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HARDY TYPE INEQUALITIES FOR AHARONOV-BOHM MAGNETIC POTENTIALS WITH MULTIPLE SINGULARITIES

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Abstract. The aim of this article is to extend the Laptev-Weidl inequalities to the case of Aharonov-Bohm magnetic potentials with multiple singularities.

1. Introduction

It is known that the classical Hardy inequality

$$\int_{\mathbb{R}^n} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}\, d\mathbf{x} \;\le\; \text{const}\cdot \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^2 d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \tag{1.1}$$

does not hold for $n = 2$. In [3] Laptev and Weidl discovered that introducing a magnetic field can improve this situation. In particular, if the gradient ∇ is replaced by the "magnetic" gradient $\nabla + iA$, where A is a smooth magnetic vector potential in $\mathbb{R}^2 \setminus \{0\}$ with

$$B := \operatorname{curl} A = 0, \tag{1.2}$$

then, for all $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$,

$$\int_{\mathbb{R}^2} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}\, d\mathbf{x} \;\le\; C \int_{\mathbb{R}^2} |(\nabla + iA)u(\mathbf{x})|^2\, d\mathbf{x}, \tag{1.3}$$

with the sharp constant $C = \left(\min_{k\in\mathbb{Z}} |k - \Psi|\right)^{-2}$. Here Ψ is the circulation of A round the origin

$$\Psi = \frac{1}{2\pi} \oint_{\mathbb{S}^1} A d\mathbf{x}. \tag{1.4}$$

In this paper we are interested in similar inequalities for magnetic Dirichlet forms with Aharonov-Bohm vector potentials that have multiple singularities. Let $P_1 = (x_1, y_1), \dots, P_n = (x_n, y_n)$ be n different points in $\mathbb{R}^2$. We can identify $\mathbb{R}^2$ with $\mathbb{C}$ by the correspondence $(x, y) \mapsto z = x + iy$ and the points $P_1, \dots, P_n$ then correspond to the complex numbers $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$. Consider a smooth vector potential $A = (A_1(x, y), A_2(x, y))$ in the punctured

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plane $M = \mathbb{R}^2 \setminus \{P_1, \dots, P_n\}$ with zero magnetic field (1.2) and denote by ω_A the differential 1-form $A_1(x, y)dx + A_2(x, y)dy$. Then (1.2) says that ω_A is a closed differential form in M, i.e. $d\omega_A = 0$. Such a vector potential A is known as a magnetic vector potential of Aharonov-Bohm type. The condition (1.2) implies that in any simply connected, open subset of M, there exists a gauge function f such that $A = \nabla f$.

For each point P_k $(k = 1, \dots, n)$ let us define a circulation of A round P_k as

$$\Phi_k = \frac{1}{2\pi} \oint_{\gamma_k} A_1(x, y)dx + A_2(x, y)dy, \tag{1.5}$$

where γ_k is a small circle in M which winds once around P_k in an anticlockwise direction. Condition (1.2) implies that (1.5) is invariant under continuous deformations of γ_k. Furthermore, if the circulations $\Phi = (\Phi_1, \dots, \Phi_n)$ of two distinct Aharonov-Bohm type vector potentials A and A' in M are equal modulo $\mathbb{Z}^n$ then A and A' are equivalent under some gauge transformation $\phi : M \to U(1)$, i.e. $A' = A + \frac{1}{i}\phi^{-1}\nabla\phi$. Here $U(1) = \{z \in \mathbb{C} | |z| = 1\}$.

For any magnetic vector potential A satisfying (1.2) in M there exists a gauge function f such that

$$A(x, y) - \sum_{j=1}^{n} \frac{\Phi_j}{r_j^2} \cdot (-y + y_j, x - x_j) = (\nabla f)(x, y),$$

where $r_j^2 = (x - x_j)^2 + (y - y_j)^2$ and Φ_j is the circulation of A round P_j.

Given the vector potential A, we define the corresponding magnetic Dirichlet form on $C_0^\infty(M)$ by

$$\mathcal{Q}_A[u] = \int_M |(\nabla + iA)u|^2 dxdy, \qquad u \in C_0^\infty(M). \tag{1.6}$$

Our main goal in this paper is to find an estimation from below for (1.6) by a Hardy-type expression

$$\mathcal{Q}_A[u] \geq \int_M H(x, y)|u(x, y)|^2 dxdy, \qquad u \in C_0^\infty(M) \tag{1.7}$$

with a suitable nonnegative function $H(x, y)$ on M.

For any real number Ψ denote by $p(\Psi)$ the distance from Ψ to the set of integers $\mathbb{Z}$, i.e.

$$p(\Psi) = \min_{k \in \mathbb{Z}} |k - \Psi|. \tag{1.8}$$

There are many functions $H(x, y)$ that give the inequality (1.7). We are interested in those $H(x, y)$ that satisfy the following conditions.

1. $H(x, y)$ depends on A only throughout the circulations $\Phi_1, \Phi_2, \dots, \Phi_n$ and the coordinates of P_j, $j = 1, \dots, n$.

2. $H(x, y)$ behaves like
$$\frac{(p(\Phi_j))^2}{(x - x_j)^2 + (y - y_j)^2}$$
near each point P_j, $j = 1, 2, \ldots, n$, and $H(x, y)$ behaves like
$$\frac{(p(\Phi_1 + \Phi_2 + \ldots + \Phi_n))^2}{x^2 + y^2}$$
near infinity.

Under a gauge transformation $u \mapsto \phi \cdot u$ with an arbitrary smooth function $\phi : M \to U(1)$, the Dirichlet form $\mathcal{Q}_A[u]$ becomes $\mathcal{Q}_{A'}[u]$ with $A' = A + \frac{1}{i}\phi^{-1}\nabla\phi$.The right hand side of (1.7) is invariant under this gauge transform. Hence, it is sufficient to establish (1.7) for any A from a given gauge equivalent class of magnetic vector potentials.

We show in this paper that any analytic function $F(z)$ on $\mathbb{C}$ with zero set $\{P_1, P_2, \ldots, P_n\}$ and $F(\infty) = \infty$ generates a function $H(x, y)$ with properties 1 and 2 above. It is also possible to choose an analytic function $F(z)$ which gives an $H(x, y)$ with the additional property

3. If $p(\Phi_j) \to 0$ for some j then the "contribution" from P_j in H also goes to zero.

For the reader's convenience we finish this introduction by giving an example of $H(x, y)$ in the case of two points $P_1 = -1$ and $P_2 = 1$ in $\mathbb{C}$ with the circulations $c_1 \equiv \Phi_1$ and $c_2 \equiv \Phi_2$, respectively.

Example 1.1. Let $P_1 = (-1, 0)$, $P_2 = (1, 0)$ be two points in $\mathbb{R}^2$, $M = \mathbb{R}^2 \setminus \{P_1, P_2\}$ and A is a magnetic vector potential of Aharonov-Bohm type in M with the circulations c_j round P_j, $j = 1, 2$. Then the inequality (1.7) holds with
$$H(x, y) = C(x, y) \cdot \left|\frac{2z}{z^2 - 1}\right|^2, \qquad z = x + iy,$$
where $C(x, y)$ is the piecewise constant function on $\mathbb{R}^2$ shown in Figure 1. In the figure, C is the curve $(x^2 - y^2 - 1) + 4x^2y^2 = 1$ which divides the plane $\mathbb{R}^2$ into three regions Ω_1, Ω_2 and Ω_∞, where $P_1 \in \Omega_1$ and $P_2 \in \Omega_2$; $C(x, y)$ equals $(p(c_1))^2$ in Ω_1, $(p(c_2))^2$ in Ω_2 and $(p(c_1 + c_2))^2/4$ in Ω_∞.

2. Inequality for doubly connected domains

Let Ω denote a bounded doubly connected domain (i.e. the boundary of Ω is a disjoint union of two closed simple curves) with a smooth boundary in the plane $\mathbb{R}^2 = \mathbb{C}$. Ω is homeomorphic to an open annulus. Let $\Omega_{r,R}$ $(r < R)$ be an annulus in $\mathbb{C}$ with the internal radius r, the external radius R and with the center at the origin, i.e.
$$\Omega_{r,R} = \{z \in \mathbb{C} \mid r < |z| < R\}.$$
From the theory of functions of one complex variable we know (see [5, Theorem 1.2]) that any doubly connected domain can be conformally mapped onto

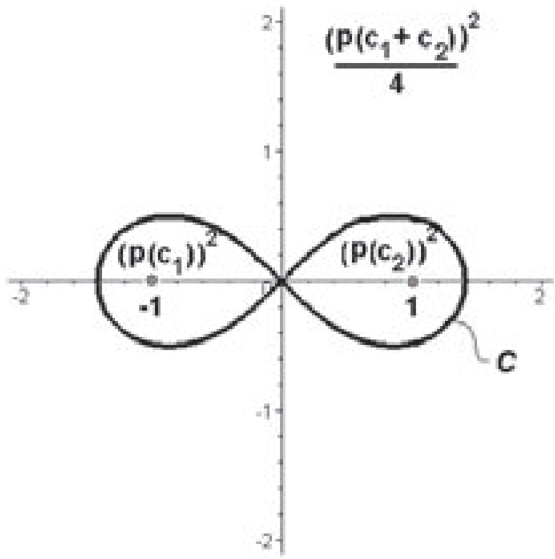

FIGURE 1. Function $C(x,y)$.

an annulus $\Omega_{r,R}$ for some r and R. For a conformal mapping $F:\Omega\to\Omega_{r,R}$ let us define a function $\mathcal{B}_{\Omega,F}(x,y)$ on Ω by

$$\mathcal{B}_{\Omega,F}(x,y)=\left|\frac{F'_z(z)}{F(z)}\right|^2, \tag{2.1}$$

where $z=x+iy$ and F'_z denote the complex derivative of F.

Lemma 2.1. *The function $\mathcal{B}_{\Omega,F}$ defined by* (2.1) *does not depend on the choice of the conformal mapping F.*

Proof. Consider any other conformal mapping $\tilde{F}$ from Ω onto $\Omega_{\tilde{r},\tilde{R}}$. From Theorem 1.3 [5] we know that

$$\frac{R}{r}=\frac{\tilde{R}}{\tilde{r}}.$$

Hence, since the right-hand side of (2.1) is invariant under scaling $F\mapsto const\cdot F$, we can assume that $r=\tilde{r}$ and $R=\tilde{R}$. $\tilde{F}\circ F^{-1}$ is a conformal automorphism of $\Omega_{r,R}$. Since any holomorphic automorphism of $\Omega_{r,R}$ is a composition of rotations and reflections (see p. 133 in [2]), we have to check that the right-hand side of (2.1) is invariant under $F\mapsto\mu\cdot F$ (for μ a unimodular constant) and under $F\mapsto\frac{r\cdot R}{F}$. This is clear and hence the proof is completed. □

We shall use the notation $\mathcal{B}_\Omega$ instead of $\mathcal{B}_{\Omega,F}$.

Let $A=(A_1(x,y),A_2(x,y))$ be a smooth magnetic vector potential in $\overline{\Omega}$ with zero magnetic field (1.2). Recall that a circulation Φ of A in the doubly connected domain Ω is

$$\Phi = \frac{1}{2\pi} \oint_{\sigma} A_1(x,y)dx + A_2(x,y)dy,$$

where σ is a closed path which parameterizes the "internal" component of the boundary of Ω. The last integral is invariant under continuous deformations of σ.

Theorem 2.2. *Let Ω be a bounded doubly connected domain in $\mathbb{R}^2$ with a smooth boundary. For any smooth function $f \in C^\infty(\overline{\Omega})$ we have*

$$\int_{\Omega} |(\nabla + iA)f|^2 dxdy \;\geq\; (p(\Phi))^2 \int_{\Omega} \mathcal{B}_\Omega(x,y)|f(x,y)|^2 dxdy, \tag{2.2}$$

where $\mathcal{B}_\Omega(x,y)$ is defined by (2.1) *and $p(\Phi)$ is defined by (1.8).*

Proof. First let us prove that for $\Omega_{r,R}$ and $\tilde{A} = \frac{\Phi}{x^2+y^2}(-y,x)$ the following inequality holds

$$\int_{\Omega_{r,R}} |(\nabla + i\tilde{A})f|^2 dxdy \;\geq\; (p(\Phi))^2 \int_{\Omega_{r,R}} \frac{|f(x,y)|^2}{x^2+y^2} dxdy, \tag{2.3}$$

for any $f \in C^\infty(\overline{\Omega_{r,R}})$ (see [3] for more general results). The left-hand side and the right-hand side are both invariant under rotation of $\mathbb{R}^2$ around the origin. So, it is sufficient to establish (2.3) for spherical functions $f(r)e^{in\theta}$, $n \in \mathbb{Z}$ and $r = \sqrt{x^2+y^2}$. For such functions

$$\int_{\Omega_{r,R}} |(\nabla + i\tilde{A})f(r)e^{in\theta}|^2 dxdy = \int_{\Omega_{r,R}} (|f'_r|^2 + \frac{1}{r^2}|f(r)|^2 \cdot (n+\Phi)^2) dxdy$$

$$\geq \int_{\Omega_{r,R}} \frac{1}{r^2}|f(r)|^2 \cdot (n+\Phi)^2 dxdy \geq (p(\Phi))^2 \int_{\Omega_{r,R}} \frac{|f(r)e^{in\theta}|^2}{x^2+y^2} dxdy.$$

Now, let $F : \Omega \to \Omega_{r,R}$ be a conformal mapping, $F(x,y) = (u(x,y), v(x,y))$. Denote by $A^F(u,v) = (A_1^F(u,v), A_2^F(u,v))$ a magnetic vector potential in $\Omega_{r,R}$ such that $F^*(\omega_{A^F}) = \omega_A$, i.e.

$$A_1^F(u,v)du + A_2^F(u,v)dv = A_1(x,y)dx + A_2(x,y)dy.$$

The magnetic vector potential A^F also has zero magnetic field and the same circulation Φ as A.

Since F is a conformal mapping, the reader will have no difficulty in showing that for any $f \in C^\infty(\overline{\Omega_{r,R}})$ we have

$$\begin{aligned}(2.4)\qquad \int\limits_{\Omega_{r,R}} &(|(\nabla_u + iA_1^F(u,v))f(u,v)|^2 + |(\nabla_v + iA_2^F(u,v))f(u,v)|^2)dudv \\ &= \int\limits_{\Omega} (|(\nabla_x + iA_1(x,y))f(u(x,y),v(x,y))|^2 \\ &\qquad + |(\nabla_y + iA_2(x,y))f(u(x,y),v(x,y))|^2)dxdy.\end{aligned}$$

Since A^F is gauge equivalent to the magnetic vector potential $\frac{\Phi}{u^2+v^2}(-v,u)$ and the inequality (2.3) is also invariant under gauge transformations, we have that

$$(2.5)\qquad \int\limits_{\Omega} |(\nabla + i\tilde{A})f(u(x,y),v(x,y))|^2 dxdy \ \geq\ (p(\Phi))^2 \int\limits_{\Omega_{r,R}} \frac{|f(u,v)|^2}{u^2+v^2} dudv.$$

Taking into account that

$$\int\limits_{\Omega_{r,R}} \frac{|f(u,v)|^2}{u^2+v^2} dudv = \int\limits_{\Omega} |f(u(x,y),v(x,y))|^2 \frac{|F_z'|^2}{|F|^2} dxdy,$$

we obtain from (2.5) the inequality (2.2). This completes the proof. □

3. Inequalities for punctured plane

Theorem 2.2 gives us a Hardy-type inequality for a magnetic vector potential of Aharonov-Bohm type in a bounded doubly connected domain. For a more general domain Ω, e.g. a Riemann surface or multiply connected domain, our strategy is:

a) find a decomposition, up to a zero measure set, of the domain Ω into doubly connected domains;

b) find conformal mappings of these doubly connected domains into annuli and apply Theorem 2.2.

The classical Morse theory (see e.g. [4]) is the most natural tool for constructing decompositions into doubly connected domains. But, in general, finding a conformal mapping from a doubly connected domain into $\Omega_{r,R}$ is a difficult problem. Our idea in this article is to use a function $|F|$ for constructing a Morse complex, where $F : \Omega \to \mathbb{C}$ is an holomorphic function. In this case, the function $|F|$ provides a decomposition and F provides the conformal mappings. We are going to apply this idea in the case of the punctured plane $M = \mathbb{C} \setminus \{P_1, P_2, \dots, P_n\}$ with a smooth magnetic vector potential A of Aharonov-Bohm type.

Let $F : \mathbb{C} \to \mathbb{C}$ be an analytic function with zero set $\{P_1, P_2, \dots, P_n\}$, i.e. $F^{-1}(0) = \{P_1, P_2, \dots, P_n\}$, and $F(\infty) = \infty$. Denote by $\mathrm{ord}_{P_j} F$ the order of zero of F at P_j. Let $\{Q_1, Q_2, \dots, Q_l\}$ be a zero set of the complex derivative F_z'

of the function F, i.e. $\{Q_1, Q_2, \dots, Q_l\} = (F'_z)^{-1}(0)$, and denote by Crit_F the following subset of $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$:

$$\mathrm{Crit}_F = \{0, |F(Q_1)|, \dots, |F(Q_l)|\}.$$

Under the map $|F| : \mathbb{C} \to \mathbb{R}_+$ the pre-image of Crit_F is a zero measure set $\mathcal{F}_c$.

Let us define a piecewise constant function C_F on $\mathbb{R}^2$. For any $(x, y) \in \mathbb{R}^2$, $x + iy \notin \mathcal{F}_c$, the set $|F|^{-1}(|F|(x + iy))$ is a disjoint union of smooth simple curves in $\mathbb{C}$ and let $\gamma_{(x,y)}$ denote one of them that goes through the point (x, y). This $\gamma_{(x,y)}$ divides $\mathbb{C}$ into two domains, a bounded domain $\Omega_{\mathrm{int}}(\gamma_{(x,y)})$ and an unbounded domain $\Omega_{\mathrm{ext}}(\gamma_{(x,y)})$. Then

$$C_F(x, y) := \frac{\left(p\Big(\sum\limits_{P_k \in \Omega_{\mathrm{int}}(\gamma_{(x,y)})} \Phi_k\Big)\right)^2}{(\mathrm{ord}_{\gamma_{(x,y)}} F)^2}, \tag{3.1}$$

where Φ_k is a circulation of A round P_k and

$$\mathrm{ord}_{\gamma_{(x,y)}} F = \sum_{P_k \in \Omega_{\mathrm{int}}(\gamma_{(x,y)})} \mathrm{ord}_{P_k} F. \tag{3.2}$$

We can now state our main result.

Theorem 3.1. *Let C_F be defined in* (3.1) *for the analytic function F. For any $u \in C_0^\infty(M)$ the following inequality holds*

$$\int_M |(\nabla + iA)u|^2 dxdy \geq \int_M C_F(x, y) \left|\frac{F'_z(x + iy)}{F(x + iy)}\right|^2 |u(x, y)|^2 dxdy. \tag{3.3}$$

Proof. Let $\mathbb{R}_+ \setminus \mathrm{Crit}_F = \bigcup\limits_{m \in \mathbb{Z}} [a_m, b_m]$ such that $(a_m, b_m) \cap (a_{m'}, b_{m'}) = \emptyset$ for $m \neq m'$. From Morse theory we know that $|F|^{-1}((a_m, b_m))$ is a disjoint union of doubly connected domains. Let Ω_0 be any connected component of $|F|^{-1}((a_m, b_m))$. Then $F|_{\Omega_0} : \Omega_0 \to \Omega_{a_m, b_m}$ is a holomorphic function from Ω_0 onto an annulus Ω_{a_m, b_m}. Since we are away from the critical set of F, the holomorphic map $F|_{\Omega_0}$ is a covering map from Ω_0 onto the annulus Ω_{a_m, b_m}. From the Argument Principle (see, e.g. [1]) the degree of $F|_{\Omega_0}$ equals to $\mathrm{ord}_{\gamma_{(x,y)}} F$ defined in (3.2), where $(x, y) \in \Omega_0$. Therefore the function $(F|_{\Omega_0})^{1/\mathrm{ord}_{\gamma_{(x,y)}} F}$ is well defined and is a conformal mapping from Ω_0 onto an annulus Ω_{r_m, R_m}, where

$$r_m = (a_m)^{1/\mathrm{ord}_{\gamma_{(x,y)}} F} \quad \text{and} \quad R_m = (b_m)^{1/\mathrm{ord}_{\gamma_{(x,y)}} F}.$$

From Theorem 2.2

$$\int_{\Omega_0} |(\nabla + iA)u|^2 dxdy \tag{3.4}$$

$$\geq \int_{\Omega_0} \Big(p\big(\sum_{P_k \in \Omega_{\text{int}}(\gamma_{(x,y)})} \Phi_k \big) \Big)^2 \left| \frac{((F|_{\Omega_0})^{1/\text{ord}_{\gamma_{(x,y)}} F})'_z}{(F|_{\Omega_0})^{1/\text{ord}_{\gamma_{(x,y)}} F}} \right|^2 |u|^2 dxdy$$

$$= \int_{\Omega_0} C_F(x,y) \left| \frac{F'_z(x+iy)}{F(x+iy)} \right|^2 |u(x,y)|^2 dxdy.$$

Summing (3.4) over all connected component of $|F|^{-1}((a_m, b_m))$ and over all $m \in \mathbb{Z}$, we obtain Theorem 3.1. This conclude the proof. □

Remark 3.2. Choosing $F(z) = \prod_{j=1}^{n} (z - z_j)$ we obtain a function

$$H_F(x,y) = C_F(x,y) \left| \frac{F'_z(x+iy)}{F(x+iy)} \right|^2$$

which satisfies the conditions 1 and 2 from the Introduction. Choosing

$$F(z) = \frac{1}{\sum_{j=1}^{n} \frac{p(\Phi_j)}{(z-z_j)}}$$

gives $H_F(x,y)$ with the additional property 3 from the Introduction.

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