Mathematical Research Letters **9**, 473–491 (2002)

THE RATIONAL COHOMOLOGY RING OF THE MODULI SPACE OF ABELIAN 3-FOLDS

Richard Hain

1. Introduction

Suppose that R is a commutative ring with unit. Denote by $Sp_n(R)$ the group of automorphisms of R^{2n} that preserve the unimodular alternating form given by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

In this note we compute the rational cohomology ring of $Sp_3(\mathbb{Z})$, or equivalently, of $\mathcal{A}_3$, the moduli space of principally polarized abelian 3-folds.

Denote the $\mathbb{Q}$-Hodge structure of dimension 1 and weight $-2n$ by $\mathbb{Q}(n)$. (For those not interested in Hodge theory, just interpret this as one copy of $\mathbb{Q}$.) Denote by λ the first Chern class in $H^2(\mathcal{A}_g;\mathbb{Q})$ of the Hodge bundle $\mathcal{H}_g := \pi_*\Omega^1_{\mathcal{X}_g/\mathcal{A}_g}$ associated to the projection $\pi : \mathcal{X}_g \to \mathcal{A}_g$ of the universal abelian g-fold.

Theorem 1. *The cohomology groups of $\mathcal{A}_3$ are given by*

$$H^j(\mathcal{A}_3;\mathbb{Q}) \cong H^j(Sp_3(\mathbb{Z});\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & j = 0; \\ \mathbb{Q}(-1) & j = 2; \\ \mathbb{Q}(-2) & j = 4; \\ E & j = 6; \\ 0 & \textit{otherwise}, \end{cases}$$

where E is a two-dimensional mixed Hodge structure that is an extension

$$0 \to \mathbb{Q}(-3) \to E \to \mathbb{Q}(-6) \to 0.$$

The ring structure is determined by the condition that $\lambda^3 \neq 0$.

I do not know whether the mixed Hodge structure (MHS) E on H^6 is split. Since $\mathcal{A}_3$ is a smooth stack over $\operatorname{Spec}\mathbb{Z}$, I expect it to be a multiple (possibly trivial) of the class

$$\zeta(3) \in \mathbb{C}/i\pi^3\mathbb{Q} \cong \operatorname{Ext}^1_{\mathrm{Hodge}}(\mathbb{Q},\mathbb{Q}(3))$$

Received March 8, 2002.
Supported in part by grant DMS-0103667 from the National Science Foundation.

given by the value of the Riemann zeta function at 3. Determining this class would be interesting.

As a corollary, we deduce the rational cohomology of $\overline{\mathcal{A}}_3$, the Satake compactification of $\mathcal{A}_3$.

Theorem 2. *The rational cohomology ring of $\overline{\mathcal{A}}_3$ is given by*

$$H^j(\mathcal{A}_3;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-n) & j = 2n, n \in \{0,1,2,4,5,6\}; \\ B & j = 6; \\ 0 & \textit{otherwise,} \end{cases}$$

where B is a 3-dimensional mixed Hodge structure that is an extension

$$0 \to \mathbb{Q}(0) \to B \to \mathbb{Q}(-3)^2 \to 0.$$

The ring structure is determined by the condition that the cohomology ring contain the graded ring $\mathbb{Q}[\lambda]/(\lambda^7)$, where λ has degree 2 and type $(1,1)$.

Along the way, we compute the rational cohomology of $\overline{\mathcal{A}}_2$, the Satake compactification of $\mathcal{A}_2$, as well.

Proposition 3. *The rational cohomology ring of $\overline{\mathcal{A}}_2$ is given by*

$$H^\bullet(\overline{\mathcal{A}}_2;\mathbb{Q}) \cong \mathbb{Q}[\lambda]/(\lambda^4).$$

where $\lambda \in H^2(\mathcal{A}_2\mathbb{Q})$ is the first Chern class of the Hodge bundle $\mathcal{H}_2$ over $\mathcal{A}_2$.

Denote the moduli space of smooth projective curves over the complex numbers by $\mathcal{M}_g$. Another consequence of the proof is the surjectivity of the homomorphism

$$H^\bullet(\mathcal{A}_3;\mathbb{Q}) \to H^\bullet(\mathcal{M}_3;\mathbb{Q})$$

induced by the period mapping $\mathcal{M}_3 \to \mathcal{A}_3$.

The computation of the rational cohomology of $\mathcal{A}_1$ is classical[1] and follows from the fact that the quotient of the upper half plane by $Sp_1(\mathbb{Z}) = SL_2(\mathbb{Z})$ is a copy of the affine line. The computation of the rational cohomology of $\mathcal{A}_2$ is (essentially) due to Igusa [13].[2] Brownstein and Lee [2, 3] have computed the integral cohomology of $Sp_2(\mathbb{Z})$.

Suppose that $g \geq 2$. Recall that that the *mapping class group* Γ_g in genus g is the group of isotopy classes of orientation preserving diffeomorphisms of a closed, oriented surface S of genus g. Its rational cohomology is isomorphic to that of $\mathcal{M}_g$. The *Torelli group* T_g is defined to be the kernel of the natural homomorphism

$$\Gamma_g \to Sp(H_1(S;\mathbb{Z})) \tag{1}$$

where Sp denotes the symplectic group, and where $H_1(S;\mathbb{Z})$ is regarded as a symplectic module via its intersection form. Choosing a symplectic basis of

[1] It is trivial except in degree 0.

[2] It is 1-dimensional in degrees 0 and 2, and trivial elsewhere.

$H_1(S;\mathbb{Z})$ gives an isomorphism $Sp_g(\mathbb{Z}) \cong Sp(H_1(S;\mathbb{Z}))$. One then obtains the well known extension

$$1 \to T_g \to \Gamma_g \to Sp_g(\mathbb{Z}) \to 1. \tag{2}$$

The *extended Torelli group* $\widehat{T}_g$ is the preimage of the center $\{\pm I\}$ of $Sp(H_1(S;\mathbb{Z}))$ under (1). Equivalently, it is the group of isotopy classes of diffeomorphisms of S that act as $\pm I$ on $H_1(S)$. One has the extensions

$$1 \to T_g \to \widehat{T}_g \to \{\pm I\} \to 1$$

and

$$1 \to \widehat{T}_g \to \Gamma_g \to PSp_g(\mathbb{Z}) \to 1 \tag{3}$$

where $PSp_g(\mathbb{Z})$ denotes the integral projective symplectic group $Sp_g(\mathbb{Z})/\{\pm I\}$.

Our approach to computing the cohomology of $Sp_3(\mathbb{Z})$ is to analyze the spectral sequence of the extension (3). This entails knowing the cohomology of Γ_3 (or equivalently, $\mathcal{M}_3$) and of $\widehat{T}_3$. Looijenga [14] computed the cohomology of $\mathcal{M}_3$ using the theory of Del-Pezzo surfaces:

$$H^j(\mathcal{M}_3;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & j = 0; \\ \mathbb{Q}(-1) & j = 1; \\ \mathbb{Q}(-6) & j = 6; \\ 0 & \text{otherwise.} \end{cases}$$

The cohomology of $\widehat{T}_3$ is computed in Section 3 using the stratified Morse theory of Goresky and MacPherson [8]. We use their theory of non-proper Morse functions (see their Part II, Chapter 10) applied to the square of the distance to a point function restricted to the jacobian locus. This is an elaboration of a trick of Geoff Mess [15] which he used to show that the Torelli group in genus 2 is free of countable rank, and which was also used by Johnson and Millson (cf. [15]) to show that Torelli space in genus 3 does not have the homotopy type of a finite complex. Our use of Morse theory is analogous to Goresky and MacPherson's treatment [8, Part III] of complements of affine subspaces of euclidean spaces.

It seems to be a curious fact that, in low genus ($g = 2, 3$ so far), it is easier to compute the rational cohomology groups of $\mathcal{M}_g$ than of $\mathcal{A}_g$. This is perhaps a reflection of the richness of curve theory — that it is a more powerful tool for understanding the geometry of $\mathcal{M}_g$ than the theory of abelian varieties is for understanding the geometry of $\mathcal{A}_g$. It will be interesting to know if this trend persists when $g \geq 4$, when $\mathcal{M}_g$ is no longer dense in $\mathcal{A}_g$. The cohomology of $\mathcal{M}_4$ is not yet know, nor does it seem tractable to compute the cohomology of the extended Torelli group in genus 4.

2. Preliminaries

General references for this section are [9, Chapt. 3] and [11].

A *polarized abelian variety* is a compact complex torus A together with a cohomology class

$$\theta \in H^{1,1}(A) \cap H^2(A;\mathbb{Z})$$

whose translation invariant representative is positive. The corresponding complex line bundle is ample. The polarization θ can be regarded as a skew symmetric bilinear form on $H_1(A;\mathbb{Z})$. The polarization is *principal* if this form is unimodular.

Jacobians of curves are polarized by the intersection pairing on $H_1(C;\mathbb{Z}) \cong H_1(\operatorname{Jac} C;\mathbb{Z})$. This form is unimodular, and so jacobians are canonically principally polarized abelian varieties.

A *framed* principally polarized abelian variety is a principally polarized abelian variety A together with a symplectic basis of $H_1(A;\mathbb{Z})$ with respect to the polarization θ.

Suppose that $g \geq 1$. The maximal compact subgroup of $Sp_g(\mathbb{R})$ is $U(g)$. The symmetric space $Sp_g(\mathbb{R})/U(g)$ is isomorphic to the rank g Siegel upper half space

$$\mathfrak{h}_g = \left\{ \begin{matrix} \text{symmetric } g \times g \text{ complex matrices} \\ \text{with positive definite imaginary part} \end{matrix} \right\}$$

(See, [12, C, p. 398], for example.) It has dimension $g(g+1)/2$. Taking a framed principally polarized abelian variety $(A; a_1, \dots, a_g, b_1, \dots, b_g)$ to the corresponding period matrix gives a bijection

$$\mathfrak{h}_g \cong \left\{ \begin{matrix} \text{isomorphism classes of framed princi-} \\ \text{pally polarized abelian varieties} \end{matrix} \right\} \tag{4}$$

We will regard $\mathfrak{h}_g$ as the (fine) moduli space of framed principally polarized abelian varieties of dimension g.

Changing symplectic bases gives a natural left action of $Sp_g(\mathbb{Z})$ on the moduli space of framed principally polarized abelian varieties. There is clearly a natural left $Sp_g(\mathbb{Z})$ action on $\mathfrak{h}_g = Sp_g(\mathbb{R})/U(g)$. The bijection (4) is equivariant with respect to these actions.

The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g is the quotient $Sp_g(\mathbb{Z})\backslash\mathfrak{h}_g$. Since $\mathfrak{h}_g$ is contractible and since $Sp_g(\mathbb{Z})$ acts discontinuously and virtually freely on $\mathfrak{h}_g$, it follows that there is a natural isomorphism

$$H^\bullet(Sp_g(\mathbb{Z});\mathbb{Q}) \cong H^\bullet(\mathcal{A}_g;\mathbb{Q}).$$

Taking a curve C to its jacobian $\operatorname{Jac} C$ defines a morphism $\mathcal{M}_g \to \mathcal{A}_g$ which is called the *period mapping*.

Now suppose that $g \geq 2$. Denote Teichmüller space in genus g by $\mathcal{X}_g$. This is the moduli space of marked, compact Riemann surfaces[3] of genus g. The

[3] A *marked Riemann surface* of genus g is a Riemann surface together with an isotopy class of orientation preserving diffeomorphisms with a fixed, oriented, genus g reference surface. A brief introduction to Teichmüller space can be found in [10], for example.

mapping class group Γ_g acts properly discontinuously and virtually freely on $\mathcal{X}_g$ with quotient $\mathcal{M}_g$. It follows that there is a natural isomorphism

$$H^\bullet(\Gamma_g;\mathbb{Q}) \cong H^\bullet(\mathcal{M}_g;\mathbb{Q}).$$

We shall need several moduli spaces that sit between $\mathcal{X}_g$ and $\mathcal{M}_g$. Denote the quotient of $\mathcal{X}_g$ by T_g by $\mathcal{T}_g$. This space is known as *Torelli space*. Since $\mathcal{X}_g$ is contractible and T_g is torsion free, T_g acts freely on $\mathcal{X}_g$ and Torelli space is an Eilenberg-MacLane space with fundamental group T_g. Consequently,

$$H^\bullet(T_g;\mathbb{Z}) \cong H^\bullet(\mathcal{T}_g;\mathbb{Z}).$$

By a *framed Riemann surface* of genus g we shall mean a compact Riemann surface C together with a symplectic basis $a_1,\dots,a_g,b_1,\dots,b_g$ of $H_1(C;\mathbb{Z})$ with respect to the intersection form. The Torelli space $\mathcal{T}_g$ is the moduli space of framed Riemann surfaces of genus g; its points correspond to isomorphism classes of framed, genus g Riemann surfaces. The symplectic group $Sp_g(\mathbb{Z})$ acts on the framings in the natural way; the quotient $Sp_g(\mathbb{Z})\backslash\mathcal{T}_g$ is $\mathcal{M}_g$.

Denote the locus in $\mathfrak{h}_g$ consisting of jacobians of smooth curves by $\mathcal{J}_g$. (Note that this is not closed in $\mathcal{A}_g$.) The period mapping $\mathcal{T}_g \to \mathcal{J}_g$ is surjective by definition. Since minus the identity is an automorphism of every polarized abelian variety A,

$$(A;a_1,\dots,b_g) \cong (A;-a_1,\dots,-b_g).$$

But if C is a genus g curve, then

$$(C;a_1,\dots,b_g) \cong (C;-a_1,\dots,-b_g)$$

if and only if C is hyperelliptic. It follows that, when $g \geq 3$, the period mapping

$$\mathcal{T}_g \to \mathcal{J}_g$$

that takes a framed curve to its jacobian with the same framing is surjective and 2:1 except along the hyperelliptic locus, where it is 1:1. It follows that, when $g \geq 3$,

$$\mathcal{J}_g = \widehat{T}_g\backslash\mathcal{X}_g \text{ and } \mathcal{M}_g = PSp_g(\mathbb{Z})\backslash\mathcal{J}_g.$$

The following diagram shows the coverings and their Galois group when $g \geq 3$:

(5)
$$\begin{tikzcd}
\mathcal{X}_g \arrow[r, "T_g"'] \arrow[rr, bend left, "\hat{T}_g"] \arrow[rrr, bend left=40, "\Gamma_g"] & \mathcal{T}_g \arrow[r, "\mathbb{Z}/2\mathbb{Z}"'] \arrow[rr, bend left, "Sp_g(\mathbb{Z})"] \arrow[dr] & \mathcal{J}_g \arrow[r, "PSp_g(\mathbb{Z})"'] \arrow[d, hook] & \mathcal{M}_g \arrow[d] \\
 & & \mathfrak{h}_g \arrow[r, "PSp_g(\mathbb{Z})"] & \mathcal{A}_g
\end{tikzcd}$$

Lemma 4. *If $1 \to \mathbb{Z}/2\mathbb{Z} \to E \to G \to 1$ is a group extension, then the projection $E \to G$ induces an isomorphism on homology and cohomology with 2-divisible coefficients. In particular*

$$H^\bullet(PSp_g(\mathbb{Z}); \mathbb{Z}[1/2]) \to H^\bullet(Sp_g(\mathbb{Z}); \mathbb{Z}[1/2])$$

is an isomorphism.

Proof. This follows from the fact that $H^j(\mathbb{Z}/2; \mathbb{Z}[1/2]) = 0$ when $j > 0$ using the Hochschild-Serre spectral sequence of the group extension. □

Proposition 5. *For all $g \geq 2$, there is a natural isomorphism*

$$H_\bullet(\mathcal{J}_g; \mathbb{Z}[1/2]) \cong H_\bullet(\widehat{T}_g; \mathbb{Z}[1/2]) \cong H_\bullet(T_g; \mathbb{Z}[1/2])^{\mathbb{Z}/2\mathbb{Z}}.$$

There are similar isomorphisms for cohomology.

Proof. Since T_g is torsion free, T_g acts fixed point freely on Teichmüller space, and $\mathcal{T}_g$ is a model of the classifying space of T_g. Recall that if X is a simplicial complex on which $\mathbb{Z}/2$ acts simplicially (but not necessarily fixed point freely), then the map

$$p_* : H_\bullet(X/(\mathbb{Z}/2); \mathbb{Z}[1/2]) \to H_\bullet(X; \mathbb{Z}[1/2])^{\mathbb{Z}/2}$$

induced by the projection p is an isomorphism, whose inverse is half the pullback map p^*. Applying this twice gives isomorphisms

$$H^\bullet(\mathcal{J}_g; \mathbb{Z}[1/2]) \cong H^\bullet(\mathcal{T}_g; \mathbb{Z}[1/2])^{\mathbb{Z}/2} \cong H^\bullet(T_g; \mathbb{Z}[1/2])^{\mathbb{Z}/2} \cong H^\bullet(\widehat{T}_g; \mathbb{Z}[1/2]).$$

□

A theta divisor of a principally polarized abelian variety A is a divisor Θ whose Poincaré dual is the polarization and which satisfies $i^*\Theta = \Theta$, where $i : x \mapsto -x$. Any two such divisors differ by translation by a point of order 2, and can be given as the zero locus of a theta function associated to a period matrix of A. A principally polarized abelian variety A is *reducible* if it is isomorphic (as a polarized abelian variety) to the product of two proper abelian subvarieties. If $A = A_1 \times A_2$, then any theta divisor of A is reducible:

$$\Theta_A = \big(\Theta_{A_1} \times A_2\big) \cup \big(A_1 \times \Theta_{A_2}\big).$$

Denote the locus of reducible abelian varieties in $\mathcal{A}_g$ by $\mathcal{A}_g^{\mathrm{red}}$ and in $\mathfrak{h}_g$ by $\mathfrak{h}_g^{\mathrm{red}}$. Elements of $\mathfrak{h}_g^{\mathrm{red}}$ are precisely those period matrices Ω that can be written as a direct sum of two smaller period matrices.

Proposition 6. *Denote the closure of $\mathcal{J}_g$ in $\mathfrak{h}_g$ by $\overline{\mathcal{J}}_g$. If $g \geq 2$, then $\mathcal{J}_g = \overline{\mathcal{J}}_g - (\overline{\mathcal{J}}_g \cap \mathfrak{h}_g^{\mathrm{red}})$.*

Proof. The period mapping $\mathcal{M}_g \to \mathcal{A}_g$ extends to a morphism $\overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g$ from the Deligne-Mumford compactification of $\mathcal{M}_g$ to the Satake compactification of $\mathcal{A}_g$. The inverse image of the boundary $\overline{\mathcal{A}}_g - \mathcal{A}_g$ of $\overline{\mathcal{A}}_g$ is the boundary divisor Δ_0 of $\overline{\mathcal{M}}_g$, whose generic point is an irreducible stable curve of genus g with one node. Denote the moduli space of curves of compact type $\overline{\mathcal{M}}_g - \Delta_0$ by $\widetilde{\mathcal{M}}_g$.

Since $\overline{\mathcal{M}}_g$ is complete, it follows that the period mapping $\widetilde{\mathcal{M}}_g \to \mathcal{A}_g$ is proper and therefore has closed image in $\mathcal{A}_g$. Since $\mathcal{M}_g$ is dense in $\overline{\mathcal{M}}_g$ and has image $Sp_g(\mathbb{Z})\backslash\mathcal{J}_g$ under the period mapping, it follows that the image of $\widetilde{\mathcal{M}}_g$ in $\mathcal{A}_g$ is $Sp_g(\mathbb{Z})\backslash\overline{\mathcal{J}}_g$.

Recall that the theta divisor $\Theta_C \subset \operatorname{Jac} C$ of a smooth genus g curve C is (up to a translate by a point of order 2) the image of mapping

$$C^{g-1} \to \operatorname{Pic}^{g-1} C \to \operatorname{Jac} C$$

that takes $(x_1, \dots, x_{g-1})$ to $x_1 + \cdots + x_{g-1} - \alpha$, where α is a square root of the canonical bundle of C. (See, for example, [9, p. 338].) It follows that Θ_C is irreducible. On the other hand, if C is a reducible, stable, curve of compact type, its jacobian is the product of the components of its irreducible components, and is therefore reducible. The result follows. □

Corollary 7. *We have* $\mathcal{J}_3 = \mathfrak{h}_3 - \mathfrak{h}_3^{\mathrm{red}}$.

Proof. Both $\mathcal{M}_3$ and $\mathcal{A}_3$ have dimension 6. Since the period mapping $\mathcal{M}_3 \to \mathcal{A}_3$ is generically of maximal rank, $\widetilde{\mathcal{M}}_3 \to \mathcal{A}_3$ is surjective. This implies that $\overline{\mathcal{J}}_3 = \mathfrak{h}_3$, from which the result follows. □

3. The Homology of $\mathcal{J}_3$ and $\widehat{T}_3$

Denote the singular locus of an analytic variety Z by Z^{sing}. We shall compute the homology of $\mathcal{J}_3$ by applying stratified Morse theory to the the stratification

$$\mathfrak{h}_3 \supseteq \mathfrak{h}_3^{\mathrm{red}} \supset \mathfrak{h}_3^{\mathrm{red,sing}} \tag{6}$$

of $\mathfrak{h}_3$. The top stratum is, by Corollary 7, $\mathcal{J}_3$. Note that there are natural inclusions

$$\mathfrak{h}_1 \times \mathfrak{h}_2 \hookrightarrow \mathfrak{h}_3^{\mathrm{red}} \text{ and } \mathfrak{h}_1 \times \mathfrak{h}_1 \times \mathfrak{h}_1 \hookrightarrow \mathfrak{h}_3^{\mathrm{red,sing}}$$

defined by

$$(\tau, \Omega) \mapsto \begin{pmatrix} \tau & 0 \\ 0 & \Omega \end{pmatrix} \text{ and } (\tau_1, \tau_2, \tau_3) \mapsto \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}$$

respectively. Note that the image of $\mathfrak{h}_1 \times \mathfrak{h}_2$ is stabilized in $Sp_3(\mathbb{R})$ by $SL_2(\mathbb{R}) \times Sp_2(\mathbb{R})$, and the image of $(\mathfrak{h}_1)^3$ by $\Sigma_3 \ltimes SL_2(\mathbb{R})^3$, where Σ_3 is identified with the subgroup

$$\{a_j \mapsto a_{\sigma(j)} \text{ and } b_j \mapsto b_{\sigma(j)} : \sigma \text{ is a permutation of } \{1,2,3\}\}$$

of $Sp_3(\mathbb{R})$. Here $a_1, \dots, b_6$ is the distinguished framing of the first homology of the corresponding abelian variety.

Lemma 8. *The stabilizer of* $\mathfrak{h}_1 \times \mathfrak{h}_2$ *in* $Sp_3(\mathbb{Z})$ *is* $SL_2(\mathbb{Z}) \times Sp_2(\mathbb{Z})$. *The stabilizer of* $(\mathfrak{h}_1)^3$ *in* $Sp_3(\mathbb{Z})$ *is* $\Sigma_3 \ltimes SL_2(\mathbb{Z})^3$.

Proof. First, suppose that $g \in Sp_3(\mathbb{Z})$ stabilizes $\mathfrak{h}_1 \times \mathfrak{h}_2$. A point of $\mathfrak{h}_1 \times \mathfrak{h}_2$ is a pair $((E, \mathbf{f}_E), (A, \mathbf{f}_A))$, where

- E is an elliptic curve and $\mathbf{f}_E$ is a symplectic framing of $H_1(E:\mathbb{Z})$, and
- A is an abelian surface and $\mathbf{f}_A$ a symplectic framing of $H_1(A;\mathbb{Z})$.

Fix such a point where A cannot be written as the product of two elliptic curves (as a polarized abelian variety) and neither A nor E have any automorphisms other than $\pm$ the identity. Then, since the decomposition, as polarized abelian varieties, of $E\times A$ into the product of an elliptic curve and an abelian surface is unique, we have

$$g((E,\mathbf{f}_E),(A,\mathbf{f}_A)) = ((E,g(\mathbf{f}_E)),(A,g(\mathbf{f}_A))).$$

Therefore there exist $g_E\in SL_2(\mathbb{Z})$ and $g_A\in Sp_2(\mathbb{Z})$ such that

$$g(\mathbf{f}_E)=g_E(\mathbf{f}_E) \text{ and } g(\mathbf{f}_A)=g_A(\mathbf{f}_A).$$

it follows that $g=g_E\times g_A$ in $Sp_3(\mathbb{Z})$, so that the stabilizer of $\mathfrak{h}_1\times\mathfrak{h}_2$ is $SL_2(\mathbb{Z})\times Sp_2(\mathbb{Z})$.

The proof in the case of $(\mathfrak{h}_1)^3$ is similar. Suppose that $g\in Sp_g(\mathbb{Z})$ stabilizes $(\mathfrak{h}_1)^3$. Consider the point $((E_1,\mathbf{f}_1),(E_2,\mathbf{f}_2),(E_3,\mathbf{f}_3))$, where E_1,E_2,E_3 are pairwise non-isomorphic elliptic curves, none of which has any non-trivial automorphisms apart from minus the identity. Since the decomposition, as a polarized abelian variety, of $E_1\times E_2\times E_3$ into a product of three elliptic curves is unique up to a permutation of the factors, there is a permutation σ in Σ_3 such that

$$g((E_1,\mathbf{f}_1),(E_2,\mathbf{f}_2),(E_3,\mathbf{f}_3)) = ((E_{\sigma(1)},g(\mathbf{f}_{\sigma(1)})),(E_{\sigma(2)},g(\mathbf{f}_{\sigma(2)}))(E_{\sigma(3)},g(\mathbf{f}_{\sigma(3)}))).$$

There are thus elements g_1,g_2,g_3 of $SL_2(\mathbb{Z})$ such that $g_j(\mathbf{f}_j)=\mathbf{f}_{\sigma(j)}$. It follows that,

$$g=\sigma\circ(g_1\times g_1\times g_3)\in\Sigma_3\ltimes SL_2(\mathbb{Z})^3\subset Sp_3(\mathbb{Z}).$$

The second statement follows $\square$

Remark 9. A similar argument can be used to prove a more general result. Namely, if we write $g=\lambda_1+2\lambda_2+\cdots+m\lambda_m$, where the λ_j are positive integers, then the stabilizer of $\mathfrak{h}_1^{\lambda_1}\times\cdots\times\mathfrak{h}_m^{\lambda_m}$ in $Sp_g(\mathbb{Z})$ is

$$\prod_{j=1}^m\big(\Sigma_{\lambda_j}\ltimes Sp_j(\mathbb{Z})^{\lambda_j}\big).$$

Proposition 10. *We have*

$$\mathfrak{h}_3^{\mathrm{red}} = \bigcup_{g\in Sp_3(\mathbb{Z})} g(\mathfrak{h}_1\times\mathfrak{h}_2) = \bigcup_{g\in Sp_3(\mathbb{Z})/(SL_2(\mathbb{Z})\times Sp_2(\mathbb{Z}))} g(\mathfrak{h}_1\times\mathfrak{h}_2) \tag{7}$$

and

$$\mathfrak{h}_3^{\mathrm{red,sing}} = \bigcup_{g\in Sp_3(\mathbb{Z})} g(\mathfrak{h}_1\times\mathfrak{h}_1\times\mathfrak{h}_1) = \coprod_{g\in Sp_3(\mathbb{Z})/(\Sigma_3\ltimes SL_2(\mathbb{Z})^3)} g(\mathfrak{h}_1\times\mathfrak{h}_1\times\mathfrak{h}_1) \tag{8}$$

In particular, $\mathfrak{h}_3^{\mathrm{red}}$ is a locally finite union of totally geodesic complex submanifolds of $\mathfrak{h}_3$ of complex codimension 2 and $\mathfrak{h}_3^{\mathrm{sing,red}}$ is a countable disjoint union of totally geodesic complex submanifolds of $\mathfrak{h}_3$ of complex codimension 3.

Proof. Since every reducible abelian variety is the product (as polarized abelian varieties) of an elliptic curve and an abelian surface, $\mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_3^{\mathrm{red}}$ is surjective. Lifting to $\mathfrak{h}_3$, this implies that the $Sp_3(\mathbb{Z})$ acts transitively on the components of $\mathfrak{h}_3^{\mathrm{red}}$, and that $\mathfrak{h}_3^{\mathrm{red}}$ is the $Sp_3(\mathbb{Z})$-orbit of $\mathfrak{h}_1 \times \mathfrak{h}_2$ in $\mathfrak{h}_3$. Assertion (7) now follows from the first statement of Lemma 8.

The components of $\mathfrak{h}_3^{\mathrm{red}}$ are smooth, so that $\mathfrak{h}_3^{\mathrm{red,sing}}$ is the locus where two or more components of $\mathfrak{h}_3^{\mathrm{red}}$ intersect. This is precisely the preimage of the locus in $\mathcal{A}_3$ of products of 3 elliptic curves. Since this locus is irreducible (it is the image of $(\mathcal{A}_1)^3 \to \mathcal{A}_3$), $\mathfrak{h}_3^{\mathrm{red,sing}}$ is the $Sp_3(\mathbb{Z})$-orbit of $(\mathfrak{h}_1)^3$. Assertion (8) now follows from the second statement of Lemma 8. □

Lemma 11. *About each point of $\mathfrak{h}_3^{\mathrm{red,sing}}$, there is a neighbourhood with holomorphic coordinates $(\tau_1, \tau_2, \tau_3, z_1, z_2, z_3)$ such that $\mathfrak{h}_3^{\mathrm{red}}$ has three components with equations*

$$z_2 = z_3 = 0, \quad z_1 = z_3 = 0, \quad z_1 = z_2 = 0.$$

Proof. There are three obvious ways to deform the product $A = E \times E' \times E''$ of three elliptic curves, preserving the polarization, into $\mathfrak{h}_3^{\mathrm{red}}$. Namely, one can deform one of the elliptic curves in $\mathfrak{h}_1$, and deform the product of the other two into $\mathfrak{h}_2 - \mathfrak{h}_2^{\mathrm{red}}$. The semi-simplicity of abelian varieties implies that each component of $\mathfrak{h}_3^{\mathrm{red,sing}}$ is smooth and there are no other ways to deform A into $\mathfrak{h}_3^{\mathrm{red}}$. It follows that 3 components of $\mathfrak{h}_3^{\mathrm{red}}$ intersect at each point of $\mathfrak{h}_3^{\mathrm{red,sing}}$.

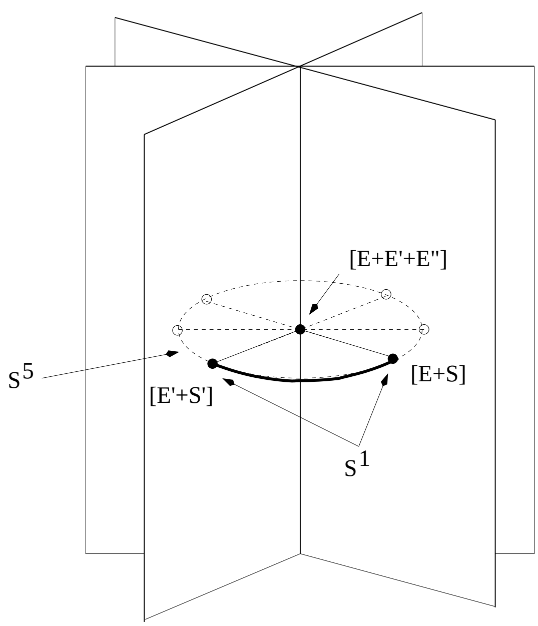

FIGURE 1. Local structure of $\mathfrak{h}_3^{\mathrm{red}}$ near $\mathfrak{h}_3^{\mathrm{red,sing}}$

Since each component of $\mathfrak{h}_3^{\mathrm{red,sing}}$ is locally homogeneous, and since all are conjugate under the action of $Sp_3(\mathbb{Z})$, to determine the local structure of $\mathfrak{h}_3^{\mathrm{red}}$

near the point $(\tau_1^o, \tau_2^o, \tau_3^o)$ of $\mathfrak{h}_3^{\mathrm{red,sing}}$, it suffices to write down local equations along the component $(\mathfrak{h}_1)^3$ of $\mathfrak{h}_3^{\mathrm{red}}$. Here we can use the coordinates

$$\begin{pmatrix} \tau_1 & z_3 & z_2 \\ z_3 & \tau_2 & z_1 \\ z_2 & z_1 & \tau_3 \end{pmatrix}$$

where each τ_j lies in a relatively compact neighbourhood V_j of τ_j^o in $\mathfrak{h}_1$, and (z_1, z_2, z_3) lies in a neighbourhood of the origin in $\mathbb{C}^3$ small enough to guarantee that this matrix has positive definite imaginary part when $\tau_j \in V_j$, $1 \le j \le 3$. In these coordinates, the three components of $\mathfrak{h}_3^{\mathrm{red}}$ have equations:

$$z_2 = z_3 = 0, \quad z_1 = z_3 = 0, \quad z_1 = z_2 = 0.$$

$\square$

Whitney's conditions ([8, p. 37]) are an immediate consequence.

Corollary 12. *This stratification satisfies Whitney's conditions (A) and (B).* $\square$

Stratified Morse theory (SMT) is a tool for understanding the topology of stratified spaces. Knowing the topology of a stratified space (as a stratified space) entails understanding the topology of various unions of strata; in particular, the top stratum. We will apply SMT in our case to determine the topology of the top stratum $\mathcal{J}_3$. A clear explanation of the basic ideas of stratified Morse theory, including how to use it to determine the topology of the top stratum using a "non-proper Morse function," is given in the first chapter of [8].

As in classical Morse theory, one needs a Morse function. Integrating an $Sp_3(\mathbb{R})$-invariant Riemannian metric along geodesics gives an $Sp_g(\mathbb{R})$-invariant distance function d on $\mathfrak{h}_3$. For a point $p \in \mathfrak{h}_3$, let $D_p : \mathfrak{h}_3 \to \mathbb{R}$ be the square of the distance to p:

$$D_p(x) = d(x,p)^2.$$

This will be a Morse function in the sense of Goresky and MacPherson [8, p. 52] when its restriction to the closure of each stratum has only non-degenerate critical points which lie at distinct distances from p.

Since $\mathfrak{h}_3$, with the symmetric space metric, is a complete non-positively curved manifold, and since the closure of each component of each stratum is totally geodesic, [6, Prop. 1.6.3] implies that the restriction of D_p to each component of each stratum has a unique critical point, which is necessarily a minimum. For D_p to be a Morse function, its critical values must be distinct. Thus D_p will be a Morse function for p in the open dense subset

$$U = \mathfrak{h}_3 - \{p \in \mathfrak{h}_3 : p \text{ is equidistant from 2 different components of } \mathfrak{h}_3^{\mathrm{red}}\}$$

of $\mathfrak{h}_3$. This is summarized in the following:

Proposition 13. *There is an open dense subset U of $\mathfrak{h}_3$ such that for all $p \in U$, $D_p : \mathfrak{h}_3 \to \mathbb{R}$ is a Morse function in the sense of Goresky and MacPherson, all* *of whose critical points are "non-depraved". Moreover, D_p has a unique critical point (a minimum) on each connected component of each stratum of $\mathfrak{h}_3^{\mathrm{red}}$.* $\square$

In SMT, just as in classical Morse theory, each time one passes a critical point, one attaches a stratified space (the Morse data). This is the product of *tangential* and *normal* Morse data at the critical point [8, p. 61]. The tangential Morse data at a critical point is the classical Morse data for the restriction of D_p to the stratum containing it. The normal Morse data at a critical point summarizes how the topology changes on a slice through the critical point, transverse and of complementary dimension to the stratum containing it. In our case, the tangential Morse data is contractible as each critical point is a local minimum of the restriction of D_p to the stratum on which it lies. So, to determine the homotopy type of the top stratum, we need only compute the homotopy type of the top stratum of the normal Morse data at each critical point. The situation considered here is analogous to the situation in [8, Chapt. III.3] where the homology of a union of affine subspaces in affine space is computed using SMT. It is important to realize that, in SMT, a critical point may contribute to the topology of the top stratum, even if it lies in a lower stratum.

Fix $p \in \mathfrak{h}_3 - \mathfrak{h}_3^{\mathrm{red}}$ such that D_p is a stratified Morse function. In view of Proposition 10, we know all of the critical points of D_p. There are two types of these: those that lie in $\mathfrak{h}_3^{\mathrm{red}} - \mathfrak{h}_3^{\mathrm{red,sing}}$, and those that lie in $\mathfrak{h}_3^{\mathrm{red}}$. The normal Morse data at each critical point depends only on its type.

Proposition 14. *If $x \in \mathfrak{h}_3^{\mathrm{red}} - \mathfrak{h}_3^{\mathrm{red,sing}}$ is a critical point of D_p, then the top stratum of the normal Morse data at x is homotopy equivalent to $(S^3, *)$.*

Proof. Since the normal slice at a smooth point of $\mathfrak{h}_3^{\mathrm{red}}$ is a complex 2-ball, the normal Morse data at x is $(S^3, *)$. $\square$

View S^5 as the unit sphere in $\mathbb{C}^3$. The intersection of each coordinate axis with S^5 is a linearly imbedded S^1. Together these give an imbedding

$$k : S^1 \amalg S^1 \amalg S^1 \hookrightarrow S^5$$

where $\amalg$ denotes disjoint union. The boundary of a small tubular neighbourhood of each S^1 is a trivial S^3 bundle over S^1. Taking one fiber of each and connecting them to an arbitrary point of $S^5 - \big(S^1 \cup S^1 \cup S^1\big)$ gives an imbedding

$$i : S^3 \vee S^3 \vee S^3 \hookrightarrow S^5 - \big(S^1 \cup S^1 \cup S^1\big).$$

Proposition 15. *If $x \in \mathfrak{h}_3^{\mathrm{red,sing}}$ is a critical point of D_p, then the top stratum of the normal Morse data at x is homotopy equivalent to*

$$\big(S^5 - k\big(S^1 \amalg S^1 \amalg S^1\big), i\big(S^3 \vee S^3 \vee S^3\big)\big).$$

Proof. By Lemma 11, [8, I.3.8] and [8, I.3.9.3], the normal Morse data is the same as that for

$$\mathbb{C}^3 - \text{ the union of the 3 coordinate axes}$$

with any Morse function, and it is of the form (N, ℓ^-) where N is the normal slice and ℓ^- is the lower half link. The normal slice is

$$\big(\text{unit ball in } \mathbb{C}^3\big) - \text{ the union of the 3 coordinate axes,}$$

which retracts onto S^5 minus 3 disjoint, linearly imbedded S^1s. Taking the Morse function to be the distance from a generic real hypersurface, we see that ℓ^- is homeomorphic to

$$\big(\text{unit ball in } \mathbb{R}^5\big) - (S^1 \amalg S^1 \amalg S^1).$$

This imbeds into N, and projects to $S^5 - (S^1 \amalg S^1 \amalg S^1)$ via radial projection. We can choose the inclusion $i : S^3 \vee S^3 \vee S^3 \hookrightarrow S^5$ so that it factors through $\mathbb{R}^5 - (S^1 \amalg S^1 \amalg S^1)$. It is straightforward to show that the inclusion

$$S^3 \vee S^3 \vee S^3 \hookrightarrow \mathbb{R}^5 - (S^1 \amalg S^1 \amalg S^1)$$

is a homotopy equivalence as both spaces are simply connected, and as the inclusion induces an isomorphism on homology. The result follows. □

To compute the homology of $\mathcal{J}_3$, we will need to know the homology of the two kinds of normal Morse data. This is trivial in the first case. For the second, denote the quotient of $\mathbb{Z}^3$ by the diagonal subgroup by V. It is isomorphic to $\mathbb{Z}^2$. The proof of the following result is elementary.

Proposition 16. *The homology of the Morse data at a critical point of* $\mathfrak{h}_3^{\mathrm{red,sing}}$ *is*

$$H_j\big(S^5 - k\big(S^1 \amalg S^1 \amalg S^1\big), i\big(S^3 \vee S^3 \vee S^3\big); \mathbb{Z}\big) = \begin{cases} V & j = 4; \\ 0 & \textit{otherwise.} \end{cases}$$

The homology is generated by the boundaries of tubular neighbourhoods of the three imbedded S^1*s, which are subject to the relation that their sum is zero.* □

The natural action of the symmetric group Σ_3 on $\mathbb{Z}^3$ (by permuting the coordinates) preserves the diagonal subgroup and therefore descends to an action on V. We view V as a $\Sigma_3 \ltimes SL_2(\mathbb{Z})^3$-module via the projection $\Sigma_3 \ltimes SL_2(\mathbb{Z})^3 \to \Sigma_3$.

Recall that if R is a commutative ring, K a subgroup of G, and M a RK module, then the G-module induced from M is defined by

$$\mathrm{Ind}_K^G M := RG \otimes_{RK} M.$$

Theorem 17. *We have,*

$$H_j(\mathcal{J}_3; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j = 0; \\ \mathrm{Ind}_{SL_2(\mathbb{Z}) \times Sp_2(\mathbb{Z})}^{Sp_3(\mathbb{Z})} \mathbb{Z} & j = 3; \\ \mathrm{Ind}_{\Sigma_3 \ltimes SL_2(\mathbb{Z})^3}^{Sp_3(\mathbb{Z})} V & j = 4; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since each critical point is a local minimum of D_p restricted to the stratum on which it lies, each critical point has trivial tangential Morse data. It follows from [8, I.3.7] that the top stratum of the Morse data at each critical point is homotopy equivalent to the top stratum of its normal Morse data. Since the homology of the corresponding normal Morse data is generated by absolute cycles, the Morse function is perfect on the top stratum. It follows that the homology of $\mathcal{J}_3$ is the sum of the relative homologies of the normal Morse data at each critical point. The action of $Sp_3(\mathbb{Z})$ follows from the description of the $Sp_3(\mathbb{Z})$-action on the strata given in Proposition 10. $\square$

For a subgroup G of $Sp_g(\mathbb{Z})$ that contains $-I$, define PG to be the subgroup $G/(\pm I)$ of $PSp_g(\mathbb{Z})$.

Corollary 18. *For all $j \geq 0$,*

$$H_j(\widehat{T}_3;\mathbb{Z}[1/2]) \cong H_k(T_3;\mathbb{Z}[1/2])^{\mathbb{Z}/2\mathbb{Z}} \cong \begin{cases} \mathbb{Z}[1/2] & j=0; \\ \operatorname{Ind}_{SL_2(\mathbb{Z})\times Sp_2(\mathbb{Z})}^{Sp_3(\mathbb{Z})} \mathbb{Z}[1/2] & j=3; \\ \operatorname{Ind}_{\Sigma_3 \ltimes SL_2(\mathbb{Z})^3}^{Sp_3(\mathbb{Z})} V \otimes \mathbb{Z}[1/2] & j=4; \\ 0 & \textit{otherwise;} \end{cases}$$

$$\cong \begin{cases} \mathbb{Z}[1/2] & j=0; \\ \operatorname{Ind}_{P(SL_2(\mathbb{Z})\times Sp_2(\mathbb{Z}))}^{PSp_3(\mathbb{Z})} \mathbb{Z}[1/2] & j=3; \\ \operatorname{Ind}_{P(\Sigma_3 \ltimes SL_2(\mathbb{Z})^3)}^{PSp_3(\mathbb{Z})} V \otimes \mathbb{Z}[1/2] & j=4; \\ 0 & \textit{otherwise.} \end{cases}$$

4. The Spectral Sequence

We shall compute the homology spectral sequence

$$H_i(PSp_3(\mathbb{Z}); H_j(\widehat{T}_3;\mathbb{Q})) \implies H_{i+j}(\Gamma_3;\mathbb{Q}). \tag{9}$$

Thanks to Shapiro's Lemma (see, for example, [1, p. 73]), which asserts that for each K-module, there is a natural isomorphism

$$H_\bullet(G;\operatorname{Ind}_K^G M) \cong H_\bullet(K,M),$$

the E^2 term of the spectral sequence can be computed.

Lemma 19. *We have*

$$H_i(PSp_3(\mathbb{Z}); H_j(\widehat{T}_3;\mathbb{Q})) \cong \begin{cases} \mathbb{Q} & i=0,2 \textit{ and } j=3; \\ 0 & j>0 \textit{ and } j\neq 3. \end{cases}$$

Proof. Applying Shapiro's Lemma, Lemma 4, the Kunneth Theorem, and the fact that the rational homology of $SL_2(\mathbb{Z})$ is that of a point, we have:

$$\begin{aligned} H_i\big(PSp_3(\mathbb{Z}); \operatorname{Ind}_{P(SL_2(\mathbb{Z})\times Sp_2(\mathbb{Z}))}^{PSp_3(\mathbb{Z})} \mathbb{Q}\big) &\cong H_i(P(SL_2(\mathbb{Z})\times Sp_2(\mathbb{Z}));\mathbb{Q}) \\ &\cong H_i(SL_2(\mathbb{Z})\times Sp_2(\mathbb{Z});\mathbb{Q}) \\ &\cong H_i(Sp_2(\mathbb{Z});\mathbb{Q}). \end{aligned}$$

This is $\mathbb{Q}$ when $i = 0, 2$ and 0 otherwise by Igusa's computation.

Let $V_\mathbb{Q} = V \otimes_\mathbb{Z} \mathbb{Q}$. This is the unique 2-dimensional irreducible representation of Σ_3. Since $V_\mathbb{Q}$ is divisible and has no coinvariants, $H_\bullet(\Sigma_3; V_\mathbb{Q})$ vanishes in all degrees. Arguing as above, we have

$$\begin{aligned} H_i\big(PSp_3(\mathbb{Z}); \mathrm{Ind}_{P(\Sigma_3 \ltimes SL_2(\mathbb{Z})^3)}^{PSp_3(\mathbb{Z})} V_\mathbb{Q}\big) &\cong H_i(P(\Sigma_3 \ltimes SL_2(\mathbb{Z})^3); V_\mathbb{Q}) \\ &\cong H_i(\Sigma_3 \ltimes SL_2(\mathbb{Z})^3; V_\mathbb{Q}) \\ &\cong H_i(\Sigma_3; V_\mathbb{Q}) \\ &\cong 0. \end{aligned}$$

□

Proposition 20. *For $2 \leq r \leq 4$, the E^r-term of the spectral sequence (9) is*

4	0	0	0	0	0	0	0	0
3	$\mathbb{Q}$	0	$\mathbb{Q}$	0	0	0	0	0
2	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}$	0	$\mathbb{Q}^2$	0
deg	0	1	2	3	4	5	6	7

(All terms not shown are zero.) In addition, the differentials

$$d^4 : E^4_{4,0} \to E^4_{0,3} \text{ and } d^4 : E^4_{6,0} \to E^4_{2,3}$$

are both surjective, and $E^5 = E^\infty$.

Proof. By Corollary 18, $E^2_{s,t}$ vanishes when $t > 0$ and $t \neq 3, 4$. The computation of $E^2_{s,3}$ and $E^2_{s,4}$ follows from Lemma 19. Together, these imply that $d^r = 0$ when $2 \leq r < 4$, and that $E^5 = E^\infty$. Our remaining task is to determine $E^2_{s,0}$. By Looijenga's computation of the rational homology of Γ_3, we know that $H_3(\Gamma_3; \mathbb{Q})$ and $H_5(\Gamma_3; \mathbb{Q})$ both vanish. This implies that the differentials

$$d^4 : E^4_{4,0} \to E^4_{0,3} \text{ and } d^4 : E^4_{6,0} \to E^4_{2,3}$$

must be surjective. Since $H_4(\Gamma_3; \mathbb{Q}) = 0$, and since $H_6(\Gamma_3; \mathbb{Q})$ is one dimensional, the first of these is an isomorphism and the second has one dimensional kernel. The result follows. □

Proof of Theorem 1. The computation of the rational homology (and therefore the rational cohomology) of $\mathcal{A}_3$ follows from Proposition 20.

Denote the first Chern class in $H^2(\mathcal{A}_3; \mathbb{Q})$ of the Hodge bundle $\mathcal{H}_3$ by λ. It is the class of an ample line bundle. A standard argument, that uses the fact that the Satake compactification of $\mathcal{A}_3$ has boundary of codimension 3, shows that there is a complete surface in A_3. This implies that $\lambda^2 \neq 0$ in $H^4(\mathcal{A}_g; \mathbb{Q})$.

The proof that λ^3 does not vanish in $H^6(\mathcal{A}_3; \mathbb{Q})$ is more subtle, and is due to van der Geer [7]. We will give a topological proof of this fact in the next section. The key point in van der Geer's argument is that there is a complete subvariety of the characteristic p version $\mathcal{A}_{3/\overline{\mathbb{F}}_p}$ of $\mathcal{A}_3$. This implies that λ^3 is not zero in $H^6_{\text{ét}}(\mathcal{A}_{3/\overline{\mathbb{F}}_p};\mathbb{Q}_\ell)$, where p is a prime where $\mathcal{A}_3$ has good reduction and ℓ is a prime distinct from p. Standard comparison theorems imply that this last group is isomorphic to $H^6(\mathcal{A}_3;\mathbb{Q}_\ell)$, which gives the desired non-vanishing of λ^3.

The statement about weights follows as λ is of type $(1,1)$. Since $\lambda^3 \neq 0$, this implies that $H^4(\mathcal{A}_3;\mathbb{Q})$ is generated by λ^2 and has type $(2,2)$, and that $H^6(\mathcal{A}_3;\mathbb{Q})$ contains a copy of $\mathbb{Q}(-3)$ spanned by λ^3. On the other hand, the spectral sequence in Proposition 20 implies that the restriction mapping

$$H^6(\mathcal{A}_3;\mathbb{Q}) \to H^6(\mathcal{M}_3;\mathbb{Q}) \cong \mathbb{Q}(-6)$$

is surjective, which completes the proof. □

5. Cycles

In this section, we give a topological proof that λ^3 is non-zero in $H^6(\mathcal{A}_3;\mathbb{Q})$. The approach is to construct a topological 6-cycle in $\mathcal{A}_3$ and then show that the value of λ^3 on it is non-zero.

Fix an imbedding of the Satake compactification $\overline{\mathcal{A}}_3$ of $\mathcal{A}_3$ in some projective space. Take a generic codimension 3 linear section of $\overline{\mathcal{A}}_3$ that avoids the boundary $\overline{\mathcal{A}}_3^{\text{red}} - \mathcal{A}_3^{\text{red}}$, is transverse to $\mathcal{A}_3^{\text{red}}$, and intersects $\mathcal{A}_3^{\text{red,sing}}$ transversally. This section is a complete curve X in $\mathcal{A}_3^{\text{red}}$, smooth (in the orbifold sense) away from its intersection with $\mathcal{A}_3^{\text{red,sing}}$. At each point x where it intersects $\mathcal{A}_3^{\text{red,sing}}$, it has three branches. Set $X' = X - (X \cap \mathcal{A}_3^{\text{red,sing}})$.

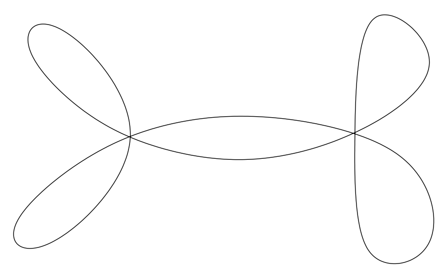

FIGURE 2

The next step is to construct a 5-cycle in $\mathcal{M}_3$ which is an S^3 bundle over X away from the triple points, and where the three branches of this bundle at each triple point x are plumbed together using the normal Morse data at x.

Denote the moduli space of principally polarized abelian 3-folds with a level ℓ structure by $\mathcal{A}_3[\ell]$. This is the quotient of $\mathfrak{h}_3$ by the level ℓ subgroup $Sp_3(\mathbb{Z})[\ell]$ of $Sp_3(\mathbb{Z})$, and is smooth when $\ell \geq 3$. Fix an $\ell \geq 3$. The symmetric space metric on $\mathfrak{h}_3$ descends to $\mathcal{A}_3[\ell]$. With the help of the metric, the normal bundle of any stratum can be viewed as a subbundle of the tangent bundle of $\mathcal{A}_3[\ell]$. Using the exponential mapping, we can identify a neighbourhood of the zero section of the normal bundle as being imbedded in $\mathcal{A}_3[\ell]$. Let Y and Y' be the inverse images of X and X' in $\mathcal{A}_3[\ell]$.

Choose a positive real number ϵ such that the exponential mapping is an imbedding on the $\epsilon/4$-ball B of the normal bundle of $\mathcal{A}_3^{\mathrm{red}}[\ell]$ restricted to Y', and also on the ϵ-ball B' of the normal bundle of $\mathcal{A}_3^{\mathrm{red,sing}}[\ell]$ at each point of $Y - Y'$. Set $\widetilde{D} = B \cup B'$ and $\widetilde{W} = \partial\widetilde{D}$. Denote their pushforwards to $\mathcal{A}_3$ by D and W. Then W is a 5-cycle in $\mathcal{M}_3$ which is generically an S^3-bundle over X'. Note that D is a 6-chain in $\mathcal{A}_3$ with $\partial D = W$.

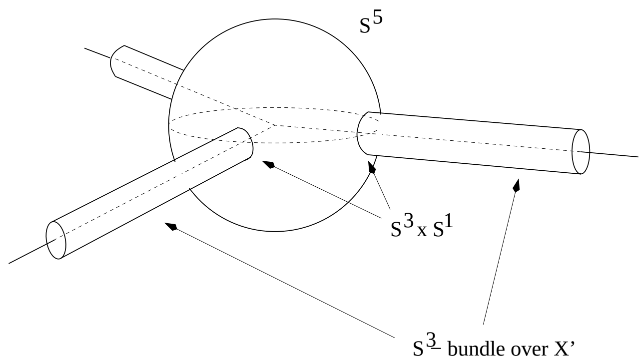

FIGURE 3. Picture of W near a triple point

On the other hand, Looijenga's computation of the rational homology of $\mathcal{M}_3$, implies that W bounds a rational 6-chain E in $\mathcal{M}_3$. Set $Z = D - E$. This is a rational 6-cycle in $\mathcal{A}_3$. By construction, we have:

Proposition 21. *The cycle Z intersects $\mathcal{A}_3^{\mathrm{red,sing}}$ transversally (in the orbifold sense), and the intersection number of Z with $\mathcal{A}_3^{\mathrm{red,sing}}$ is non-zero.* □

Corollary 22. *The class of Z is non-trivial in $H_6(\mathcal{A}_3;\mathbb{Q})$ and the class of $\mathcal{A}_3^{\mathrm{red,sing}}$ is non-trivial in $H^6(\mathcal{A}_3;\mathbb{Q})$.* □

The proof of the non-triviality of λ^3 is completed by the following result.

Proposition 23. *The class of $\mathcal{A}_3^{\mathrm{red,sing}}$ in $H^6(\mathcal{A}_3;\mathbb{Q})$ is a non-zero multiple of λ^3.*

Proof. Denote the closure in $\overline{\mathcal{A}}_3$ of a subvariety X of $\mathcal{A}_3$ by $\overline{X}$. Set $\partial X = \overline{X} - X$. Since $\mathcal{A}_3$ is a rational homology manifold, the sequence

$$H_8(\overline{\mathcal{A}}_3^{\mathrm{red}}, \partial\mathcal{A}_3^{\mathrm{red}};\mathbb{Q}) \to H^4(\mathcal{A}_3;\mathbb{Q}) \to H^4(\mathcal{M}_3;\mathbb{Q})$$

is exact. Since $\mathcal{A}_3^{\mathrm{red}}$ is irreducible of dimension 4, the left hand group is one-dimensional and spanned by the fundamental class of $\mathcal{A}_3^{\mathrm{red}}$. Since the middle group is one-dimensional and the right hand group trivial, we see that the class of $\mathcal{A}_3^{\mathrm{red}}$ spans $H^4(\mathcal{A}_3;\mathbb{Q})$. On the other hand, since λ is ample, and $\mathcal{A}_3$ contains a complete surface, λ^2 also spans $H^4(\mathcal{A}_3;\mathbb{Q})$. It follows that there is a non-zero rational number c such that

$$\lambda^2 = c[\mathcal{A}_3^{\mathrm{red}}] \text{ in } H^4(\mathcal{A}_3;\mathbb{Q}).$$

Denote the determinant of the Hodge bundle $\mathcal{H}_g$ by $\mathcal{L}$. The class λ^3 is represented by the divisor of a section of the restriction of $\mathcal{L}$ to $\mathcal{A}_3^{\mathrm{red}}$. This can be computed by pulling back along the mapping $\mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_3^{\mathrm{red}}$. Since the Picard group of $\mathcal{A}_1$ is torsion, we see that the pullback of $\mathcal{L}$ is represented mod-torsion by $\mathcal{A}_1 \times D$, where D is a cycle representing λ on $\mathcal{A}_2$. But the cusp form χ_{10} of weight 10 on $\mathcal{A}_2$ is a section of $\mathcal{L}^{10}$, and has divisor supported on $\mathcal{A}_2^{\mathrm{red}}$. So we see that λ is represented by a non-zero rational multiple of the cycle $\mathcal{A}_1 \times \mathcal{A}_2^{\mathrm{red}}$ on $\mathcal{A}_1 \times \mathcal{A}_2$, and by a non-zero multiple of $\mathcal{A}_3^{\mathrm{red,sing}}$ in $H^4(\mathcal{A}_3^{\mathrm{red}};\mathbb{Q})$. This implies that λ^3 is represented by a non-zero rational multiple of $\mathcal{A}_3^{\mathrm{red,sing}}$ in $H^4(\mathcal{A}_3;\mathbb{Q})$. $\square$

6. The rational cohomology of $\overline{\mathcal{A}}_2$ and $\overline{\mathcal{A}}_3$

Note that $\overline{\mathcal{A}}_0$ is just a point. Suppose now that $g > 0$. Then

$$\overline{\mathcal{A}}_g = \mathcal{A}_g \amalg \overline{\mathcal{A}}_{g-1}.$$

Choose a triangulation of $\overline{\mathcal{A}}_g$ such that $\overline{\mathcal{A}}_{g-1}$ is a subcomplex. Let U_g be a regular PL neighbourhood of $\overline{\mathcal{A}}_{g-1}$ in $\overline{\mathcal{A}}_g$. Set $U_g^* = U_g - \overline{\mathcal{A}}_{g-1}$. This has the homotopy type of ∂U_g, which is a rational homology manifold (as $\mathcal{A}_g[3]$ is smooth) of dimension $2d_g - 1$, where $d_g = g(g+1)/2$ is the dimension of $\mathcal{A}_g$. The cohomology of U_g^* has a mixed Hodge structure and the cup product

$$H^{k-1}(U_g^*;\mathbb{Q}) \otimes H^{2d_g-k}(U_g^*;\mathbb{Q}) \to H^{2d_g-1}(U_g^*;\mathbb{Q}) \cong \mathbb{Q}(-d_g)$$

is a perfect pairing of mixed Hodge structures (MHSs) (see [5], for example).

Since $\mathcal{A}_g$ is a rational homology manifold, Lefschetz duality gives an isomorphism

$$H_c^k(\mathcal{A}_g,\mathbb{Q}) \cong \mathrm{Hom}(H^{2d_g-k}(\mathcal{A}_g), \mathbb{Q}(-d_g)).$$

The standard long exact sequence

$$(10) \qquad \cdots \to H^{k-1}(U_g^*) \to H_c^k(\mathcal{A}_g;\mathbb{Q}) \to H^k(\mathcal{A}_g;\mathbb{Q}) \to H^k(U_g^*;\mathbb{Q}) \to \cdots$$

is exact in the category of MHSs. These facts will allow us to compute the cohomology of U_2^* and U_3^*.

The second step in the computation will be to use the Mayer-Vietoris sequence

$$(11) \quad \cdots \to H^k(\overline{\mathcal{A}}_g) \to H^k(\mathcal{A}_g) \oplus H^k(\overline{\mathcal{A}}_{g-1}) \to H^k(U_g^*) \to H^{k+1}(\overline{\mathcal{A}}_g) \to \cdots$$

associated to the covering $\overline{\mathcal{A}}_g = \mathcal{A}_g \cup U_g$, which is exact in the category of MHS, to compute the cohomology of $\overline{\mathcal{A}}_2$ and $\overline{\mathcal{A}}_3$.

For determining the ring structure and also for seeing that some maps in these long exact sequences are non-trivial, it is useful to observe that since λ is the class of an ample line bundle on $\overline{\mathcal{A}}_g$, the rational cohomology ring of $\overline{\mathcal{A}}_g$ contains the ring $\mathbb{Q}[\lambda]/(\lambda^{d_g+1})$.

6.1. Proof of Proposition 3. Since $H^\bullet(\mathcal{A}_2;\mathbb{Q})$ is $\mathbb{Q}$ in degree 0, $\mathbb{Q}(-1)$ in degree 2, and 0 otherwise, $H_c^\bullet(\mathcal{A}_2;\mathbb{Q})$ is $\mathbb{Q}(-2)$ in degree 4, $\mathbb{Q}(-3)$ in degree 6, and 0 otherwise. Using the sequence (10) and the fact that the cohomology of U_2^* satisfies Poincaré duality, we have

$$H^j(U_2^*;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & j=0; \\ \mathbb{Q}(-1) & j=2; \\ \mathbb{Q}(-2) & j=3; \\ \mathbb{Q}(-3) & j=5. \end{cases}$$

Putting this into the Mayer-Vietoris sequence (11), and using the fact that $\overline{\mathcal{A}}_1$ is $\mathbb{P}^1$, we obtain the result.

6.2. Proof of Proposition 2. By duality, $H_c^\bullet(\mathcal{A}_3;\mathbb{Q})$ is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$ in degree 6, $\mathbb{Q}(-4)$ in degree 8, $\mathbb{Q}(-5)$ in degree 10, and $\mathbb{Q}(-6)$ in degree 12. Using the sequence (10) and the facts that $H^j(\mathcal{A}_3;\mathbb{Q})$ vanishes when $j \geq 7$ and $H_c^j(\mathcal{A}_3;\mathbb{Q})$ vanishes when $j \leq 5$, we have

$$H^j(U_3^*;\mathbb{Q}) \cong \begin{cases} H^j(\mathcal{A}_3;\mathbb{Q}) & j<5; \\ H_c^{j+1}(\mathcal{A}_3;\mathbb{Q}) & j \geq 7. \end{cases}$$

In degrees 5 and 6 we have the exact sequence

$$0 \to H^5(U_3^*;\mathbb{Q}) \to H_c^6(\mathcal{A}_3;\mathbb{Q}) \xrightarrow{\alpha} H^6(\mathcal{A}_3;\mathbb{Q}) \to H^6(U_3^*;\mathbb{Q}) \to 0.$$

It follows from [4, 8.2.2] that the image of α is all of $W_6H^6(\mathcal{A}_3;\mathbb{Q})$, so that α is non-zero. Since the sequence is exact in the category of MHSs, it follows that $H^5(U_3^*;\mathbb{Q})$ is $\mathbb{Q}(0)$ and $H^6(U_3^*;\mathbb{Q})$ is $\mathbb{Q}(-6)$.

Since $\overline{\mathcal{A}}_3$ is projective and λ is the class of a projective imbedding, the restriction of λ^j to U_3 is non-zero when $j=1,2,3$. It follows from this and the computations above that λ and λ^2 restrict to non-trivial classes in the rational cohomology of U_3^*.

Putting all of this into the Mayer-Vietoris sequence (11) easily gives the computation of $H^j(\overline{\mathcal{A}}_3;\mathbb{Q})$ when $j \neq 6$. The computation of $H^6(\overline{\mathcal{A}}_3;\mathbb{Q})$ follows as the sequence

$$0 \to H^5(U_3^*;\mathbb{Q}) \to H^6(\overline{\mathcal{A}}_3;\mathbb{Q}) \to H^6(\mathcal{A}_3;\mathbb{Q}) \oplus H^6(\overline{\mathcal{A}}_2;\mathbb{Q}) \to H^6(U_3^*;\mathbb{Q}) \to 0$$

is exact in the category of MHSs.

Acknowledgment

I would like to thank the referee for his/her detailed comments and suggestions, which lead to considerable improvements in the exposition.

References

[1] K. Brown, *Cohomology of groups,* Graduate Texts in Mathematics, 87, Springer-Verlag, New York-Berlin, 1982.

[2] A. Brownstein, R. Lee, *Cohomology of the symplectic group* $\mathrm{Sp}_4(\mathbb{Z})$*. I. The odd torsion case,* Trans. Amer. Math. Soc. **334** (1992), 575–596.

[3] A. Brownstein, R. Lee, *Cohomology of the symplectic group* $\mathrm{Sp}(4,\mathbb{Z})$*. II. Computations at the prime* 2*,* Michigan Math. J. **41** (1994), 181–208.

[4] P. Deligne, *Théorie de Hodge. III.* Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 5–77.

[5] A. Durfee, R. Hain, *Mixed Hodge structures on the homotopy of links,* Math. Ann. **280** (1988), 69–83.

[6] P. Eberlein, *Geometry of nonpositively curved manifolds,* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.

[7] G. van der Geer, *The Chow ring of the moduli space of abelian threefolds,* J. Algebraic Geom. **7** (1998), 753–770.

[8] M. Goresky, R. MacPherson, *Stratified Morse theory,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 14. Springer-Verlag, Berlin, 1988.

[9] P. Griffiths, J. Harris, *Principles of algebraic geometry,* Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.

[10] R. Hain, *Moduli of Riemann surfaces, transcendental aspects,* School on Algebraic Geometry (Trieste, 1999), 293–353, ICTP Lect. Notes, 1, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000.

[11] R. Hain, E. Looijenga, *Mapping class groups and moduli spaces of curves,* in Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., 62, Part 2 (1997), 97–142.

[12] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces,* Pure and Applied Mathematics, 80. Academic Press, Inc., New York-London, 1978.

[13] J. Igusa, *On Siegel modular forms of genus two,* Amer. J. Math. **84** (1962), 175–200.

[14] E. Looijenga, *Cohomology of* $\mathcal{M}_3$ *and* $\mathcal{M}_3^1$*,* Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., 150, 1993, 205–228,

[15] G. Mess, *The Torelli groups for genus* 2 *and* 3 *surfaces,* Topology **31** (1992), 775–790.

Department of Mathematics, Duke University, Durham, NC 27708-0320, U.S.A.
E-mail address: `hain@math.duke.edu`