–Macaulay ring.*

Proof. 1 $\Leftrightarrow$ 2: The $\mathbb{Z}^d$-degree $\mathbf{0}$ part of Lemma 3.2 says that ZE^1 has $\bigoplus_{\dim G=q} H^p_G(\Delta)$ in cohomological degree $-q$. The equivalence is now immediate from Definition 4.1.

1 $\Rightarrow$ 3: The E^1 term in question is the complex $H_\delta D(\Delta)$, with the differential ∂ in Lemma 3.2. That lemma together with Definition 4.1 implies that the horizontal cohomology $H_\delta D(\Delta)$ has one column (indexed by $\dim \Delta$), that must therefore be a resolution of something having the same Hilbert series as $k[Q]/I_\Delta$ by Theorem 3.4. Set $n = \dim \Delta$. Since $H^n_F(\Delta) = kF$ for facets $F \in \Delta$, it is enough to check that the diagonal embedding $k[Q]/I_\Delta \hookrightarrow \bigoplus_{\text{facets } F\in\Delta} k[F]$ is contained in the kernel of the first map of $(H_\delta D(\Delta), \partial)$. If $\dim G = n-1$ for some face $G \in \Delta$, then

$$H^n_G(\Delta) \quad = \quad \bigl(\bigoplus kF\bigr)/\langle \textstyle\sum \varepsilon(G,F)F\rangle,$$

both sums being over all facets $F \in \Delta$ containing G. Now calculate

$$\begin{aligned}(4)\qquad \partial\Bigl(\sum_{\dim F=n} F\otimes e_F\Bigr) &= \sum_{\dim F=n}\sum_{F\supset G} F\otimes\varepsilon(G,F)e_G \\ &= \sum_{\dim G=n-1}\Bigl(\sum_{F\supset G}\varepsilon(G,F)F\Bigr)\otimes e_G \;=\; 0.\end{aligned}$$

3 $\Rightarrow$ 4: Trivial.

4 $\Rightarrow$ 5: If M is any module having a linear irreducible resolution $\overline{W}^\bullet$ in which each summand of $\overline{W}^i$ is isomorphic to $k[F]$ for some face F, such as $M = k[Q]/I$ as in Corollary 3.5, then M is Cohen–Macaulay. This can be seen by induction on $\dim(M)$ via the long exact sequence for local cohomology $H^i_{\mathfrak{m}}$, where $\mathfrak{m}$ is the graded maximal ideal. The induction requires the modules $\overline{W}^i$ to be Cohen–Macaulay themselves, which holds because Q and hence all of its faces F are saturated.

$5 \Rightarrow 1$: $k[Q]/I$ being Cohen–Macaulay implies that $\underline{\mathrm{Ext}}^i_{k[Q]}(M, \omega_{k[Q]})$ is zero for $i \neq d-n$, where $n = \dim \Delta$. In particular, if $\Omega^\bullet$ is the Q-graded part of the minimal injective resolution of $\omega_{k[Q]}$, then the i^{th} cohomology of $\underline{\mathrm{Hom}}\,(k[Q]/I, \Omega^\bullet)$ is zero unless $i = d-n$. The complex $\Omega^\bullet$ is the linear irreducible resolution of $\omega_{k[Q]}$ in which each quotient $k[F]$ for $F \in \mathbb{R}_{\geq 0}Q$ appears precisely once; see (3). Since $\mathrm{Hom}(k[Q]/I, k[F]) = k[F]$ if $F \in \Delta$ and zero otherwise, $\underline{\mathrm{Hom}}\,(k[Q]/I, \Omega^\bullet)$ is

$$0 \to \bigoplus_{\substack{F\in\Delta \\ \dim F = n}} k[F] \to \cdots \to \bigoplus_{\substack{F\in\Delta \\ \dim F = \ell}} k[F] \to \cdots \to k \to 0.$$

If $a \in Q$ is in the relative interior of $G \in \Delta$, then the $\mathbb{Z}^d$-graded degree a component of this complex is the homological shift of $C^\bullet(\Delta_G)$ whose i^{th} cohomology is $H_G^{d-i}(\Delta)$. $\square$

Remark 4.3. Note the interaction of Theorem 4.2 with the characteristic of k: the horizontal cohomology of the Zeeman double complex $D(\Delta)$ can depend on $\mathrm{char}(k)$, just as the other parts of the theorem can.

When the semigroup Q is $\mathbb{N}^d$, so that $k[Q]$ is just the polynomial ring in d variables $z_1, \ldots, z_d$ over k, the polyhedral complex Δ becomes a simplicial complex. Thinking of Δ as an order ideal in the lattice $2^{[d]}$ of subsets of $[d] := \{1, \ldots, d\}$, the *Alexander dual* simplicial complex $\Delta^\star$ is the complement of Δ in $2^{[d]}$, but with the partial order reversed. Another way to say this is that $\Delta^\star = \{[d] \smallsetminus F \mid F \notin \Delta\}$.

Theorem 4.2 can be thought of as the extension to arbitrary normal semigroup rings of the Eagon–Reiner theorem [ER98], which concerns the case $Q = \mathbb{N}^d$, via the Alexander duality functors defined in [Mil00, Röm01]. To see how, recall that a $\mathbb{Z}$-graded $k[\mathbb{N}^d]$-module is said to have *linear free resolution* if its minimal $\mathbb{Z}$-graded free resolution over $k[z_1, \ldots, z_d]$ can be written using matrices filled with linear forms.

Corollary 4.4 (Eagon–Reiner)**.** *If Δ is a simplicial complex on $\{1, \ldots, d\}$, then Δ is Cohen–Macaulay if and only if $I_{\Delta^\star}$ has linear free resolution.*

Proof. The minimal free resolution of $I_{\Delta^\star}$ is the functorial Alexander dual (see [Röm01, Definition 1.9] or [Mil00, Theorem 2.6] with $\mathbf{a} = \mathbf{1}$) of the minimal irreducible resolution of $k[\mathbb{N}^d]/I_\Delta$ guaranteed by Theorem 4.2. Linearity of the irreducible resolution translates directly into linearity of the free resolution of $I_{\Delta^\star}$. $\square$

Remark 4.5. If M is a 'squarefree module' in the sense of Yanagawa [Yan01a], then a minimal irreducible resolution of M is a minimal injective resolution of M in the category of squarefree modules. The equivalence of parts 4 and 5 in Theorem 4.2 therefore holds with an arbitrary squarefree module in place of $k[Q]/I$, by [Yan01a, Corollary 4.17]. More generally, the analogue of Terai's theorem [Ter99], which measures the difference between depth and dimension, holds for squarefree modules by [Yan01a, Theorem 4.15].

Remark 4.6. Is there a generalization of Theorem 4.2 to the sequential Cohen–Macaulay case that works for arbitrary saturated semigroups, analogous and Alexander dual to the generalization [HRW99] of Corollary 4.4? Probably; and if so, it will likely say that the ordinary and $\mathbb{Z}^d$-graded Zeeman spectral sequences collapse at E^2 (i.e. all differentials in $E^{\geq 3}$ vanish).

Remark 4.7. The Alexander dual of the complex $\mathbb{Z}E^1(\Delta) = (H_\delta D(\Delta), \partial)$, which provides a linear free resolution of $I_{\Delta^\star}$ in the Cohen–Macaulay case, also provides the "linear part" of the free resolution of $I_{\Delta^\star}$ when Δ is arbitrary [RW01]. It is possible to give an apropos proof of this fact using the Alexander dual of the Zeeman spectral sequence for a Stanley–Reisner ring along with an argument due to J. Eagon [Eag90] concerning how to make spectral sequences into minimal free resolutions.

5. Remarks and further directions

Zeeman's original spectral sequence appears verbatim as the ordinary Zeeman spectral sequence $ZE^\bullet_{pq}$ in Definition 3.6, with $Q = \mathbb{N}^d$. Zeeman used his double complex and spectral sequence to provide an extension of Poincaré duality for singular triangulated topological spaces [Zee62a, Zee62b, Zee63]. When the topological space is a manifold, of course, usual Poincaré duality results. In the present context, Zeeman's version of the Poincaré duality isomorphism should glue two complexes of irreducible quotients of $k[\mathbb{N}^d]$ together to form the minimal irreducible resolution for the Stanley–Reisner ring of any Buchsbaum simplicial complex—these simplicial complexes behave much like manifolds. This gluing procedure should work also for the more general Buchsbaum polyhedral complexes Δ obtained by considering arbitrary saturated affine semigroups Q.

Theorem 4.2 is likely capable of providing a combinatorial construction of the "canonical Čech complex" for $I_{\Delta^\star}$ [Mil00, Yan01b] when Δ is Cohen–Macaulay, or even Buchsbaum (if the previous paragraph works). Although $\Delta^\star$ has only been defined a priori for simplicial complexes Δ, when $Q = \mathbb{N}^d$, the definition of functorial squarefree Alexander duality extends easily to the case of arbitrary saturated semigroups [Yan01a, Remark 4.18]. The catch is that $I_{\Delta^\star}$ is not an ideal in $k[Q]$, but rather an ideal in the semigroup ring $k[Q^\star]$ for the cone $Q^\star$ dual to Q. Combinatorially speaking, the face poset of Q is not usually self-dual, as it is when $Q = \mathbb{N}^d$, so the process of "reversing the partial order" geometrically forces the switch to $Q^\star$. The functorial part of Alexander duality follows the same pattern as the case $Q = \mathbb{N}^d$: quotients $k[F]$ of $k[Q]$ are dual to prime ideals $P_{F^\star}$ inside $k[Q^\star]$, where $F^\star$ is the face of $Q^\star$ dual to F. See also [Yan01b, Section 6].

In general, irreducible resolutions—and perhaps other resolutions by structure sheaves of subschemes—can be useful for computing the K-homology classes of reduced subschemes that are unions of transverally intersecting components. When the ambient scheme is regular, this method is an alternative to calculating free resolutions, which produce K-*co*homology classes. In particular, this holds for subspace arrangements in projective spaces. This philosophy underlies the application of irreducible resolutions in [KM01, Appendix A.3] to the definition of "multidegrees".

Note that when $Q \not\cong \mathbb{N}^d$, irreducible resolutions are the only finite resolutions to be had: free and injective resolutions of finitely generated modules rarely terminate. In particular, an understanding of the Hilbert series of irreducible quotients of $k[Q]$—a polyhedral problem—would give rise to formulae for Hilbert series of Q-graded modules. Similarly, algorithmic computations with irreducible resolutions can allow explicit computation of injective resolutions, local cohomology, and perhaps other homological invariants in the $\mathbb{Z}^d$-graded setting over semigroup rings.

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