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A SATURATED STATIONARY SUBSET OF $\mathcal{P}_\kappa\kappa^+$

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Abstract. Somewhere saturation of the club filter on $\mathcal{P}_\kappa\kappa^+$ is shown consistent relative to a κ^+-supercompact cardinal κ.

1. Introduction

Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. In [15] Solovay established that a stationary subset of κ splits into κ stationary sets. Solovay's theorem led Menas [13] to conjecture that a stationary subset S of $\mathcal{P}_\kappa\lambda$ splits into $\lambda^{<\kappa}$ stationary sets. Menas' conjecture holds in the constructible universe L (see [12]). In fact L satisfies $\diamondsuit^*_{\kappa,\lambda}$ in the sense of [5] (see [10]), which implies the conjecture via $\diamondsuit_{\kappa,\lambda}(S)$. On the other hand the case $\lambda = \kappa^+$ of the conjecture implies $(\kappa^+)^{<\kappa} = \kappa^+$, hence is independent of ZFC, since some stationary subset of $\mathcal{P}_\kappa\kappa^+$ has size κ^+ (see [4]).

The last observation suggests that $\lambda^{<\kappa}$ in the conjecture should be replaced by the smallest size of $S \cap C$ for C a club subset of $\mathcal{P}_\kappa\lambda$. Two results support the revised conjecture: First an unbounded subset X of $\mathcal{P}_\kappa\lambda$ splits into $\min\{|X \cap C| : C \subset \mathcal{P}_\kappa\lambda \text{ is cobounded}\}$ unbounded sets (see [1]). Second $\mathcal{P}_\kappa\lambda$ splits into λ^ω stationary sets (see [14]). By [3] and [11], λ^ω is the best lower bound of the size of club sets in terms of cardinal arithmetic.

In [6] Gitik constructed a model in which even the revised conjecture fails, assuming a κ^{+3}-supercompact cardinal κ exists. Indeed some stationary subset of $\mathcal{P}_\kappa\kappa^+$ cannot split into κ^+ stationary sets in his model. It was basically a modification of the Jech–Woodin model [8], in which the club filter on κ is somewhere κ^+-saturated. Unfortunately Gitik's proof involved a difficult construction of the intermediate model. Incorporating a variation of the Jech–Shelah forcing [7] for shooting a club subset of $\mathcal{P}_\kappa\kappa^+$, we resolve the difficulty as well as reduce the assumption:

Theorem. *Assume κ is κ^+-supercompact and GCH. Then some poset with the κ^+-cc forces that κ is Mahlo and the club filter on $\mathcal{P}_\kappa\kappa^+$ is somewhere κ^+-saturated.*

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2. Preliminaries

Our reference book is [9]. See [2] for the basics of iterated forcing. We understand that a limit ordinal is even. We call a set x of ordinals σ-closed if $\sup a \in x$ for $a \in [x]^\omega$. An infinite set of ordinals has the σ-closure, i.e. the smallest σ-closed superset, of the same size.

Throughout the paper κ denotes a regular uncountable cardinal. Let S_κ be the set $\{x \in \mathcal{P}_\kappa\kappa^+ : x \text{ is } \sigma\text{-closed} \wedge x \cap \kappa \text{ is a limit ordinal} \wedge \operatorname{ot} x = (x \cap \kappa)^+\}$. By an argument of [3], S_κ agrees with the set $\{x \in \mathcal{P}_\kappa\kappa^+ : \operatorname{cf}(x \cap \kappa) > \omega \wedge \operatorname{ot} x = (x \cap \kappa)^+\}$ on some club set.

For a poset P let $\mathcal{G}_\kappa(P)$ be the following game: Players I and II play conditions of P at odd and even stages respectively, where the move at stage 0 is trivial, and produce a descending sequence as long as they can. II wins iff the resulting sequence has length at least κ. We call P weakly κ-closed if II has a winning strategy in $\mathcal{G}_\kappa(P)$. A weakly κ-closed poset is κ-Baire, preserves the stationarity of subsets of κ, and has an M-generic filter if M has at most κ maximal antichains. Assume $\langle P_\alpha : \alpha \leq \gamma \rangle$ is a forcing iteration with Easton support, $\kappa < \gamma$, P_κ has the κ-cc and P_α forces that $P_{\alpha+1}/\dot{G}_\alpha$ is weakly κ-closed for $\kappa \leq \alpha < \gamma$, where $\dot{G}_\alpha$ is the P_α-name for the generic filter. Then P_κ forces that $P_\gamma/\dot{G}_\kappa$ is weakly κ-closed.

3. Shooting club subsets of $\mathcal{P}_\kappa\kappa^+$

This section is devoted to our variation of the Jech–Shelah forcing [7].

Let κ be an inaccessible cardinal. Order $R = \{p : d \times (d \cap \kappa) \to d : d \in \mathcal{P}_\kappa\kappa^+ \text{ is } \sigma\text{-closed} \wedge d \cap \kappa \in \kappa \wedge \forall\alpha \in d(p``\{\alpha\} \times (d \cap \kappa) = d \cap (\alpha \cup \kappa))\}$ by reverse inclusion. For a condition $p : d \times (d \cap \kappa) \to d$, set $d(p) = d$ and $C(p) = \{x \subset d : \forall\alpha \in x(p``\{\alpha\} \times (x \cap \kappa) = x \cap (\alpha \cup \kappa))\}$. For $d \in \mathcal{P}_\kappa\kappa^+$ σ-closed with $d \cap \kappa \in \kappa$, $d = d(p)$ for some $p \in R$ iff $\operatorname{ot} d \leq (d \cap \kappa)^+$. For $X \subset S_\kappa$ let $R(X)$ be the suborder $\{p \in R : S_\kappa \cap C(p) \subset X\}$ of R.

Lemma 1. *$\{q \in R(X) : z \subset d(q)\}$ is dense for $z \in \mathcal{P}_\kappa\kappa^+$.*

Proof. We can assume z is σ-closed. Let $p \in R(X)$. Take a cardinal $\delta < \kappa$ greater than $\sup(z \cap \kappa)$, $|z|$ and $d(p) \cap \kappa$. We have $p \subset q \in R$ such that $d(q) = (\delta+1) \cup z \cup d(p)$ and $q(\alpha, 0) = \delta$ for $\alpha \in d(q) - d(p)$. We show $S_\kappa \cap C(q) \subset C(p)$, which implies $q \in R(X)$.

Fix $x \in S_\kappa \cap C(q)$. Then $x \subset d(p)$: Otherwise we would have $\alpha \in x - d(p)$. Then $q(\alpha, 0) = \delta$ by the choice of q. Hence $\delta \in x \cap \kappa \subset d(q) \cap \kappa = \delta + 1$, since $\alpha \in x$, $0 \in x$ by $x \in S_\kappa$, and $x \in C(q)$. Thus $x \cap \kappa$ is not a limit ordinal, which contradicts $x \in S_\kappa$, as desired. Hence $x \in C(p)$ by $x \in C(q)$ and $p \subset q$. $\square$

Lemma 2. *$R(X)$ has the κ^+-cc.*

Proof. Let $A \subset R(X)$ have size κ^+. We have $B \subset A$ of size κ^+ such that $\{d(p) : p \in B\}$ forms a Δ-system with root $d \in \mathcal{P}_\kappa\kappa^+$ and $d(p) \cap \kappa = \delta \in \kappa$ for

$p \in B$. Then $p``d \times \delta \subset \sup d < \kappa^+$ for $p \in B$. Take $p \neq q$ from B which agree on $d \times \delta$. We claim $p \parallel q$.

We have $p \cup q \subset r \in R$ such that $d(r) = (\delta + \delta + 1) \cup d(p) \cup d(q)$ and $r(\alpha, 0) = \delta + \delta$ for $\alpha \in d(r) - (d(p) \cup d(q))$. We show $S_\kappa \cap C(r) \subset C(p) \cup C(q)$, which implies $r \in R(X)$.

Fix $x \in S_\kappa \cap C(r)$. Then $x \subset d(p) \cup d(q)$ by the proof of Lemma 1. Hence $x \cap d(p)$ or $x \cap d(q)$ is unbounded in x. We can assume the former by symmetry. Then $x \subset d(p)$: Fix $\gamma \in x$. Take $\gamma < \alpha \in x \cap d(p)$. Then $\gamma = r(\alpha, \beta)$ for some $\beta \in x \cap \kappa$ by $x \in C(r)$. Hence $\gamma = p(\alpha, \beta) \in d(p)$, since $\alpha \in d(p)$, $\beta \in x \cap \kappa \subset (d(p) \cup d(q)) \cap \kappa = d(p) \cap \kappa$ and $p \subset r$, as desired. Thus $x \in C(p)$ by $x \in C(r)$ and $p \subset r$. □

Lemma 3. *$R(X)$ is weakly κ-closed.*

Proof. We give a strategy τ for II in the game $\mathcal{G}_\kappa(R)$, which works uniformly for $\mathcal{G}_\kappa(R(X))$. Let $\langle p_\zeta : \zeta < \xi \rangle$ be a partial run of $\mathcal{G}_\kappa(R)$ according to τ with $0 < \xi < \kappa$ even. Set $\tau(\langle p_\zeta : \zeta < \xi \rangle) = p$ as follows:

When $\xi = \zeta + 1$, take $p_\zeta \subset p \in R$ as in the proof of Lemma 1 so that $|d(p_\zeta)| < |d(p) \cap \kappa|$. Then $p \in R(X)$ if $p_\zeta \in R(X)$ by the proof of Lemma 1.

If ξ is limit, then $|\bigcup_{\zeta<\xi} d(p_\zeta)| = \bigcup_{\zeta<\xi} |d(p_\zeta)| = \bigcup_{\zeta<\xi} d(p_\zeta) \cap \kappa$, since $|d(p_\zeta)|$, $d(p_\zeta) \cap \kappa < |d(p_\zeta)|^+ \leq |d(p_{\zeta+1}) \cap \kappa| \leq |d(p_{\zeta+1})|$, $d(p_{\zeta+1}) \cap \kappa$ for $\zeta < \xi$ odd.

When $\operatorname{cf} \xi = \omega$, let d be the σ-closure of $\bigcup_{\zeta<\xi} d(p_\zeta)$ and $\delta = |d|$. Then $\delta = |\bigcup_{\zeta<\xi} d(p_\zeta)| = \bigcup_{\zeta<\xi} d(p_\zeta) \cap \kappa$. Hence $d \cap \kappa = \delta + 1$ by $\operatorname{cf} \delta = \omega$. We have $\bigcup_{\zeta<\xi} p_\zeta \subset p \in R$ such that $d(p) = (\delta + \delta + 1) \cup d$ and $p(\alpha, 0) = \delta + \delta$ for $\alpha \in d(p) - \bigcup_{\zeta<\xi} d(p_\zeta)$. We show $S_\kappa \cap C(p) \subset \bigcup_{\zeta<\xi} C(p_\zeta)$, which implies $p \in R(X)$ if $\{p_\zeta : \zeta < \xi\} \subset R(X)$.

Fix $x \in S_\kappa \cap C(p)$. Then $x \subset \bigcup_{\zeta<\xi} d(p_\zeta)$ by the proof of Lemma 1. Hence $x \cap \kappa < \operatorname{ot} x = |x| \leq |\bigcup_{\zeta<\xi} d(p_\zeta)| = \bigcup_{\zeta<\xi} d(p_\zeta) \cap \kappa$ by $x \in S_\kappa$. Thus we have $\zeta < \xi$ such that $x \cap \kappa \subset d(p_\zeta) \cap \kappa$ and $x \cap d(p_\zeta)$ is unbounded in x, since $x \subset \bigcup_{\zeta<\xi} d(p_\zeta)$, and $\operatorname{cf} \xi = \omega < \operatorname{cf} \operatorname{ot} x$ by $x \in S_\kappa$. Hence $x \subset d(p_\zeta)$ and $x \in C(p_\zeta)$ by the proof of Lemma 2.

When $\operatorname{cf} \xi > \omega$, $\bigcup_{\zeta<\xi} d(p_\zeta)$ is σ-closed. Set $p = \bigcup_{\zeta<\xi} p_\zeta \in R$. We show $S_\kappa \cap C(p) \subset \bigcup_{\zeta<\xi} C(p_\zeta)$, which implies $p \in R(X)$ if $\{p_\zeta : \zeta < \xi\} \subset R(X)$.

Fix $x \in S_\kappa \cap C(p)$. Then $x \subset d(p) = \bigcup_{\zeta<\xi} d(p_\zeta)$. Hence $x \cap \kappa < \bigcup_{\zeta<\xi} d(p_\zeta) \cap \kappa$ as in the last case. Thus $x \cap \kappa \subset d(p_{\zeta_0}) \cap \kappa$ for some $\zeta_0 < \xi$. We have $\zeta < \xi$ such that $x \cap d(p_\zeta)$ is unbounded in x: Otherwise we would have increasing sequences $\{\zeta_n : n < \omega\} \subset \xi$ and $\{\alpha_n : n < \omega\} \subset x$ such that $\sup(x \cap d(p_{\zeta_n})) < \alpha_n \in d(p_{\zeta_{n+1}})$. Then $\zeta = \sup_{n<\omega} \zeta_n < \xi$ by $\operatorname{cf} \xi > \omega$. Since $\alpha_n \in x \cap d(p_{\zeta_{n+1}}) \subset x \cap d(p_\zeta)$ for $n < \omega$, and $x \cap d(p_\zeta)$ is σ-closed by $x \in S_\kappa$ and $p_\zeta \in R$, $\alpha = \sup_{n<\omega} \alpha_n \in x \cap d(p_\zeta)$. Since $\sup(x \cap d(p_{\zeta_n})) < \alpha_n < \alpha \in x$ for $n < \omega$, $\alpha \notin \bigcup_{n<\omega} d(p_{\zeta_n})$. Set $\delta = \bigcup_{n<\omega} d(p_{\zeta_n}) \cap \kappa \in \kappa$. Then $p_\zeta(\alpha, 0) = \delta + \delta$ by the choice of p_ζ. Hence $\delta + \delta = p(\alpha, 0) \in x \cap \kappa$ by $p_\zeta \subset p$ and $\alpha, 0 \in x \in C(p)$, which contradicts $x \cap \kappa \subset d(p_{\zeta_0}) \cap \kappa \subset \delta$, as desired. Thus $x \cap \kappa \subset d(p_\zeta) \cap \kappa$ and

$x \cap d(p_\zeta)$ is unbounded in x for some $\zeta < \xi$. Hence $x \subset d(p_\zeta)$ and $x \in C(p_\zeta)$ by the proof of Lemma 2. $\square$

Let $\dot{H}$ be the $R(X)$-name for the generic filter. Then $R(X)$ forces that $\bigcup \dot{H} : \kappa^+ \times \kappa \to \kappa^+$ satisfies $(\bigcup \dot{H})``\{\alpha\} \times \kappa = \alpha \cup \kappa$ for $\alpha < \kappa^+$ and generates the club set $\{x \in \mathcal{P}_\kappa \kappa^+ : \forall \alpha \in x((\bigcup \dot{H})``\{\alpha\} \times (x \cap \kappa) = x \cap (\alpha \cup \kappa))\}$ disjoint from $S_\kappa - X$.

By induction on γ, we build a suborder Q_γ of the product of γ copies of R with $< \kappa$-support and take a Q_γ-name $\dot{X}_\gamma$ for a subset of S_κ so that $Q_\gamma = \{p : b \to R : b \in \mathcal{P}_\kappa \gamma \wedge \forall \beta \in b(p|\beta \Vdash_{Q_\beta} p(\beta) \in R(\dot{X}_\beta))\}$. If neccessary, we understand that $p \in Q_\gamma$ maps $\beta \notin \mathrm{dom}(p)$ to $\emptyset$.

Lemma 4. *Q_γ has the κ^+-cc.*

Proof. Let $A \subset Q_\gamma$ have size κ^+. We have $B \subset A$ of size κ^+, $b \in \mathcal{P}_\kappa \gamma$ and $\{d_\beta : \beta \in b\} \subset \mathcal{P}_\kappa \kappa^+$ such that $\{\bigcup_{\beta \in \mathrm{dom}(p)} \{\beta\} \times d(p(\beta)) : p \in B\}$ forms a Δ-system with root $\bigcup_{\beta \in b} \{\beta\} \times d_\beta$ and $d(p(\beta)) \cap \kappa = d_\beta \cap \kappa \in \kappa$ for $p \in B$ and $\beta \in b$ as in the proof of Lemma 2. Take $p \neq q$ from B so that $p(\beta)$ and $q(\beta)$ agree on $d_\beta \times (d_\beta \cap \kappa)$ for $\beta \in b$. We claim $p \parallel q$.

We have $r : \mathrm{dom}(p) \cup \mathrm{dom}(q) \to R$ such that $r(\beta)$ extends $p(\beta)$ and $q(\beta)$ as in the proof of Lemma 2 if $\beta \in b$, and is $p(\beta) \cup q(\beta)$ otherwise. Then $r|\beta \in Q_\beta$ extends $p|\beta$ and $q|\beta$ by induction on $\beta \leq \gamma$. $\square$

Lemma 5. *Q_γ is weakly κ-closed.*

Proof. We give a strategy τ_γ for II in the game $\mathcal{G}_\kappa(Q_\gamma)$. Let $\langle p_\zeta : \zeta < \xi \rangle$ be a partial run of $\mathcal{G}_\kappa(Q_\gamma)$ according to τ_γ with $\xi < \kappa$ even. We have $p : \bigcup_{\zeta < \xi} \mathrm{dom}(p_\zeta) \to R$ such that $p(\beta) = \tau(\langle p_\zeta(\beta) : \zeta < \xi \rangle)$. Then we can set $\tau_\gamma(\langle p_\zeta : \zeta < \xi \rangle) = p$, since $p|\beta \in Q_\beta$ extends $p_\zeta|\beta$ for $\zeta < \xi$ by induction on $\beta \leq \gamma$. $\square$

We can identify a Q_γ-name $\dot{X}$ for a subset of S_κ with a subset "$\dot{X}$" of $S_\kappa \times Q_\gamma$ such that $\{p \in Q_\gamma : (x, p) \in \text{"}\dot{X}\text{"}\}$ is an antichain for $x \in S_\kappa$. Set $\mathrm{supp}(\dot{X}) = \bigcup\{\mathrm{dom}(p) : \exists x \in S_\kappa((x, p) \in \text{"}\dot{X}\text{"})\}$. Let $s \subset \gamma$ and $\mathrm{supp}(\dot{X}_\beta) \subset s \cap \beta$ for $\beta \in s$. Then $Q_s = \{p \in Q_\gamma : \mathrm{dom}(p) \subset s\}$ is a complete suborder of Q_γ, hence has the κ^+-cc. In fact $p|s$ is a reduction of $p \in Q_\gamma$ to Q_s. The winning strategy τ_γ witnesses that Q_s is weakly κ-closed.

Finally assume further $2^{2^\kappa} = \kappa^{++}$. Then $Q_{\kappa^{++}} = \bigcup_{\gamma < \kappa^{++}} Q_\gamma$ has size κ^{++}, since $\{Q_\gamma : \gamma < \kappa^{++}\} \subset H(\kappa^{++})$. Hence we can arrange the construction so that a $Q_{\kappa^{++}}$-name for a subset of S_κ agrees with $\dot{X}_\gamma$ for κ^{++} many γ's.

4. The main forcing

In this section the Jech–Woodin model [8] is modified for our

Proof of Theorem. Let U be a normal ultrafilter on $\mathcal{P}_\kappa \kappa^+$ and $j_U : V \to M = \mathrm{Ult}(V, U)$ the induced embedding. We have $S^* \subset S_\kappa$ in U such that ot $x < y \cap \kappa$

for $x \subsetneq y$ both in S^* by the partition property of U and $S_\kappa \in U$. We perform a forcing iteration $P = \langle P_\alpha : \alpha \leq \kappa \rangle$ with Easton support so that $P_{\alpha+1}$ is $P_\alpha * \dot{Q}_{\alpha^{++}}$ if $\alpha < \kappa$ is Mahlo, and $P_\alpha * \{\emptyset\}$ otherwise. For $\alpha \leq \kappa$ inaccessible, P_α has size α by $P``\alpha \subset V_\alpha$, hence forces $2^\alpha = \alpha^+$ by GCH. Thus for $\alpha \leq \kappa$ Mahlo, P_α has the α-cc and forces that α is inaccessible and $2^{2^\alpha} = \alpha^{++}$. Hence for $\alpha < \kappa$ Mahlo, P_κ forces that α is inaccessible. Thus P_κ forces that κ is Mahlo, since $\{\alpha < \kappa : \alpha \text{ is Mahlo}\}$ is stationary and P_κ has the κ-cc.

Let $G \subset P_\kappa$ be V-generic and work in $V[G]$. By the first paragraph, κ is Mahlo and $2^{2^\kappa} = \kappa^{++}$. Since $P|\kappa = j_U(P)|\kappa$, $P_\kappa = j_U(P)_\kappa \in M$. $M[G]$ is closed under the κ^+-sequences, since M is closed under the κ^+-sequences and P_κ has the κ-cc in V. Hence κ is inaccessible and $2^{2^\kappa} = \kappa^{++}$ in $M[G]$. Set $Q = (j_U(\dot{Q})_{\kappa^{++}})_G \in M[G]$. Then we have Q-names $\{\dot{X}_\gamma : \gamma < \kappa^{++}\}$ for subsets of S_κ with which Q is built in $M[G]$. We can regard Q and $\{\dot{X}_\gamma : \gamma < \kappa^{++}\}$ as built in $V[G]$, since $M[G]$ is closed under the κ^+-sequences. We show Q forces that j_U lifts to $j : V[G] \to M[j(G)]$ and $M[j(G)]$ is closed under the κ^+-sequences.

Let $H \subset Q$ be $V[G]$-generic and work in $V[G][H]$. Since $j_U(P_\kappa)$ has size $j_U(\kappa)$ and the $j_U(\kappa)$-cc, and $j_U(P)_{\kappa+1} = P_\kappa * j_U(\dot{Q})_{\kappa^{++}}$ has the κ^+-cc in M, $j_U(P_\kappa)/(G * H)$ has size at most $j_U(\kappa)$ and the $j_U(\kappa)$-cc in $M[G][H]$. Since $j_U(\kappa) > |j_U(P)_{\kappa+1}|$ is inaccessible in M, $j_U(\kappa)$ remains inaccessible in $M[G][H]$. Hence $M[G][H]$ has at most $|j_U(\kappa)| = \kappa^{++}$ maximal antichains of $j_U(P_\kappa)/(G * H)$. $M[G][H]$ is closed under the κ^+-sequences, since M is closed under the κ^+-sequences and $j_U(P)_{\kappa+1}$ has the κ^+-cc in V. Hence $j_U(P_\kappa)/(G * H)$ is weakly κ^{++}-closed, since it is the case in $M[G][H]$. Thus we have $K \subset j_U(P_\kappa)/(G * H)$ which is $M[G][H]$-generic. Since $j_U``G \subset G * H * K$, j_U lifts to $j : V[G] \to M[j(G)] = M[G][H][K]$. $M[j(G)]$ is closed under the κ^+-sequences, since the κ^+-sequences of ordinals are in $M[G][H] \subset M[j(G)]$.

Work in $V[G]$. Since Q has size κ^{++} and the κ^+-cc, Q forces that in $M[j(G)]$, $j(Q)$ has size $j(\kappa^{++})$ and the $j(\kappa^+)$-cc. Hence we can let Q force that $\langle \dot{A}_\gamma : \gamma < \kappa^{++} \rangle$ lists the maximal antichains of $j(Q)$ in $M[j(G)]$ with κ^{++} many repetitions, since Q forces $|(j(\kappa^{++})^{j(\kappa)})^{M[j(G)]}| = |j((\kappa^{++})^\kappa)| = |j(\kappa^{++})| = \kappa^{++}$. Let $\dot{H}$ be the Q-name for the generic filter and Q force that $\dot{H}_\gamma = \{p(\gamma) : p \in \dot{H}\}$ for $\gamma < \kappa^{++}$. Then Q forces $\bigcup j``\dot{H}_\gamma : j``\kappa^+ \times \kappa \to j``\kappa^+ \in j(R)$, since $Q$ forces $\bigcup \dot{H}_\gamma : \kappa^+ \times \kappa \to \kappa^+$, $(\bigcup \dot{H}_\gamma)``\{\alpha\} \times \kappa = \alpha \cup \kappa$ for $\alpha < \kappa^+$, $j``\dot{H}_\gamma = \{j``p(\gamma) : p \in \dot{H}\}$ and $M[j(G)]$ is closed under the κ^+-sequences. Recall $\tau_{\kappa^{++}}$ is the winning strategy for II in the game $\mathcal{G}_\kappa(Q)$.

We build $s \subset \kappa^{++}$ and Q-names $\{\dot{q}_\xi : \xi < \kappa^{++}\}$ inductively so that $\mathrm{supp}(\dot{X}_\gamma) \subset s \cap \gamma$ if $\gamma \in s$, and Q forces that in $M[j(G)]$, $\langle \dot{q}_\xi : \xi \leq 2\gamma \rangle$ is a partial run of the game $j(\mathcal{G}_\kappa(Q_{s\cap\gamma}))$ according to $j(\tau_{\kappa^{++}})$, $\dot{q}_{2\gamma+1} \in j(Q_{s\cap(\gamma+1)})$ extends some condition of $\dot{A}_\gamma$ if $\dot{A}_\gamma$ is a maximal antichain of $j(Q_{s\cap\gamma})$, and $\dot{q}_{2\gamma+1}(j(\gamma)) = \bigcup j``\dot{H}_\gamma$ if $\gamma \in s$. Assume we have $s \cap \gamma$ and $\{\dot{q}_\xi : \xi < 2\gamma\}$ with $\gamma < \kappa^{++}$.

We can let Q force $\dot{q}_{2\gamma} = j(\tau_{\kappa^{++}})(\langle \dot{q}_\xi : \xi < 2\gamma \rangle)$, since Q forces that in $M[j(G)]$, $\langle \dot{q}_\xi : \xi < 2\gamma \rangle$ is a partial run of $j(\mathcal{G}_\kappa(Q_{s\cap\gamma}))$. Let $\gamma \in s$ iff $\mathrm{supp}(\dot{X}_\gamma) \subset$

$s \cap \gamma$ and Q forces that $\dot{X}_\gamma \subset S^*$ and in $M[j(G)]$, $\dot{q}_{2\gamma}$ forces $j``\kappa^+ \in j(\dot{X}_\gamma)$. We have a Q-name $\dot{q}$ such that Q forces $\dot{q} \in j(Q_{s\cap\gamma})$ extends $\dot{q}_{2\gamma}$, and some condition of $\dot{A}_\gamma$ if $\dot{A}_\gamma$ is a maximal antichain of $j(Q_{s\cap\gamma})$. Let Q force that $\dot{q}_{2\gamma+1}$ is $\dot{q} \cup \{(j(\gamma), \bigcup j``\dot{H}_\gamma)\}$ if $\gamma \in s$, and $\dot{q}$ otherwise. In the former case, we show $Q$ forces that in $M[j(G)]$, $\dot{q}_{2\gamma}$ forces $\bigcup j``\dot{H}_\gamma \in j(R)(j(\dot{X}_\gamma))$, which implies Q forces $\dot{q}_{2\gamma+1} \in j(Q_{s\cap(\gamma+1)})$.

Let $H \subset Q$ be $V[G]$-generic and work in $V[G][H]$. Then $\bigcup j``H_\gamma = \bigcup\{j(p(\gamma)) : p \in H\} \in j(R)$. In $V[G]$, $Q_{s\cap\gamma}$ forces $\dot{X}_\gamma \subset S^*$, since $\operatorname{supp}(\dot{X}_\gamma) \subset s \cap \gamma$ and Q forces $\dot{X}_\gamma \subset S^*$. Hence it suffices to show in $M[j(G)]$, $(\dot{q}_{2\gamma})_H$ forces $j(S^*) \cap j(C)(\bigcup j``H_\gamma) \subset j(\dot{X}_\gamma)$, since $j(Q_{s\cap\gamma})$ forces $j(\dot{X}_\gamma) \subset j(S^*)$. Fix $z \in j(S^*) \cap j(C)(\bigcup j``H_\gamma)$. Then $z \subset j``\kappa^+$ by $\bigcup j``H_\gamma : j``\kappa^+ \times \kappa \to j``\kappa^+$. If $z = j``\kappa^+$, then in $M[j(G)]$, $(\dot{q}_{2\gamma})_H$ forces $z = j``\kappa^+ \in j(\dot{X}_\gamma)$ by $\gamma \in s$, as desired. If $z \subsetneq j``\kappa^+$, then $\operatorname{ot} z < (j``\kappa^+) \cap j(\kappa) = \kappa$, since $z \in j(S^*)$, and $j``\kappa^+ \in j(S^*)$ by $S^* \in U$. Hence $z = j``x = j(x)$ for some $x \in \mathcal{P}_\kappa\kappa^+$. Take $p \in H$ with $z \in j(S^*) \cap j(C)(j(p(\gamma)))$. Then $x \in S^* \cap C(p(\gamma))$ by $z = j(x)$. Hence in $V[G]$, $p|\gamma \in Q_\gamma$ forces $x \in \dot{X}_\gamma$, since $p|\gamma$ forces $p(\gamma) \in R(\dot{X}_\gamma)$. Thus in $V[G]$, $p|(s\cap\gamma) \in Q_{s\cap\gamma}$ forces $x \in \dot{X}_\gamma$, since $\operatorname{supp}(\dot{X}_\gamma) \subset s\cap\gamma$. If $r \in H \cap Q_{s\cap\gamma}$, then $(\dot{q}_{2\gamma})_H \le j(r)$, since $\operatorname{dom}(j(r)) = j``\operatorname{dom}(r)$ and $(\dot{q}_{2\gamma})_H(j(\beta)) \le (\dot{q}_{2\beta+1})_H(j(\beta)) = \bigcup j``H_\beta \le j(r(\beta)) = j(r)(j(\beta))$ for $\beta \in \operatorname{dom}(r)$. Hence in $M[j(G)]$, $(\dot{q}_{2\gamma})_H \le j(p|(s\cap\gamma))$ forces $z = j(x) \in j(\dot{X}_\gamma)$, since $p|(s\cap\gamma) \in H \cap Q_{s\cap\gamma}$, as desired.

Work in $V[G]$. Let Q force $\dot{K} = \{q \in j(Q_s) : \exists\xi < \kappa^{++}(\dot{q}_\xi \le q)\}$. By the κ^+-cc of Q_s, Q forces that in $M[j(G)]$, $j(Q_s)$ has the $j(\kappa^+)$-cc. Hence Q forces that a maximal antichain of $j(Q_s)$ in $M[j(G)]$ agrees with some $\dot{A}_\gamma$ which is a maximal antichain of $j(Q_{s\cap\gamma})$. Thus Q forces that $\dot{K}$ is $M[j(G)]$-generic. Q forces $j``(\dot{H} \cap Q_s) \subset \dot{K}$, since Q forces that $\dot{q}_{2\gamma} \le j(r)$ if $r \in \dot{H} \cap Q_{s\cap\gamma}$ for $\gamma < \kappa^{++}$. Hence Q forces that j lifts to $j^* : V[G][\dot{H} \cap Q_s] \to M[j(G)][\dot{K}]$.

Now let $H_s \subset Q_s$ be $V[G]$-generic and work in $V[G][H_s]$. Since Q_s is weakly κ-closed in $V[G]$, κ remains Mahlo. By the last paragraph, Q/H_s forces that j lifts to $j^* : V[G][H_s] \to M[j(G)][\dot{K}]$. By the κ^+-cc of Q and Q_s in $V[G]$, Q/H_s has the κ^+-cc. Let F be the filter $\{X \subset \mathcal{P}_\kappa\kappa^+ : \Vdash_{Q/H_s} j^*``\kappa^+ \in j^*(X)\}$. We have a complete embedding $X \mapsto \|j^*``\kappa^+ \in j^*(X)\|$ from F^+ into the completion of Q/H_s. Hence F is κ^+-saturated. Finally we claim F is generated by S^* over the club filter.

First $\{x \in \mathcal{P}_\kappa\kappa^+ : f``x^{<\omega} \subset x\} \in F$ for $f : (\kappa^+)^{<\omega} \to \kappa^+$, and $S^* \in F$. Next fix $X \in F$. We show $S^* - X$ is nonstationary. Take a Q_s-name $\dot{X} \in V[G]$ with $\dot{X}_{H_s} = S^* \cap X \in F$ and work in $V[G]$. We can assume Q forces that $\dot{X} \subset S^*$ and in $M[j(G)][\dot{K}]$, $j``\kappa^+ \in j(\dot{X})_{\dot{K}}$. Hence $Q$ forces that in $M[j(G)]$ some $\dot{q}_\xi$ forces $j``\kappa^+ \in j(\dot{X})$. The κ^+-cc of Q gives $\xi < \kappa^{++}$ such that Q forces that in $M[j(G)]$, $\dot{q}_\xi$ forces $j``\kappa^+ \in j(\dot{X})$. By the κ^+-cc of Q_s, $\operatorname{supp}(\dot{X}) \subset s \cap \beta$ for some $\beta < \kappa^{++}$. Hence we have $\gamma < \kappa^{++}$ such that $\dot{X}$ agrees with $\dot{X}_\gamma$, $\operatorname{supp}(\dot{X}) \subset s\cap\gamma$ and Q forces that in $M[j(G)]$, $\dot{q}_{2\gamma}$ forces $j``\kappa^+ \in j(\dot{X})$. Thus $\gamma \in s$. Now in

$V[G][H_s]$, $\bigcup\{p(\gamma) : p \in H_s\} : \kappa^+ \times \kappa \to \kappa^+$ generates a club set disjoint from $S^* - (\dot{X}_\gamma)_{H_s} = S^* - \dot{X}_{H_s} = S^* - X$, as desired. □

5. Remarks

In the final model $V[G][H_s]$, F^+ is κ-Baire, since it is completely embeddable into a κ-Baire poset. For F a κ^+-saturated normal filter on $\mathcal{P}_\kappa\kappa^+$ in general, $S_\kappa \in F$ if F^+ is ω_1-Baire.

Finally we extend Solovay's theorem as in [6] by an elementary proof:

Proposition. *A stationary subset of $\mathcal{P}_\kappa\lambda$ splits into κ stationary sets.*

Proof. Let $S \subset \mathcal{P}_\kappa\lambda$ be stationary. We can assume $x \cap \kappa \in \kappa$ for $x \in S$. Then $T = \{y \in S : \exists g : y^{<\omega} \to y(S \cap \{x \subset y : \operatorname{ot} x < \operatorname{ot} y \wedge g``x^{<\omega} \subset x\} = \emptyset)\}$ is stationary: Fix $f : \lambda^{<\omega} \to \lambda$. Take $y \in S$ closed under f so that $\operatorname{ot} y$ is as small as possible. Then $f|y^{<\omega}$ witnesses $y \in T$, as desired. For $y \in T$ take a witness $g_y : y^{<\omega} \to y$. Set $z_a = \{\gamma < \lambda : \{y \in T : g_y(a) = \gamma\}$ is stationary$\}$ for $a \in \lambda^{<\omega}$. Then S splits into stationary sets $\{y \in T : g_y(a) = \gamma\}$ for $\gamma \in z_a$. We claim $|z_a| \geq \kappa$ for some $a \in \lambda^{<\omega}$.

Suppose to the contrary $z_a \in \mathcal{P}_\kappa\lambda$ for $a \in \lambda^{<\omega}$. Take $C_{a,\gamma} \subset \mathcal{P}_\kappa\lambda$ club and disjoint from $\{y \in T : g_y(a) = \gamma\}$ for $a \in \lambda^{<\omega}$ and $\gamma \in \lambda - z_a$. Then $C = \{x \in \mathcal{P}_\kappa\lambda : \forall a \in x^{<\omega}(z_a \subset x \wedge \forall \gamma \in x - z_a(x \in C_{a,\gamma}))\}$ is club. Take $x \subset y$ from $T \cap C$ so that $\operatorname{ot} x < \operatorname{ot} y$. We have $a \in x^{<\omega}$ with $g_y(a) \notin x$, since $x \subset y$ are in T and $\operatorname{ot} x < \operatorname{ot} y$. Then $\gamma = g_y(a) \notin z_a$, since $z_a \subset x$ by $x \in C$ and $a \in x^{<\omega}$. Hence $y \in C_{a,\gamma}$ by $y \in C$, $a \in y^{<\omega}$ and $\gamma \in y$, which contradicts $y \in T$ and $g_y(a) = \gamma$, as desired. □

References

[1] Y. Abe, *Saturation of fundamental ideals on $\mathcal{P}_\kappa\lambda$*, J. Math. Soc. Japan **48** (1996), 511–524.

[2] J. Baumgartner, *Iterated forcing*, Surveys in set theory, 1–59, London Math. Soc. Lecture Note Ser., 87, Cambridge Univ. Press, Cambridge, 1983.

[3] ———, *On the size of closed unbounded sets*, Ann. Pure Appl. Logic **54** (1991), 195–227.

[4] J. Baumgartner, A. Taylor, *Saturation properties of ideals in generic extensions. I*, Trans. Amer. Math. Soc. **270** (1982), 557–574.

[5] H.-D. Donder, P. Matet, *Two cardinal versions of diamond*, Israel J. Math. **83** (1993), 1–43.

[6] M. Gitik, *Nonsplitting subset of $\mathcal{P}_\kappa(\kappa^+)$*, J. Symbolic Logic **50** (1985), 881–894.

[7] T. Jech, S. Shelah, *On reflection of stationary sets in $\mathcal{P}_\kappa\lambda$*, Trans. Amer. Math. Soc. **352** (2000), 2507–2515.

[8] T. Jech, W. Woodin, *Saturation of the closed unbounded filter on the set of regular cardinals*, Trans. Amer. Math. Soc. **292** (1985), 345–356.

[9] A. Kanamori, *The higher infinite*, Large cardinals in set theory from their beginnings. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994.

[10] ———, Handwritten notes, 1997.

[11] M. Magidor, *Representing sets of ordinals as countable unions of sets in the core model*, Trans. Amer. Math. Soc. **317** (1990), 91–126.

[12] Y. Matsubara, *Consistency of Menas' conjecture*, J. Math. Soc. Japan **42** (1990), 259–263.

[13] T. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic **7** (1974/75), 327–359.

[14] M. Shioya, *Splitting $\mathcal{P}_\kappa\lambda$ into maximally many stationary sets*, Israel J. Math. **114** (1999), 347–357.

[15] R. Solovay, *Real-valued measurable cardinals*, Axiomatic set theory. Proc. Sympos. Pure Math., Vol. XIII, Part I, 397–428. Amer. Math. Soc., Providence, RI, 1971.

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