

REMARKS ON THE BOX PROBLEM

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0. Introduction

In [CCW], in connection with estimates on Fourier integral operators, the following question was raised. Let

$$Q = [0, 1]^n \subset \mathbb{R}^n,$$

be the unit cube in n dimensions. Let f be a C^n function on Q satisfying

$$\frac{\partial^n f}{\partial x_1 \partial x_2 \dots \partial x_n} > \Lambda.$$

What upper bound can be given on the measure of the set

$$S = \{x \in Q : |f(x)| < 1\}.$$

One can attempt to bound the measure of the set using combinatorial means. By a simple application of the fundamental theorem of calculus - one sees there is constant K_n depending only on the dimension so that for no sequence of pairs

$$(x_{11}, x_{12}), (x_{21}, x_{22}), \dots, (x_{n1}, x_{n2}),$$

satisfying

$$\prod_{i=1}^n |x_{i1} - x_{i2}| \geq \frac{1}{K_n \Lambda},$$

is it the case that the points $(x_{1\alpha_1}, \dots, x_{n\alpha_n})$ are elements of S for every choice of $\alpha_1, \dots, \alpha_n$ running from 1 to 2. Another way of saying this is that S does not contain the corners of a large box with sides in the coordinate directions. We will say throughout this note that any set satisfying this restriction satisfies the (CCW)-condition with constants (K_n, Λ) .

A few applications of Cauchy Schwarz show that a set satisfying the (CCW) condition has measure at most $O(\Lambda^{\frac{1}{2^n-1}-\epsilon})$ for any $\epsilon > 0$. On the other hand,

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this bound may not be sharp - no one knows how to find a (CCW) set that large. A very easy example of a (CCW) set is as follows, however. Let

$$f = \frac{\Lambda}{n!} (x_1 + x_2 + \dots x_n - \frac{n}{2})^N.$$

The N th mixed partial of f is Λ . Let

$$S = \{(x_1, \dots, x_n) : |f(x_1, \dots, x_n)| \leq 1.\}$$

Clearly S is the set of points $O(\Lambda^{\frac{1}{n}})$ away from a plane which cuts the cube generically. Thus clearly $|S| = O(\Lambda^{\frac{1}{n}})$. This index was observed in [CCW].

Thus we have an upper bound a little worse than $O(\Lambda^{\frac{1}{2^{n-1}}})$ and a lower bound of $O(\Lambda^{\frac{1}{n}})$. However $\frac{1}{2^{n-1}}$ becomes a lot smaller than $\frac{1}{n}$ as n grows bigger than 2. The first interesting case is $n = 3$ where the former is $\frac{1}{4}$ and the latter $\frac{1}{3}$.

One can make discrete analogues of this question. Let I_{ijk} be an $N \times N \times N$ tensor of 1's and 0's. We say I satisfies the (E) condition provided it does not contain 1's at all the corners of *any* nontrivial box. That is, I satisfies the (E) condition, if for any $i_1 \neq i_2, j_1 \neq j_2, k_1 \neq k_2$, we have that

$$\Pi_{l=1}^2 \Pi_{m=1}^2 \Pi_{n=1}^2 I_{i_l j_m k_n} = 0.$$

We can ask: what is the best upper bound that can be proven for the number of 1's in any tensor I satisfying the (E) condition? We refer to this as the box problem. It was first stated by Erdős [E] in the context of hypergraphs. We are interested in it mainly because of its relation to the work in [CCW].

A few applications of the Cauchy Schwarz inequality show that the number of 1's is at most $O(N^{\frac{11}{4}})$. Erdős [E] conjectured it to be sharp and it is the analogue of the $\frac{1}{4}$ above. One might ask - does one have an analogue of the lower bound with $\frac{1}{3}$? Simple numerology shows that this should consist of an example with $O(N^{\frac{8}{3}})$ many 1's. Unfortunately, the best published example seems to have only $O(N^{\frac{13}{5}})$ many 1's, see ([GRS], Lemma 2.4). This state of affairs is bad - since it would seem to indicate that such examples are far from providing new results on sets with the (CCW) condition. The [GRS] construction is inefficient in that it uses probabilistic methods. In its defense, one should note that in 5 and higher dimensions ([GRS], Lemma 2.5), it starts to beat the (CCW) index (and ties it in 4 dimensions) and it generally provides evidence for Erdős' conjecture.

We try to rectify the three dimensional situation.

Theorem. *Let p be any prime. There is a $p^3 - 1 \times p^3 - 1 \times p^3 - 1$ tensor I satisfying the (E) condition containing $p^2(p^3 - 1)^2$ many 1's.*

This does not provide new results about sets with the (CCW) condition but any improvement in the index would.

We prove the theorem in Section 1.

1. Proof of Theorem

We begin by recalling the standard construction of a sharp example for the two dimensional problem. It is simply part of the finite projective plane.

We consider the finite field F_q with q elements. (It must be that $q = p^n$ for some prime p and some integer n .) We define a $q^2 - 1 \times q^2 - 1$ matrix I_{ij} consisting of 1's and 0's so that for no $i_1 \neq i_2$ and $j_1 \neq j_2$ is it the case that

$$(1) \quad I_{i_1 j_1} I_{i_1 j_2} I_{i_2 j_1} I_{i_2 j_2} = 1.$$

We shall index the $q^2 - 1$ rows i by $F_q^2 \setminus \{(0,0)\}$, which is the plane over F_q excluding the origin. We shall index the columns j by the set of lines in F_q^2 not containing the origin. There are $q^2 - 1$ different such lines, given by the equations

$$ax + by = 1,$$

where $(a, b) \in F_q^2 \setminus \{(0,0)\}$. (Thus we notice a duality between the set of lines we consider and the set of points we consider.) Now we define $I_{ij} = 1$ if the point indexed by i is contained in the line indexed by j . Now since the intersection between two different lines is at most a point, we cannot have (1) whenever $j_1 \neq j_2$ and $i_1 \neq i_2$. We produce such an I with $q(q^2 - 1)$ many 1's, which is close to the most possible.

We now attempt to generalize this idea to three dimensions in order to prove the theorem. We shall fix a prime p and do geometry in $F_p^3 \setminus \{0\}$. A plane in F_p^3 is an equation of the form

$$ax + by + cz = 1,$$

with

$$(a, b, c) \in F_p^3 \setminus \{0\}.$$

Notice that here we have point-plane duality. Occasionally, we shall consider F_p^3 to be identified with F_{p^3} which is the extension of F_p by an irreducible cubic. We let r be a root of this cubic. Then an element of F_{p^3} can be written uniquely as

$$\alpha = a + rb + r^2c,$$

with $a, b, c \in F_p$. We say α is real if $b = c = 0$. Otherwise, we say α is complex.

We are now ready to define our construction of I_{ijk} a $(p^3 - 1) \times (p^3 - 1) \times (p^3 - 1)$ tensor of 1's and 0's. We index i and j by $F_{p^3} \setminus \{0\}$ and we index k by $F_p^3 \setminus \{(0,0,0)\}$. We will denote multiplication in F_{p^3} by $*$. To each element $\alpha \in F_{p^3} \setminus \{0\}$, we assign a plane P_α in $F_p^3 \setminus \{(0,0,0)\}$ by saying that P_α is given by the equation

$$ax + by + cz = 0,$$

when

$$\alpha = a + rb + r^2c.$$

Now we define I_{ijk} . We let $I_{ijk} = 1$ if $k \in P_{i*j}$, and we let $I_{ijk} = 0$ otherwise.

Clearly I_{ijk} has $p^2(p^3 - 1)^2$ many 1's. To prove the theorem, it now suffices to show that I satisfies the (E)-condition. We need only show that if $i_1 \neq i_2$ and $j_1 \neq j_2$ then the cardinality of

$$(2) \quad A_{i_1 i_2 j_1 j_2} = P_{i_1 * j_1} \cap P_{i_2 * j_1} \cap P_{i_1 * j_2} \cap P_{i_2 * j_2},$$

is at most one. Now since $A_{i_1 i_2 j_1 j_2}$ is an intersection of four planes, it is clear that it is either a plane, a line, a point, or empty. We need only exclude the first two possibilities.

Since $i_1 \neq i_2$, it must be that

$$P_{i_1 * j_1} \cap P_{i_2 * j_1} = L_{j_1},$$

is either empty or a line since it is the intersection of two different planes. Thus we may assume it is a line. Similarly, we may assume that

$$P_{i_1 * j_2} \cap P_{i_2 * j_2} = L_{j_2},$$

is a line.

Now (2) is exactly $L_{j_1} \cap L_{j_2}$. Thus to show it has cardinality at most 1, we need only show that L_{j_1} and L_{j_2} are not the same line. Let us suppose that they are. We call the line L .

Let K_1 be the set of all i so that $P_{i * j_1}$ contains the line L . The set of α so that P_α contains L is identified to a line K in $F_p^3 \setminus \{(0, 0, 0)\}$. Thus

$$K_1 = M_{j_1} K,$$

is also a line where M_{j_1} is the linear operator on F_p^3 induced by multiplication in F_{p^3} by j_1 . Similarly, we define K_2 to be the set of all i so that $P_{i * j_2}$ contains the line L and

$$K_2 = M_{j_2} K.$$

By our assumption $i_1, i_2 \in K_1$ and $i_1, i_2 \in K_2$. Since K_1 and K_2 are lines, this means

$$K_1 = K_2.$$

But

$$K_1 = M_{j_1} M_{j_2}^{-1} K_2 = M_{j_1 * (j_2)^{-1}} K_2 = M_{j_1 * (j_2)^{-1}} K_1.$$

Thus the linear map $M_{j_1 * (j_2)^{-1}}$ preserves the line K_1 .

Now since $j_1 \neq j_2$ it must be that $j_1 * (j_2)^{-1} \neq 1$. We claim that any M_α with $\alpha \in F_{p^3} \setminus \{0, 1\}$ cannot preserve any lines not containing the origin. This is a contradiction. Thus to prove that I satisfies the (E)-condition, we need only prove the claim.

We proceed to prove the claim. There are two cases, α is real or α is complex. If α is real, then $\alpha = d$, where $0, 1 \neq d \in F_p$. Then M_α takes the line

$$ax + by + cz = 1,$$

into

$$ax + by + cz = d.$$

These are clearly different lines.

Now suppose that α is complex and that M_α preserves the line K_1 . Then M_α preserves the two dimensional subspace of F_p^3 spanned by K_1 . Thus the characteristic polynomial of M_α factors nontrivially over F_p . But this is impossible since the characteristic polynomial of M_α is an irreducible cubic over F_p of which α is a root. (This is true since the representation $\alpha \rightarrow M_\alpha$ is faithful so that α must be a root of the characteristic polynomial of M_α . But α is a nonscalar element of F_{p^3} which has no proper subfields larger than F_p .) Thus we have proven the claim and the theorem. \square

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