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STANLEY-REISNER RINGS, SHEAVES, AND POINCARÉ-VERDIER DUALITY

Kohji Yanagawa

Abstract. A few years ago, I defined a *squarefree module* over a polynomial ring $S = k[x_1, \dots, x_n]$ generalizing the Stanley-Reisner ring $k[\Delta] = S/I_\Delta$ of a simplicial complex $\Delta \subset 2^{\{1,\cdots,n\}}$. This notion is very useful in the Stanley-Reisner ring theory. In this paper, from a squarefree S-module M, we construct the k-sheaf M^+ on an $(n-1)$ simplex B which is the geometric realization of $2^{\{1,\cdots,n\}}$. For example, $k[\Delta]^+$ is (the direct image to B of) the constant sheaf on the geometric realization $|\Delta| \subset B$. We have $H^i(B, M^+) \cong [H^{i+1}_{\mathfrak{m}}(M)]_0$ for all $i \geq 1$. The Poincaré-Verdier duality for sheaves M^+ on B corresponds to the local duality for squarefree modules over S. For example, if $|\Delta|$ is a manifold, then $k[\Delta]$ is a Buchsbaum ring and its canonical module $K_{k[\Delta]}$ is a squarefree module which gives the orientation sheaf of $|\Delta|$ with the coefficients in k.

1. Introduction

This paper presents a new geometric aspect of combinatorial commutative algebra on *normal semigroup rings.* But, in this introduction, we restrict ourselves to the polynomial ring case for simplicity. In this paper, we use the theory of sheaves on a locally compact topological space. For this theory, consult [I]. Basically, we use the same notation as [I] here.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k, and Δ a simplicial complex whose vertex set is a subset of $[n] := \{1, \dots, n\}$. Then the Stanley-Reisner ring $k[\Delta] := S/(\prod_{i \in F} x_i \mid F \subset [n] \text{ with } F \not\in \Delta)$ of Δ reflects topological properties of the geometric realization $|\Delta|$, and has been studied since 1970's (see [BH, Sta]). In [Y1], the author introduced the notion of a *squarefree module* which generalizes Stanley-Reisner rings. This notion allows us to apply homological methods (e.g., derived categories) to the theory of Stanley-Reisner rings more systematically. The purpose of this paper is to give a geometric meaning of squarefree modules.

Let B be an $(n-1)$-simplex which is the geometric realization of $2^{[n]}$. We construct the k-sheaf M^+ on B from a squarefree module M. For example, $k[\Delta]^+ \cong j_*\underline{k}_{|\Delta|}$, where $\underline{k}_{|\Delta|}$ is the constant sheaf on $|\Delta|$ and $j : |\Delta| \to B$ is the embedding map. Let Sq be the category of squarefree modules, and $\mathrm{Sh}(B)$ the category of k-sheaves on B. Then the functor $(-)^+ : \mathrm{Sq} \to \mathrm{Sh}(B)$ is exact.

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If M is a squarefree S-module, Theorem 3.3 gives an isomorphism

$$H^i(B, M^+) \cong [H^{i+1}_{\mathfrak{m}}(M)]_0 \quad \text{for all } i \geq 1,$$

and an exact sequence

$$0 \to [H^0_{\mathfrak{m}}(M)]_0 \to M_0 \to H^0(B, M^+) \to [H^1_{\mathfrak{m}}(M)]_0 \to 0,$$

where $H^i_{\mathfrak{m}}(-)$ stands for the local cohomology module with support in the maximal ideal $\mathfrak{m} := (x_1, \dots, x_n)$. So our functor $(-)^+$ is somewhat analogous to "Proj" of the scheme theory.

If $\mathbf{a} = (a_1, \dots, a_n) \notin -\mathbb{N}^n$, it is well-known that $[H^i_{\mathfrak{m}}(M)]_{\mathbf{a}} = 0$. If $0 \neq \mathbf{a} \in -\mathbb{N}^n$, $[H^i_{\mathfrak{m}}(M)]_{\mathbf{a}}$ is isomorphic to the cohomology with compact support $H^{i-1}_c(U_F, j^*M^+)$, where U_F is the open subset determined by $F := \{i \in [n] \mid a_i < 0\}$ (see Theorem 3.5 for detail) and $j : U_F \to B$ is the embedding map. These results generalize a well-known formula of Hochster on $H^i_{\mathfrak{m}}(k[\Delta])$.

Let $D^b(\mathrm{Sq})$ be the bounded derived category of Sq and $\omega^\bullet_S \in D^b(\mathrm{Sq})$ an injective resolution of $K_S[n-1]$. Set $\omega^\bullet_{k[\Delta]} := \mathrm{Hom}^\bullet_{k[\Delta]}(k[\Delta], \omega^\bullet_S)$. Then $\omega^\bullet_{k[\Delta]}$ is a complex of squarefree $k[\Delta]$-modules, and isomorphic to a (non-normalized) $\mathbb{Z}^n$-graded dualizing complex of $k[\Delta]$ in the derived category. Let $\mathcal{D}^\bullet_{|\Delta|}$ be a dualizing complex of the topological space $|\Delta|$ with the coefficients in k. Corollary 4.3 states that $\mathcal{D}^\bullet_{|\Delta|} \cong j^*(\omega^\bullet_{k[\Delta]})^+$ in $D^b(\mathrm{Sh}(|\Delta|))$, where $j : |\Delta| \to B$ is the embedding map. (Since the functors $(-)^+ : \mathrm{Sq} \to \mathrm{Sh}(B)$ and $j^* : \mathrm{Sh}(B) \to \mathrm{Sh}(|\Delta|)$ are exact, we have the functor $j^*(-)^+ : D^b(\mathrm{Sq}) \to D^b(\mathrm{Sh}(|\Delta|))$.) Moreover, if each component M^i of $M^\bullet \in D^b(\mathrm{Sq})$ is a $k[\Delta]$-module, we have

$$R\mathcal{H}om_{\mathrm{Sh}(|\Delta|)}(j^*(M^\bullet)^+, \mathcal{D}^\bullet_{|\Delta|}) \cong j^*(R\,\mathrm{Hom}_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})^+)$$

in $D^b(\mathrm{Sh}(|\Delta|))$. See Theorem 4.2. If further $[H^i(M^\bullet)]_0 = 0$ for all i (note that the sheaf M^+ does not reflect the degree 0 component M_0), we have

$$\mathrm{Ext}^i_{\mathrm{Sh}(|\Delta|)}(j^*(M^\bullet)^+, \mathcal{D}^\bullet_{|\Delta|}) \cong [\mathrm{Ext}^i_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})]_0.$$

So the Poincaré-Verdier duality for $|\Delta|$ corresponds to the local duality for $k[\Delta]$ in our context. For example, if $|\Delta|$ is a manifold, then $k[\Delta]$ is a Buchsbaum ring and the canonical module $K_{k[\Delta]}$ of $k[\Delta]$ is a squarefree module which gives the orientation sheaf of $|\Delta|$ with the coefficients in k. So, the well-known duality between $H^i_{\mathfrak{m}}(k[\Delta])$ and $H^j_{\mathfrak{m}}(K_{k[\Delta]})$ ([SV, II. Theorem 4.9]) corresponds to the Poincaré duality for $|\Delta|$.

2. Preliminaries

Let $Q \subset \mathbb{N}^n$ be an affine semigroup (i.e., a finitely generated sub-semigroup containing 0), and $k[Q] = k[\,x^{\mathbf{a}} \mid \mathbf{a} \in Q\,] \subset S := k[\,x_1, \dots, x_n\,]$ the semigroup ring of Q over a field k. Here $x^{\mathbf{a}}$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ means the monomial $\prod x_i^{a_i} \in S$. We always assume that Q is saturated (i.e., if $\mathbf{a} \in \mathbb{N}^n$ satisfies $m\mathbf{a} \in Q$ for a positive integer m, then $\mathbf{a} \in Q$) and $\mathbb{Z}Q = \mathbb{Z}^n$. Thus $k[Q]$ is a normal Cohen-Macaulay $\mathbb{Z}^n$-graded ring of dimension n with the graded maximal ideal $\mathfrak{m} = (\, x^{\mathbf{a}} \mid 0 \neq \mathbf{a} \in Q \,)$. For basic properties of ($\mathbb{Z}^n$-graded modules over) $k[Q]$ and the related notions from convex geometry, see [BH, GW].

Consider the polyhedral cone $\mathbb{R}_{\geq 0}Q \subset \mathbb{R}^n$ spanned by Q ($\subset \mathbb{N}^n \subset \mathbb{R}^n$). Let L be the set of non-empty faces of $\mathbb{R}_{\geq 0}Q$. The order by inclusion makes L a finite poset. If $p \in \mathbb{R}_{\geq 0}Q$, there is a unique face $F \in L$ such that the relative interior rel-int(F) of F contains p. We call this F the *support* of p, and denote it by supp(p).

If $\mathbb{R}_{\geq 0}Q$ is spanned by n vectors as a polyhedral cone, we say $k[Q]$ is *simplicial*. In this case, L is isomorphic to the boolean lattice $2^{[n]}$ as a poset. For example, the polynomial ring $k[\mathbb{N}^n] = k[x_1, \dots, x_n]$ is a simplicial semigroup ring.

Let H be a hyperplane of $\mathbb{R}^n$ which intersects the cone $\mathbb{R}_{\geq 0}Q$ transversally. Consider the $(n-1)$-dimensional polytope $B := H \cap \mathbb{R}_{\geq 0}Q$. Of course, B is homeomorphic to a closed ball of dimension $n-1$. If $k[Q]$ is simplicial, then B is a simplex. For a face $F \in L$, set $|F| := F \cap H$ to be a face of B, and $|F|^\circ :=$ rel-int$(|F|)$ its relative interior. If $\Delta \subset L$ is an *order ideal* (i.e., $F \in \Delta$, $G \in L$ and $F \supset G \Rightarrow G \in \Delta$), then $|\Delta| := \coprod_{F \in \Delta} |F|^\circ$ is a finite regular cell complex.

For a $\mathbb{Z}^n$-graded $k[Q]$-module M and $\mathbf{a} \in \mathbb{Z}^n$, $M_{\mathbf{a}}$ denotes the degree $\mathbf{a}$ component of M. Let *Mod be the category of $\mathbb{Z}^n$-graded $k[Q]$-modules. Here a morphism f in *Mod is a $k[Q]$-homomorphism $f : M \to N$ with $f(M_{\mathbf{a}}) \subset N_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n$.

We assign an order ideal $\Delta \subset L$ to the ideal

$$I_\Delta := (\, x^{\mathbf{a}} \mid \mathbf{a} \in Q \text{ and } \operatorname{supp}(\mathbf{a}) \notin \Delta \,)$$

of $k[Q]$. Set $k[\Delta] := k[Q]/I_\Delta$. Clearly,

$$k[\Delta]_{\mathbf{a}} \cong \begin{cases} k & \text{if } \mathbf{a} \in Q \text{ and } \operatorname{supp}(\mathbf{a}) \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\Delta = L$ (resp. $\Delta = \emptyset$), then $I_\Delta = 0$ (resp. $I_\Delta = k[Q]$) and $k[\Delta] = k[Q]$ (resp. $k[\Delta] = 0$). If $\Delta \neq \emptyset$ or $\{\, \{0\} \,\}$, then $\dim k[\Delta] = \dim |\Delta| + 1$, where $\dim |\Delta|$ is the dimension as a cell complex. When $k[Q]$ is a polynomial ring, $k[\Delta]$ is nothing other than the Stanley-Reisner ring of a simplicial complex Δ. (If $k[Q]$ is simplicial, Δ can be seen as a simplicial complex, and $|\Delta| = \coprod_{F \in \Delta} |F|^\circ$ is homeomorphic to the geometric realization of Δ as a simplicial complex.)

We now recall the definition of *squarefree* $k[\Delta]$-modules.

Definition 2.1 ([Y1, Y2])**.** A $\mathbb{Z}^n$-graded $k[Q]$-module $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ is *squarefree* if the following two conditions are satisfied:

(1) M is finitely generated and Q-graded (i.e., $M_{\mathbf{a}} = 0$ for all $\mathbf{a} \notin Q$).
(2) The multiplication map $M_{\mathbf{a}} \ni y \mapsto x^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in Q$ with $\operatorname{supp}(\mathbf{a}+\mathbf{b}) = \operatorname{supp}(\mathbf{a})$.

The $\mathbb{Z}^n$-graded canonical module $K_{k[Q]}$ of $k[Q]$ is a squarefree module. In fact, $K_{k[Q]}$ is isomorphic to the ideal $(\, x^{\mathbf{a}} \mid \mathbf{a} \in Q \text{ with } \operatorname{supp}(\mathbf{a}) = \mathbb{R}_{\geq 0}Q \,)$ of $k[Q]$. The quotient rings $k[\Delta]$ (in particular, $k[Q]$ itself) are also squarefree.

If M is squarefree, then $M_{\mathbf{a}} \cong M_{\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in Q$ with $\operatorname{supp}(\mathbf{a}) = \operatorname{supp}(\mathbf{b})$. In fact, since $\operatorname{supp}(\mathbf{a}) = \operatorname{supp}(\mathbf{a}+\mathbf{b}) = \operatorname{supp}(\mathbf{b})$, we have $M_{\mathbf{a}} \cong M_{\mathbf{a}+\mathbf{b}} \cong M_{\mathbf{b}}$.

Denote the full subcategory of ${}^*\mathrm{Mod}$ consisting of all squarefree $k[Q]$-modules by Sq. For $M \in {}^*\mathrm{Mod}$ and $\mathbf{a} \in \mathbb{Z}^n$, $M(\mathbf{a})$ denotes the shifted module of M with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. If $M, N \in {}^*\mathrm{Mod}$ and M is finitely generated, then $\operatorname{Hom}_{k[Q]}(M, N)$ has the natural $\mathbb{Z}^n$-grading with

$$[\operatorname{Hom}_{k[Q]}(M,N)]_{\mathbf{a}} = \operatorname{Hom}_{{}^*\mathrm{Mod}}(M, N(\mathbf{a})).$$

Lemma 2.2 ([Y2, §4])**.** (1) Sq *is a thick abelian subcategory of* ${}^*\mathrm{Mod}$ *(i.e., closed under kernels, cokernels, and extensions in* ${}^*\mathrm{Mod}$*).*

(2) Sq *is an abelian category with enough projectives and injectives. An indecomposable projective (resp. injective) object in* Sq *is isomorphic to*

$$J_F := (x^{\mathbf{a}} \mid \mathbf{a} \in Q \textit{ with } \operatorname{supp}(\mathbf{a}) \supset F) \subset k[Q]$$

$$\left(\textit{resp. } k[F] := k[Q]/(x^{\mathbf{a}} \mid \mathbf{a} \in Q \textit{ with } \operatorname{supp}(\mathbf{a}) \not\subset F) \right)$$

for some $F \in L$. *And both* $\operatorname{proj.dim}_{\mathrm{Sq}} M$ *and* $\operatorname{inj.dim}_{\mathrm{Sq}} M$ *are at most* n *for all* $M \in \mathrm{Sq}$.

(3) *The projective object* J_F *is a Cohen-Macaulay* $k[Q]$*-module of dimension* n*. And*

$$\operatorname{Hom}_{k[Q]}(J_F, K_{k[Q]}) \cong (\, x^{\mathbf{a}} \mid \mathbf{a} \in Q \textit{ such that } \operatorname{supp}(\mathbf{a}) \vee F = \mathbb{R}_{\geq 0}Q \,),$$

where $\operatorname{supp}(\mathbf{a}) \vee F \in L$ *is the smallest face containing both* $\operatorname{supp}(\mathbf{a})$ *and* F*. In particular,* $\operatorname{Hom}_{k[Q]}(J_F, K_{k[Q]})$ *is squarefree again.*

For derived categories, we use the same notation as [H] (unless otherwise specified). In particular, for a module M and an integer i, $M[i]$ means the complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M at the $(-i)^{\text{th}}$ place.

Lemma 2.3. *We have the canonical category equivalence* $D^b(\mathrm{Sq}) \cong D^b_{\mathrm{Sq}}({}^*\mathrm{Mod})$*, and* $D^b(\mathrm{Sq})$ *can be seen as a full subcategory of* $D^b({}^*\mathrm{Mod})$*.*

Proof. Let ${}^*\mathrm{Mod}_Q$ be the full subcategory of ${}^*\mathrm{Mod}$ consisting of finitely generated Q-graded modules. Then ${}^*\mathrm{Mod}_Q$ is a thick abelian subcategory of ${}^*\mathrm{Mod}$. Moreover, ${}^*\mathrm{Mod}_Q$ has enough projectives, and projective objects $k[Q](-\mathbf{a})$ with $\mathbf{a} \in Q$ are also projective in ${}^*\mathrm{Mod}$. Thus $D^b_{{}^*\mathrm{Mod}_Q}({}^*\mathrm{Mod}) \cong D^b({}^*\mathrm{Mod}_Q)$ and $D^b_{{}^*\mathrm{Mod}_Q}({}^*\mathrm{Mod})$ is a full subcategory of $D^b({}^*\mathrm{Mod})$. On the other hand, Sq is a thick abelian subcategory of ${}^*\mathrm{Mod}_Q$, and an injective object $k[F]$ of Sq is also injective in ${}^*\mathrm{Mod}_Q$ by [M, Remark 2.5]. So $D^b(\mathrm{Sq}) \cong D^b_{\mathrm{Sq}}({}^*\mathrm{Mod}_Q)$, and $D^b_{\mathrm{Sq}}({}^*\mathrm{Mod}_Q)$ is a full subcategory of $D^b({}^*\mathrm{Mod}_Q)$, which can be seen as a full subcategory of $D^b({}^*\mathrm{Mod})$. □

Next we will study $R\operatorname{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})$ for a complex $M^\bullet \in D^b({}^*\mathrm{Mod})$. Here $R\operatorname{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})$ is the "$R\operatorname{Hom}$" as complexes of (non-graded) $k[Q]$-modules. But if each $H^i(M^\bullet)$ is finitely generated, $R\operatorname{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})$ has a natural $\mathbb{Z}^n$-grading, and defines an object in $D^b({}^*\mathrm{Mod})$. In fact, if $P^\bullet$ is a $\mathbb{Z}^n$-graded finite free resolution of $M^\bullet$, then

$$R\operatorname{Hom}_{k[Q]}(M^\bullet, K_{k[Q]}) \cong \operatorname{Hom}^\bullet_{k[Q]}(P^\bullet, K_{k[Q]})$$

and each $\operatorname{Hom}^i_{k[Q]}(P^\bullet, K_{k[Q]})\,(=\operatorname{Hom}_{k[Q]}(P^{-i}, K_{k[Q]}))$ has the $\mathbb{Z}^n$-grading. We can also define $R\operatorname{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})$ using a $\mathbb{Z}^n$-graded injective resolution $I^\bullet \in D^b({}^*\mathrm{Mod})$ of $K_{k[Q]}$, but we get the same $\mathbb{Z}^n$-grading.

Lemma 2.4. *If $M^\bullet \in D^b_{\mathrm{Sq}}({}^*\mathrm{Mod})$, then $R\operatorname{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})$ is in $D^b_{\mathrm{Sq}}({}^*\mathrm{Mod})$ too. That is, $\operatorname{Ext}^i_{k[Q]}(M^\bullet, K_{k[Q]})$ is squarefree for all i.*

Proof. By Lemma 2.3, we may assume that $M^\bullet \in D^b(\mathrm{Sq})$. Then we have a projective resolution $P^\bullet \in D^b(\mathrm{Sq})$ of $M^\bullet$. By Lemma 2.2 (3), $\operatorname{Ext}^i_{k[Q]}(P^j, K_{k[Q]}) = 0$ for all $i \neq 0$ and all j. Hence we have

$$R\operatorname{Hom}_{k[Q]}(M^\bullet, K_{k[Q]}) \cong \operatorname{Hom}^\bullet_{k[Q]}(P^\bullet, K_{k[Q]}).$$

But each $\operatorname{Hom}_{k[Q]}(P^j, K_{k[Q]})$ is squarefree by Lemma 2.2 (3). □

Take some $\mathbf{a}(F) \in Q \cap \text{rel-int}(F)$ for each $F \in L$. For a squarefree module M, set $M_F := M_{\mathbf{a}(F)}$. If $F, G \in L$ and $G \supset F$, [Y2, Theorem 3.3] gives a k-linear map $\varphi^M_{G,F} : M_F \to M_G$. These maps satisfy $\varphi^M_{F,F} = \mathrm{Id}$ and $\varphi^M_{G,F} \circ \varphi^M_{F,E} = \varphi^M_{G,E}$ for all $G \supset F \supset E$. For $F \in L$, we define the complex $C^\bullet_F(M) : 0 \to C^0_F \to C^1_F \to \cdots \to C^n_F \to 0$ of k-vector spaces by

$$C^i_F = \bigoplus_{\substack{G \in L,\, G \supset F \\ \dim G = i}} M_G,$$

and the differential

$$d : C^i_F \supset M_G \ni y \longmapsto \sum_{\substack{G' \in L,\, G' \supset G \\ \dim G' = i+1}} \varepsilon(G', G) \cdot \varphi^M_{G',G}(y) \;\in \bigoplus_{\substack{G' \in L,\, G' \supset G \\ \dim G' = i+1}} M_{G'} \subset C^{i+1}_F.$$

Here ε is an incidence function on the cell complex $B = \coprod_{F \in L} |F|^\circ$. The complex $C^\bullet_F(M)$ does not depend on the particular choice of $\mathbf{a}(F)$'s up to isomorphism. By the computation of a Čech complex with supports in $\mathfrak{m}$, we have the following.

Lemma 2.5 ([Y2, Theorem 3.10])**.** *Let the notation be as above. If $\mathbf{a} \notin Q$, then $[H^i_{\mathfrak{m}}(M)]_{-\mathbf{a}} = 0$. If $\mathbf{a} \in Q$ and $\operatorname{supp}(\mathbf{a}) = F$, then $[H^i_{\mathfrak{m}}(M)]_{-\mathbf{a}} \cong H^i(C^\bullet_F(M))$.*

3. Sheaves associated with squarefree modules

We keep the same notation as above. For a squarefree module M, set

$$\operatorname{Spé}(M) := \coprod_{F \in L} |F|^\circ \times M_F.$$

Let $\pi : \operatorname{Spé}(M) \to B$ be the projection map which sends $(p, m) \in |F|^\circ \times M_F \subset \operatorname{Spé}(M)$ to $p \in |F|^\circ \subset B$. For an open subset $U \subset B$ and a map $s : U \to \operatorname{Spé}(M)$, we will consider the following conditions:

($*$) $\pi \circ s = \mathrm{Id}_U$ and $s_q = \varphi^M_{G,F}(s_p)$ for all $p, q \in U$ such that $F := \operatorname{supp}(p)$ is contained in $G := \operatorname{supp}(q)$. Here s_p (resp. s_q) is the element of M_F (resp. M_G) with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).

($**$) There is an open covering $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ such that the restriction of s to U_λ satisfies ($*$) for all $\lambda \in \Lambda$.

Now we define the k-sheaf associated to M on B, denoted by M^+, as follows. The sections $M^+(U)$ of M^+ over an open set U is

$$\{\, s \mid s : U \to \operatorname{Spé}(M) \text{ is a map satisfying } (**) \,\}$$

and the restriction map $M^+(U) \to M^+(V)$ is the natural one. (That M^+ is actually a sheaf is obvious.)

We say an open set U of B is *neat* with respect to a face $F \in L$, if U itself and $U \cap |G|^\circ$ are connected for all $G \in L$ with $G \supset F$, and $q \in U$ implies $\operatorname{supp}(q) \supset F$. For example, the open set $U_F := \coprod_{G \supset F} |G|^\circ$ is neat with respect to F. For $x \in |F|^\circ$ and sufficiently small $\varepsilon > 0$, $U_\varepsilon(x) := \{\, y \in B \mid d(x, y) < \varepsilon \,\}$ is also neat with respect to F, where $d(-,-)$ stands for the usual metric of $\mathbb{R}^n$ ($\supset B$). We can easily check the following.

(i) Assume that $U \cap |F|^\circ$ is connected, and let $s \in M^+(U)$ be a section. Then there is some $y \in M_F$ such that $s(p) = (p, y)$ for all $p \in U \cap |F|^\circ$.

(ii) Assume that U is neat with respect to F. For any $y \in M_F$, the map $s_y : U \to \operatorname{Spé}(M)$ defined by $(U \cap |G|^\circ) \ni p \mapsto (p, \varphi_{G,F}(y))$ satisfies ($*$). In particular, $s_y \in M^+(U)$.

(iii) If U is neat with respect to F, any section $s \in M^+(U)$ coincides with s_y of (ii) for some $y \in M_F$.

Hence, if U is neat with respect to F, then $M^+(U) \cong M_F$. Note that the set of neat open sets is an open base of B. Thus, for a point $p \in |F|^\circ$, the stalk $(M^+)_p$ of M^+ at p is isomorphic to M_F. So $\operatorname{Spé}(M)$ is the etale space of the sheaf M^+.

Let $\Psi \subset L$ be an order filter of the poset L, that is, $F \in \Psi$, $G \in L$, and $G \supset F$ imply $G \in \Psi$. Then $U_\Psi := \coprod_{F \in \Psi} |F|^\circ$ is an open subset of B. If M is a squarefree module, then the submodule

$$M_\Psi := \bigoplus_{\mathbf{a} \in Q,\, \operatorname{supp}_+(\mathbf{a}) \in \Psi} M_{\mathbf{a}}$$

is also squarefree. Moreover, we have the following.

Lemma 3.1. *The sheaf* $(M_\Psi)^+$ *is isomorphic to* $j_! j^* M^+$, *where* $j : U_\Psi \to B$ *is the embedding map.*

Proof. Straightforward. □

Example 3.2. (1) Let $\Delta \subset L$ be an order ideal, and j the embedding map from the closed subset $|\Delta| = \coprod_{F \in \Delta} |F|$ to B. Then the sheaf $k[\Delta]^+$ is isomorphic to $j_* \underline{k}_{|\Delta|}$, where $\underline{k}_{|\Delta|}$ is the constant sheaf on $|\Delta|$.

(2) Let J_F be the projective object in Sq associated with a face $F \in L$. Then the sheaf $(J_F)^+$ is isomorphic to $j_! \underline{k}_{U_F}$, where j is the embedding map from the open set $U_F = \coprod_{G \in L,\, G \supset F} |G|^\circ$ to B. Note that

$$U_F \cong \begin{cases} \mathbb{R}^{n-1}, & \text{if } F = \mathbb{R}_{\geq 0}Q, \\ \mathbb{R}^{n-1}_+ := \{\, (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} \mid y_{n-1} \geq 0 \,\}, & \text{if } F \neq \mathbb{R}_{\geq 0}Q, \{0\}, \\ B^{n-1} := \{\, (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1} \mid \sum_{i=1}^{n-1} y_i^2 \leq 1 \,\}, & \text{if } F = \{0\}. \end{cases}$$

(3) Let $\Delta, \Sigma \subset L$ be order ideals with $\Delta \supset \Sigma$. We have $I_\Delta \subset I_\Sigma$. Set $I_{\Delta/\Sigma} := I_\Sigma / I_\Delta$ (see [Sta, III.7]). If $\Sigma = \emptyset$ (resp. $\Delta = L$), then $I_{\Delta/\Sigma} = k[\Delta]$ (resp. $I_{\Delta/\Sigma} = I_\Sigma$). It is easy to see that $I_{\Delta/\Sigma}$ is a squarefree module with $(I_{\Delta/\Sigma})^+ \cong j_! \, \underline{k}_{|\Delta| - |\Sigma|}$, where j is the embedding map from the locally closed subset $|\Delta| - |\Sigma|$ to B.

For a topological space X, $\mathrm{Sh}(X)$ denotes the category of k-sheaves on X (i.e., the category of $\underline{k}_X$-modules).

If M is a squarefree module, $M_{>0}$ denotes the submodule $\bigoplus_{\mathbf{a} \in Q \setminus \{0\}} M_\mathbf{a}$ of M. Then $M_{>0}$ is squarefree again, and $M^+ \cong (M_{>0})^+$. For a complex $0 \to L \to M \to N \to 0$ of squarefree modules, the complex of sheaves $0 \to L^+ \to M^+ \to N^+ \to 0$ is exact if and only if $0 \to L_F \to M_F \to N_F \to 0$ is exact for all $\{0\} \neq F \in L$. Hence the functor $(-)^+ : \mathrm{Sq} \to \mathrm{Sh}(B)$ is exact. But this functor is neither full nor faithful. The degree 0 component M_0 causes this problem. Let Sq_+ be the full subcategory of Sq consisting of all M with $M_0 = 0$. It is easy to see that the functor $(-)^+ : \mathrm{Sq}_+ \to \mathrm{Sh}(B)$ is fully faithful.

Theorem 3.3. *If* M *is a squarefree* $k[Q]$*-module, we have an isomorphism*

$$H^i(B, M^+) \cong [H^{i+1}_\mathfrak{m}(M)]_0 \quad \text{for all } i \geq 1,$$

and an exact sequence

$$0 \to [H^0_\mathfrak{m}(M)]_0 \to M_0 \to H^0(B, M^+) \to [H^1_\mathfrak{m}(M)]_0 \to 0. \tag{3.1}$$

In particular, if $M \in \mathrm{Sq}_+$, *then* $H^i(B, M^+) \cong [H^{i+1}_\mathfrak{m}(M)]_0$ *for all* $i \geq 0$.

Proof. As usual, let $\Gamma_\mathfrak{m} : {}^*\mathrm{Mod} \to {}^*\mathrm{Mod}$ be the functor defined by $\Gamma_\mathfrak{m}(N) := \{y \in N \mid \mathfrak{m}^l y = 0 \text{ for } l \gg 0\}$, and $\Gamma(B, -) : \mathrm{Sh}(B) \to \mathrm{vect}_k$ the global sections functor.

Let $I^\bullet$ (resp. $\check{I}^\bullet$) be a minimal injective resolution of M in Sq (resp. in ${}^*\mathrm{Mod}$), and consider the exact sequence

$$0 \to \Gamma_\mathfrak{m}(I^\bullet) \to I^\bullet \to I^\bullet / \Gamma_\mathfrak{m}(I^\bullet) \to 0 \tag{3.2}$$

of cochain complexes. Put $J^\bullet := I^\bullet/\Gamma_{\mathfrak{m}}(I^\bullet)$. Each component of $J^\bullet$ is a direct sum of copies of $k[F]$ for various $\{0\} \neq F \in L$. Since $k[F]^+$ is the constant sheaf on $|F|$ which is homeomorphic to a closed ball, we have $H^i(B, k[F]^+) = H^i(|F|; k) = 0$ for all $i \geq 1$. Hence $(J^\bullet)^+ \, (\cong (I^\bullet)^+)$ gives a $\Gamma(B, -)$-acyclic resolution of M^+. It is easy to see that $[J^\bullet]_0 \cong \Gamma(B, (J^\bullet)^+)$. By [M, Theorem 2.4], $I^\bullet$ coincides with the Q-graded part $\bigoplus_{\mathbf{a} \in Q} [\check{I}^\bullet]_{\mathbf{a}}$ of $\check{I}^\bullet$. Thus we have $[H^i(\Gamma_{\mathfrak{m}}(I^\bullet))]_0 = [H^i(\Gamma_{\mathfrak{m}}(\check{I}^\bullet))]_0 = [H^i_{\mathfrak{m}}(M)]_0$. So the first and the second assertions follow from (3.2), since $[H^0(I^\bullet)]_0 \cong M_0$ and $H^i(I^\bullet) = 0$ for all $i \geq 1$.

To prove the last isomorphism, we may assume that $i = 0$. But the isomorphism follows from the exact sequence (3.1), since $H^0_{\mathfrak{m}}(M) = M_0 = 0$ in this case. □

Remark 3.4. Let M be a finitely generated $\mathbb{Z}$-graded module over $S = k[x_1, \dots, x_n]$. Then we have an algebraic coherent sheaf $\tilde{M}$ on $\mathbb{P}^{n-1} = \operatorname{Proj}(S)$. Like our functor $(-)^+$, if $\dim_k M < \infty$, then $\tilde{M} = 0$. Moreover, it is well-known that $H^i(\mathbb{P}^{n-1}, \tilde{M}) \cong [H^{i+1}_{\mathfrak{m}}(M)]_0$ for all $i \geq 1$, and

$$0 \to [H^0_{\mathfrak{m}}(M)]_0 \to M_0 \to H^0(\mathbb{P}^{n-1}, \tilde{M}) \to [H^1_{\mathfrak{m}}(M)]_0 \to 0 \qquad \text{(exact)},$$

(see, for example, [SV, p.38]). So Theorem 3.3 gives an analogy between Proj and our $(-)^+$.

Recall that we chose $\mathbf{a}(F) \in Q \cap F$ for each $F \in L$ in the previous section. By the graded local duality, the $\mathbb{Z}^n$-graded k-dual of $H^i_{\mathfrak{m}}(M)$ is isomorphic to the squarefree module $\operatorname{Ext}^{n-i}_{k[Q]}(M, K_{k[Q]})$. So to determine the $\mathbb{Z}^n$-graded Hilbert function of $H^i_{\mathfrak{m}}(M)$, it suffices to know $[H^i_{\mathfrak{m}}(M)]_{-\mathbf{a}(F)}$ for each F. Since Theorem 3.3 deals with the case when $F = \{0\}$ (i.e., $\mathbf{a}(F) = 0$), we may assume that $F \neq \{0\}$.

Theorem 3.5. *Let M be a squarefree $k[Q]$-module, and j the embedding map from the open set $U_F = \coprod_{G \supset F} |G|^\circ$ to B. If $F \neq \{0\}$, we have*

$$H^i_c(U_F, j^* M^+) \cong [H^{i+1}_{\mathfrak{m}}(M)]_{-\mathbf{a}(F)} \quad \textit{for all } i \geq 0,$$

where $H^i_c(-)$ stands for the cohomology with the compact support.

Proof. Let $\Psi := \{G \in L \mid G \supset F\}$ be the order filter of L. Under the same notation as Lemma 3.1, we have $U_\psi = U_F$. We have the following.

$$\begin{aligned}
[H^{i+1}_{\mathfrak{m}}(M)]_{-\mathbf{a}(F)} &\cong H^{i+1}(C^\bullet_F(M)) && \text{(by Lemma 2.5)} \\
&\cong H^{i+1}(C^\bullet_{\{0\}}(M_\Psi)) \\
&\cong [H^{i+1}_{\mathfrak{m}}(M_\Psi)]_0 && \text{(by Lemma 2.5)} \\
&\cong H^i(\,B,\, (M_\Psi)^+\,) && \text{(by Theorem 3.3. Note that } M_\Psi \in \mathrm{Sq}_+\text{)} \\
&\cong H^i(\,B,\, j_! j^* M^+\,) && \text{(by Lemma 3.1)} \\
&\cong H^i_c(\,U_F,\, j^* M^+\,) && \text{(by [I, III, Corollary 7.3])}
\end{aligned}$$

□

Remark 3.6. When $k[Q]$ is a polynomial ring $k[\mathbb{N}^n] = k[x_1, \dots, x_n]$, Theorems 3.3 and 3.5 generalize a well-known formula of Hochster ([BH, Theorem 5.3.8], see also [BH, Lemma 5.4.5]). This formula states that $[H^{i+1}_{\mathfrak{m}}(k[\Delta])]_0 \cong \tilde{H}_i(|\Delta|; k)$ for all $i \geq 0$, where the right hand side is the ith reduced homology group of $|\Delta|$. On the other hand, Theorem 3.3 states that $[H^{i+1}_{\mathfrak{m}}(k[\Delta])]_0 = H^i(B, k[\Delta]^+)$ for all $i \geq 1$ and $H^0(B, k[\Delta]^+) = [H^1_{\mathfrak{m}}(k[\Delta])]_0 \oplus k[\Delta]_0 \cong [H^1_{\mathfrak{m}}(k[\Delta])]_0 \oplus k$. But $H^i(B, k[\Delta]^+) = H^i(B,\, j_* \underline{k}_{|\Delta|}) = H^i(|\Delta|;\, k) = \tilde{H}_i(|\Delta|;\, k)$ for all $i \geq 1$, and $H^0(B, k[\Delta]^+) = H_0(|\Delta|; k) = \tilde{H}_0(|\Delta|; k) \oplus k$. So Theorem 3.3 coincides with Hochster's formula. If $0 \neq \mathbf{a} \in \mathbb{N}^n$ and $\mathrm{supp}(\mathbf{a}) = F$, Hochster's formula states that $[H^{i+1}_{\mathfrak{m}}(k[\Delta])]_{-\mathbf{a}} = H_i(|\Delta|, |\Delta| - \{p\}; k)$ for a point $p \in |F|^\circ$. Set $u_F := U_F \cap |\Delta|$. For $p \in |F|^\circ$, u_F is a cone neighbourhood of p and $|\Delta| - u_F$ is a deformation retract of $|\Delta| - \{p\}$. Hence we have

$$
\begin{aligned}
[H^{i+1}_{\mathfrak{m}}(M)]_{-\mathbf{a}} &\cong H^i_c(U_F, j^* k[\Delta]^+) && \text{(by Theorem 3.5)} \\
&\cong H^i_c(\, u_F,\, \underline{k}_{u_F}\,) \\
&\cong H^i(\,|\Delta|,\, |\Delta| - u_F;\, k\,) && \text{(see [I, IV. Definition 8.1])} \\
&\cong H^i(\,|\Delta|,\, |\Delta| - \{p\};\, k\,).
\end{aligned}
$$

So Theorem 3.5 and Hochster's formula also coincide.

4. Relation to Poincaré-Verdier Duality

Since the functor $(-)^+ : \mathrm{Sq} \to \mathrm{Sh}(B)$ is exact, it can be extended to the functor $(-)^+ : D^b(\mathrm{Sq}) \to D^b(\mathrm{Sh}(B))$. If $M^\bullet \in D^b(\mathrm{Sq})$, we have $R\,\mathrm{Hom}_{k[Q]}(M^\bullet, K_{k[Q]}) \in D^b_{\mathrm{Sq}}({}^*\mathrm{Mod})$ by Lemma 2.4. So there is a bounded complex $N^\bullet$ of squarefree modules such that $N^\bullet \cong R\,\mathrm{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})$ in $D^b({}^*\mathrm{Mod})$. We denote $(N^\bullet)^+ \in D^b(\mathrm{Sh}(B))$ by $R\,\mathrm{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})^+$. $R\,\mathrm{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})^+$ does not depend on the particular choice of $N^\bullet$ up to isomorphism in $D^b(\mathrm{Sh}(B))$, of course.

For a locally compact topological space X of finite dimension (e.g., a locally closed subset of B), $\mathcal{D}^\bullet_X$ denotes a dualizing complex of X with the coefficients in k (see [I, V. §2]). In this paper, we frequently use the isomorphism $\mathcal{D}^\bullet_Y \cong j^! \mathcal{D}^\bullet_X$ for the embedding map j from a locally closed subset Y to X (see [I, V. Theorem 5.6]). If X is a manifold (with or without boundary), we have the orientation sheaf or_X of X with the coefficients in k. In this case, we have $\mathcal{D}^\bullet_X \cong or_X[\dim X]$ (see [I, V. §3]).

Lemma 4.1. *With the above notation, we have the following.*

1. $or_B \cong (K_{k[Q]})^+$.
2. *Let J_F be the projective object in* Sq *associated with a face $F \in L$. Then* $R\mathcal{H}om_{\mathrm{Sh}(B)}((J_F)^+, or_B) \cong \mathrm{Hom}_{k[Q]}(J_F, K_{k[Q]})^+$.
3. *If $M^\bullet \in D^b(\mathrm{Sq})$, we have an isomorphism*

$$R\mathcal{H}om_{\mathrm{Sh}(B)}((M^\bullet)^+, or_B) \cong R\,\mathrm{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})^+$$

in $D^b(\mathrm{Sh}(B))$.

Proof. (1) Let $\underline{k}_{B^\circ}$ be the constant sheaf on the relative interior B° of B. If $j : B^\circ \to B$ is the embedding map, then $or_B \cong j_! \underline{k}_{B^\circ}$ by [I, VI. Proposition 3.3]. On the other hand, $(K_{k[Q]})^+ \cong j_! \underline{k}_{B^\circ}$ as we have seen in Example 3.2.

(2) Recall that if U is an open set with the embedding map $j : U \to B$ and $\mathcal{I}$ is an injective object in $\mathrm{Sh}(B)$, then $j^*\mathcal{I}$ $(= j^!\mathcal{I})$ is injective in $\mathrm{Sh}(U)$. So $\mathcal{E}xt^i_{\mathrm{Sh}(B)}((J_F)^+, or_B)$ is the sheaf associated to the presheaf which sends an open set U to $\mathrm{Ext}^i_{\mathrm{Sh}(U)}(j^*(J_F)^+, j^* or_B)$. Note that $j^* or_B \cong or_U$. By the Poincaré-Verdier duality ([I, V. 2.1]), we have

$$\mathrm{Ext}^i_{\mathrm{Sh}(U)}(j^*(J_F)^+, j^* or_B) \cong H_c^{n-1-i}(U, j^*(J_F)^+)^\vee,$$

where $(-)^\vee$ means the dual k-vector space. For any open neighbourhood V of p, there is an open set U with $p \in U \subset V$ such that $U \cap U_F \cong \mathbb{R}^{n-1}$ or $\mathbb{R}^{n-1}_+$. Then $H^i_c(U, j^*(J_F)^+) \cong H^i_c(U \cap U_F; k) = 0$ for all $i \neq n-1$. Thus $\mathcal{E}xt^i_{\mathrm{Sh}(B)}((J_F)^+, or_B) = 0$ for all $i \neq 0$. Hence we have $R\mathcal{H}om_{\mathrm{Sh}(B)}((J_F)^+, or_B) \cong \mathcal{H}om_{\mathrm{Sh}(B)}((J_F)^+, or_B)$.

Recall that $(J_F)^+$ is the constant sheaf on U_F and or_B is the constant sheaf on B°. For a point $p \in B$, the stalk $\mathcal{H}om_{\mathrm{Sh}(B)}((J_F)^+, or_B)p$ at p is nonzero (equivalently, $\mathcal{H}om_{\mathrm{Sh}(B)}((J_F)^+, or_B)p = k$) if and only if there is an open neighbourhood U_p of p such that $U_p \cap U_F \subset B^\circ$. With the same notation as Lemma 2.2 (3), the latter condition is equivalent to the condition that $\mathrm{supp}(p) \vee F = \mathbb{R}_{\geq 0} Q$. So the assertion follows from Lemma 2.2 (3).

(3) Let $P^\bullet$ be a projective resolution of $M^\bullet$ in Sq, that is, there is a quasi isomorphism $P^\bullet \to M^\bullet$ and each P^i is a direct sum of copies of J_F for various F. By (2), we can compute $R\mathcal{H}om_{Sh(B)}((M^\bullet)^+, or_B)$ by $(P^\bullet)^+$. So we have

$$\begin{aligned} R\mathcal{H}om_{\mathrm{Sh}(B)}((M^\bullet)^+, or_B) &\cong \mathcal{H}om^\bullet_{\mathrm{Sh}(B)}((P^\bullet)^+, or_B) \\ &\cong \mathrm{Hom}^\bullet_{k[Q]}(P^\bullet, K_{k[Q]})^+ \\ &\cong R\,\mathrm{Hom}_{k[Q]}(M^\bullet, K_{k[Q]})^+. \end{aligned}$$

□

The normalized $\mathbb{Z}^n$-graded dualizing complex of $k[Q]$ is a $\mathbb{Z}^n$-graded injective resolution of $K_{k[Q]}[n]$. But, in this paper, we will consider a $\mathbb{Z}^n$-graded injective resolution of $K_{k[Q]}[n-1]$, which is a non-normalized dualizing complex. The reason why we use this convention is that $k[Q]$ represents the $(n-1)$-dimensional polytope B in our context.

Let $\omega^\bullet_{k[Q]}$ be the Q-graded part of a minimal $\mathbb{Z}^n$-graded injective resolution of $K_{k[Q]}[n-1]$. The complex $\omega^\bullet_{k[Q]}$, which is a minimal injective resolution of $K_{k[Q]}[n-1]$ in Sq, is of the form

$$\omega^\bullet_{k[Q]} : 0 \longrightarrow \omega^{-n+1} \longrightarrow \omega^{-n+2} \longrightarrow \cdots \longrightarrow \omega^1 \longrightarrow 0, \tag{4.1}$$

$$\omega^i = \bigoplus_{\substack{F \in L \\ \dim F = -i+1}} k[F],$$

and the differential is composed of the maps $\varepsilon(F,G)\cdot \mathrm{nat} : k[F] \to k[G]$ for all $G \in L$ with $\dim G = \dim F - 1$, where ε is the incidence function on the cell complex $B = \coprod_{F\in L} |F|^\circ$ and $\mathrm{nat} : k[F] \to k[G]$ is the natural surjection.

For an order ideal $\Delta \subset L$, set $\omega^\bullet_{k[\Delta]} := \mathrm{Hom}_{k[Q]}(k[\Delta], \omega^\bullet_{k[Q]})$. This is a complex of squarefree $k[\Delta]$-modules with

$$\omega^i_{k[\Delta]} = \bigoplus_{\substack{F\in\Delta \\ \dim F = -i+1}} k[F].$$

Note that $\omega^\bullet_{k[\Delta]}$ is isomorphic to a non-normalized $\mathbb{Z}^n$-graded dualizing complex of $k[\Delta]$ in the derived category of $\mathbb{Z}^n$-graded $k[\Delta]$-modules.

Let $\mathrm{Sq}(\Delta)$ be the full subcategory of Sq consisting of $k[\Delta]$-modules, that is, $M \in \mathrm{Sq}(\Delta)$ if and only if M is a squarefree $k[Q]$-module whose annihilator $\mathrm{Ann}(M)$ contains I_Δ. The category $\mathrm{Sq}(\Delta)$ is a thick abelian subcategory of Sq, and $\mathrm{Sq}(\Delta)$ has enough injectives, and an indecomposable injective object is of the form $k[F]$ for some $F \in \Delta$ (c.f. [RWY]), which is also injective in Sq. Thus $D^b(\mathrm{Sq}(\Delta)) \cong D^b_{\mathrm{Sq}(\Delta)}(\mathrm{Sq}) \cong D^b_{\mathrm{Sq}(\Delta)}({}^*\mathrm{Mod})$, and $D^b(\mathrm{Sq}(\Delta))$ can be viewed as a full subcategory of $D^b({}^*\mathrm{Mod})$.

If $M^\bullet \in D^b(\mathrm{Sq}(\Delta))$, we have $R\,\mathrm{Hom}_{k[Q]}(M^\bullet, K_{k[Q]}[n-1]) \cong R\,\mathrm{Hom}_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})$ in $D^b({}^*\mathrm{Mod})$ by the local duality. In particular, $R\,\mathrm{Hom}_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})$ belongs to $D^b_{\mathrm{Sq}(\Delta)}({}^*\mathrm{Mod})$, and we can define $R\,\mathrm{Hom}_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})^+ \in D^b(\mathrm{Sh}(B))$.

If $M \in \mathrm{Sq}(\Delta)$ and $j : |\Delta| \to B$ is the embedding map, then $\mathrm{Supp}(M^+) \subset |\Delta|$ and $j_*j^*M^+ \cong M^+$. Since $j_*(= j_!) : \mathrm{Sh}(|\Delta|) \to \mathrm{Sh}(B)$ is an exact functor in this case, it can be extended to the functor $j_* : D^b(\mathrm{Sh}(|\Delta|)) \to D^b(\mathrm{Sh}(B))$.

Theorem 4.2. *With the above notation, for $M^\bullet \in D^b(\mathrm{Sq}(\Delta))$, we have*

$$R\mathcal{H}om_{\mathrm{Sh}(|\Delta|)}(\, j^*(M^\bullet)^+, \mathcal{D}^\bullet_{|\Delta|}\,) \cong j^*(R\,\mathrm{Hom}_{k[\Delta]}(\,M^\bullet, \omega^\bullet_{k[\Delta]})^+)$$

in $D^b(\mathrm{Sh}(|\Delta|))$.

Proof. In $D^b(\mathrm{Sh}(B))$, we have the following isomorphisms.

$$\begin{aligned}
& j_*R\mathcal{H}om_{\mathrm{Sh}(|\Delta|)}(\, j^*(M^\bullet)^+, \mathcal{D}^\bullet_{|\Delta|}\,) \\
\cong\ & j_*R\mathcal{H}om_{\mathrm{Sh}(|\Delta|)}(\, j^*(M^\bullet)^+, j^!\mathcal{D}^\bullet_B\,) \\
\cong\ & R\mathcal{H}om_{\mathrm{Sh}(B)}(\, j_*j^*(M^\bullet)^+, \mathcal{D}^\bullet_B\,) \qquad \text{(by [I, VII. Theorem 5.2])} \\
\cong\ & R\mathcal{H}om_{\mathrm{Sh}(B)}(\,(M^\bullet)^+, or_B[n-1]\,) \\
\cong\ & R\,\mathrm{Hom}_{k[Q]}(\,M^\bullet, K_{k[Q]}[n-1]\,)^+ \qquad \text{(by Lemma 4.1 (3))} \\
\cong\ & R\,\mathrm{Hom}_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})^+.
\end{aligned}$$

Hence $j_*R\mathcal{H}om_{\mathrm{Sh}(|\Delta|)}(\, j^*(M^\bullet)^+, \mathcal{D}^\bullet_{|\Delta|}\,) \cong R\,\mathrm{Hom}_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})^+$. Applying j^* to the both sides of this isomorphism, we have the expected isomorphism. □

Corollary 4.3. *With the above notation, we have $\mathcal{D}^\bullet_{|\Delta|} \cong j^*(\omega^\bullet_{k[\Delta]})^+$.*

Proof.

$$\mathcal{D}^\bullet_{|\Delta|} \cong R\mathcal{H}om_{\mathrm{Sh}(|\Delta|)}(\underline{k}_{|\Delta|}, \mathcal{D}^\bullet_{|\Delta|}) \cong j^*(R\operatorname{Hom}_{k[\Delta]}(k[\Delta], \omega^\bullet_{k[\Delta]})^+) \cong j^*(\omega^\bullet_{k[\Delta]})^+$$

□

Proposition 4.4. *Let $\Delta, \Sigma \subset L$ be order ideals with $\Delta \supset \Sigma$, and j the embedding map from $Z := |\Delta| - |\Sigma|$ to B. Then*

$$\mathcal{D}^\bullet_Z \cong j^*(R\mathcal{H}om_{k[Q]}(\, I_{\Delta/\Sigma},\, \omega^\bullet_{k[Q]})^+),$$

where $I_{\Delta/\Sigma} := I_\Sigma / I_\Delta$.

Proof. In $D^b(\mathrm{Sh}(B))$, we have the following isomorphisms.

$$\begin{aligned}
& R\operatorname{Hom}_{k[Q]}(\, I_{\Delta/\Sigma},\, \omega^\bullet_{k[Q]})^+ & \\
\cong\ & R\mathcal{H}om_{\mathrm{Sh}(B)}(\, j_!\underline{k}_Z,\, \mathcal{D}^\bullet_B\,) & \text{(by Theorem 4.2)} \\
\cong\ & Rj_*R\mathcal{H}om_{\mathrm{Sh}(Z)}(\, \underline{k}_Z,\, j^!\mathcal{D}^\bullet_B\,) & \text{(by [I, VII. Theorem 5.2])} \\
\cong\ & Rj_*R\mathcal{H}om_{\mathrm{Sh}(Z)}(\, \underline{k}_Z,\, \mathcal{D}^\bullet_Z\,) & \\
\cong\ & Rj_*\mathcal{D}^\bullet_Z. &
\end{aligned}$$

Hence $R\operatorname{Hom}_{k[Q]}(\, I_{\Delta/\Sigma},\, \omega^\bullet_{k[Q]})^+ \cong Rj_*\mathcal{D}^\bullet_Z$. Applying j^* to the both sides of this isomorphism, we have the expected isomorphism. In fact, since the functor $j^*j_* : \mathrm{Sh}(Z) \to \mathrm{Sh}(Z)$ is natural equivalent to the identity functor, we have $j^*Rj_* \cong j^*j_* \cong \mathrm{Id}$ as an endofunctor on $D^b(\mathrm{Sh}(Z))$. □

In our context, the notion of a Buchsbaum ring is natural and important. The original definition of a Buchsbaum ring (see [SV]) is slightly complicated, but for $k[\Delta]$, we have a simple criterion.

Lemma 4.5. *Let $A = \bigoplus_{i \geq 0} A_i$ be a noetherian $\mathbb{N}$-graded commutative ring with the graded maximal ideal $\mathfrak{m} = \bigoplus_{i>0} A_i$ (thus $A_0 = k$ is a field). Let M be a finitely generated graded A-module of dimension r. If there is some $s \in \mathbb{Z}$ such that $[H^i_\mathfrak{m}(M)]_t = 0$ for all $i < r$ and $t \neq s$, then M is a Buchsbaum A-module.*

If A is generated by A_1 as a k-algebra, the above fact is a special case of the well-known result [SV, I. Proposition 3.10]. Even in the general case, this fact was essentially pointed out in [SS].

Proof. Note that A has a graded normalized dualizing complex $I^\bullet_A$. Set $N^\bullet := \tau_{-r}\operatorname{Hom}^\bullet_A(M, I^\bullet_A)$. Here, for a complex $C^\bullet$, $\tau_{-r}C^\bullet$ is the truncated complex

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Im}(C^{-r} \to C^{-r+1}) \longrightarrow C^{-r+1} \longrightarrow C^{-r+2} \longrightarrow \cdots .$$

We have $H^i(N^\bullet) = 0$ for all $i \leq -r$, and $H^i(N^\bullet)$ is the graded k-dual of $H^{-i}_\mathfrak{m}(M)$ for all $i > -r$ by the local duality. So the cohomologies of $N^\bullet$ are concentrated in the degree $-s$ components. By [SV, II.Theorem 4.1], it suffices to prove that $N^\bullet$ is isomorphic to a complex of k-vector spaces in the derived category of graded A-modules. For a graded A-module N and an integer t, set $N_{\geq t} := \bigoplus_{i \geq t} N_i$. Then chain maps $N^\bullet_{\geq -s} \to N^\bullet$ and $N^\bullet_{\geq -s} \to N^\bullet_{\geq -s}/N^\bullet_{\geq -s+1}$ are quasi-isomorphisms.

Thus, in the derived category, $N^\bullet$ is isomorphic to $N^\bullet_{\geq -s}/N^\bullet_{\geq -s+1}$, which is a complex of k-vector spaces. □

Corollary 4.6 (c.f. [Y2, Corollary 3.8]). *Let M be a squarefree $k[Q]$-module of dimension r. Then the following are equivalent.*

(a) *M is a Buchsbaum module.*
(b) $\dim_k H^i_{\mathfrak{m}}(M) < \infty$ *for all* $i \neq r$.
(c) $[H^i_{\mathfrak{m}}(M)]_{\mathbf{a}} = 0$, *if* $i \neq r$ *and* $\mathbf{a} \neq 0$.

Proof. The implication (a) ⇒ (b) is a basic property of Buchsbaum modules. The $\mathbb{Z}^n$-graded k-dual of $H^i_{\mathfrak{m}}(M)$ is the squarefree module $\operatorname{Ext}^{n-i}_{k[Q]}(M, K_{k[Q]})$. Hence $\dim_k H^i_{\mathfrak{m}}(M) < \infty$ if and only if $H^i_{\mathfrak{m}}(M) = [H^i_{\mathfrak{m}}(M)]_0$. So we have (b) ⇔ (c). The implication (c) ⇒ (a) follows from Lemma 4.5. □

Corollary 4.7. *Let $\Delta \subset L$ be an order ideal with $d = \dim |\Delta|$ (so $\dim k[\Delta] = d+1$). The following are equivalent.*

(a) *$k[\Delta]$ is a Buchsbaum ring.*
(b) $\mathcal{H}^i(\mathcal{D}^\bullet_{|\Delta|}) = 0$ *for all* $i \neq -d$.
(c) $H_i(|\Delta|, |\Delta| - \{p\}; k) = 0$ *for all* $i < d$ *and all* $p \in |\Delta|$.

In particular, the Buchsbaum property of $k[\Delta]$ is a topological property of $|\Delta|$ (i.e., depends only on the topology of $|\Delta|$ and $\operatorname{char}(k)$).

Proof. We have $\mathcal{H}^i(\mathcal{D}^\bullet_{|\Delta|}) \cong j^*(H^i(\omega^\bullet_{k[\Delta]})^+) \cong j^*(\operatorname{Ext}^i_{k[Q]}(k[\Delta], \omega^\bullet_{k[Q]})^+)$ by Corollary 4.3, where $j : |\Delta| \to B$ is the embedding map. Thus $\mathcal{H}^i(\mathcal{D}^\bullet_{|\Delta|}) = 0$ if and only if $\dim_k \operatorname{Ext}^i_{k[Q]}(k[\Delta], \omega^\bullet_{k[Q]}) = \dim_k H^{-i+1}_{\mathfrak{m}}(k[\Delta]) < \infty$. So (b) is equivalent to (a). The equivalence (b) ⇔ (c) must be obvious for algebraic topologists. But the equivalence (c) ⇔ (a) (⇔ (b)) also follows from Theorem 3.5. In fact, if $0 \neq \mathbf{a} \in \mathbb{N}^n$, we have $[H^{i+1}_{\mathfrak{m}}(k[\Delta])]_{-\mathbf{a}} \cong H_i(|\Delta|, |\Delta| - \{p\}; k)$ for $p \in |\operatorname{supp}(\mathbf{a})|^\circ$, as we have seen in Remark 3.6. □

Remark 4.8. The implication (b) ⇒ (a) of Corollary 4.7 does not hold for a locally closed subset $Z := |\Delta| - |\Sigma|$ and its squarefree module $I_{\Delta/\Sigma} := I_\Sigma / I_\Delta$, while we have Proposition 4.4. Since

$$R\operatorname{Hom}_{k[Q]}(I_{\Delta/\Sigma}, \omega^\bullet_{k[Q]})^+ \cong Rj_*\mathcal{D}^\bullet_Z$$

by the proof of Proposition 4.4, $I_{\Delta/\Sigma}$ is a Buchsbaum module of dimension $d+1$ if and only if $R^i j_* \mathcal{D}^\bullet_Z = 0$ for all $i \neq -d$. These conditions are stronger than the condition that $\mathcal{H}^i(\mathcal{D}^\bullet_Z) = 0$ for all $i \neq -d$.

For example, consider a polynomial ring $k[x,y,z]$, and simplicial complexes $\Delta = 2^{\{x,y,z\}}$ and $\Sigma = \{\{x\}, \emptyset\}$. Then Z is a manifold with boundary (in fact, $Z \cong \mathbb{R}^2_+$), and $\mathcal{H}^i(\mathcal{D}^\bullet_Z) = 0$ for all $i \neq -2$. But $I_{\Delta/\Sigma} = (y,z)$ is not a Buchsbaum module. In this case, $R^i j_* \mathcal{D}^\bullet_Z \neq 0$ for $i = -1, -2$.

Since $\operatorname{Supp}(R^i j_* \mathcal{D}^\bullet_Z) \subset \bar{Z} = |\Delta|$, it suffices to check $R^i h_* \mathcal{D}^\bullet_Z$ to see the vanishing of $R^i j_* \mathcal{D}^\bullet_Z$, where $h : Z \to |\Delta|$ is the embedding map. That is, $I_{\Delta/\Sigma}$ is a Buchsbaum module of dimension $d+1$ if and only if $R^i h_* \mathcal{D}^\bullet_Z = 0$ for all

$i \neq -d$. Hence the Buchsbaum property of $I_{\Delta/\Sigma}$ is a topological property of the pair $(|\Delta|, |\Sigma|)$.

If $|\Delta|$ is a manifold (with or without boundary) of dimension d, then we have $\mathcal{D}^\bullet_{|\Delta|} \cong or_{|\Delta|}[d]$ and $k[\Delta]$ is a Buchsbaum ring of dimension $d+1$. By Corollary 4.3, we have $j^*(K_{k[\Delta]})^+ \cong or_{|\Delta|}$, where $K_{k[\Delta]} := \mathrm{Ext}^{n-d-1}_{k[Q]}(k[\Delta], K_{k[Q]})$ is the canonical module of $k[\Delta]$.

Let $(A, \mathfrak{m})$ be a Buchsbaum local ring of dimension $d+1$ admitting a canonical modules K_A. Then [SV, II. Theorem 4.9] states that

$$H^i_{\mathfrak{m}}(K_A) \cong \mathrm{Hom}_A(H^{d-i+2}_{\mathfrak{m}}(A),\, E(A/\mathfrak{m})) \quad \text{for all } 2 \leq i \leq d,$$

where $E(A/\mathfrak{m})$ is the injective hull of $A/\mathfrak{m}$. We will see that this duality corresponds to the Poincaré duality in our context.

Assume that $k[\Delta]$ is a Buchsbaum ring of dimension $d+1$ (thus $\dim |\Delta| = d$). Then we have

$$[H^i_{\mathfrak{m}}(\,(K_{k[\Delta]})_{>0}\,)]_0 \cong [H^{d-i+2}_{\mathfrak{m}}(\,k[\Delta]_{>0}\,)^\vee]_0 \quad \text{for all } 1 \leq i \leq d+1. \tag{4.2}$$

(When $2 \leq i \leq d$, this is just a $\mathbb{Z}^n$-graded version of [SV, II. Theorem 4.9]. We leave the case when $i = 1, d+1$ for the reader as an easy exercise.) By Theorem 3.3,

$$[\,H^i_{\mathfrak{m}}(\,(K_{k[\Delta]})_{>0}\,)\,]_0 \cong H^{i-1}(\,|\Delta|,\,(K_{k[\Delta]})^+) \cong H^{i-1}(\,|\Delta|,\,or_{|\Delta|}\,)$$

and

$$[\,H^{d-i+2}_{\mathfrak{m}}(\,k[\Delta]_{>0}\,)\,]_0 \cong H^{d-i+1}(\,|\Delta|,\,\underline{k}_{|\Delta|}\,) \cong H^{d-i+1}(\,|\Delta|\,;\,k\,)$$

for all $1 \leq i \leq d+1$. So (4.2) also follows from the Poincaré duality

$$H^i(\,|\Delta|,\,or_{|\Delta|}\,) \cong H^j(\,|\Delta|\,;\,k\,)^\vee \text{ for all } i,j \text{ with } i+j=d.$$

Note that $|\Delta|$ is an orientable manifold (i.e., a manifold with $\underline{k}_{|\Delta|} \cong or_{|\Delta|}$) if and only if $k[\Delta]$ is a Buchsbaum ring with $(K_{k[\Delta]})_{>0} \cong k[\Delta]_{>0}$. In this case, (4.2) corresponds to the most familiar form of the Poincaré duality. We also remark that if $|\Delta|$ is an orientable manifold of dimension d then $\dim_k [H^{d+1}_{\mathfrak{m}}(k[\Delta])]_0$ equals the number of the connected components of $|\Delta|$. When $|\Delta|$ is a connected manifold, $|\Delta|$ is orientable if and only if $\dim_k [H^{d+1}_{\mathfrak{m}}(k[\Delta])]_0 = 1$. In this case, $K_{k[\Delta]} \cong k[\Delta]$.

Let $\mathrm{Sq}_+(\Delta)$ be the full subcategory of Sq consisting of squarefree $k[\Delta]$-modules M with $M_0 = 0$. For a while, let $M^\bullet$ be an object of $D^b(\mathrm{Sq}_+(\Delta))$.

For $M^\bullet \in D^b(\mathrm{Sq}_+(\Delta))$, by the local duality and Theorem 3.3, we have

$$[\mathrm{Ext}^{-i}_{k[\Delta]}(M^\bullet, \omega^\bullet_{k[\Delta]})^\vee]_0 \cong [R^{i+1}\Gamma_{\mathfrak{m}}(M^\bullet)]_0 \cong R^i\,\Gamma(B, (M^\bullet)^+) \cong R^i\,\Gamma(\,|\Delta|,\,j^*(M^\bullet)^+).$$

On the other hand, we have $\mathrm{Ext}^{-i}_{\mathrm{Sh}(|\Delta|)}(\,j^*(M^\bullet)^+,\,\mathcal{D}^\bullet_{|\Delta|}\,)^\vee \cong R^i\,\Gamma(\,|\Delta|,\,j^*(M^\bullet)^+)$ by the Poincaré-Verdier duality ([I, V, 2.1]). Thus

$$\mathrm{Ext}^i_{\mathrm{Sh}(|\Delta|)}(\,j^*(M^\bullet)^+,\,\mathcal{D}^\bullet_{|\Delta|}\,) \cong [\mathrm{Ext}^i_{k[\Delta]}(\,M^\bullet,\,\omega^\bullet_{k[\Delta]}\,)]_0. \tag{4.3}$$

We can give another proof of (4.3). Let $P^\bullet \to M^\bullet$ be a projective resolution in Sq. Since $M^\bullet \in D^b(\mathrm{Sq}_+(\Delta))$, we may assume that each component of $P^\bullet$ is a direct sum of copies of J_F for various $\{0\} \neq F \in L$. If $F \neq \{0\}$, then $\mathrm{Supp}((J_F)^+) = U_F \cong \mathbb{R}^{n-1}$ or $\mathbb{R}_+^{n-1}$ and $\mathrm{Ext}^i_{\mathrm{Sh}(B)}((J_F)^+, or_B) = H^{n-1-i}(B, (J_F)^+) = H_c^{n-1-i}(U_F; k) = 0$ for all $i \neq 0$. So we can compute $\mathrm{Ext}^i_{\mathrm{Sh}(B)}(\,(M^\bullet)^+, or_B\,)$ using $(P^\bullet)^+$, and we have the following.

$$\begin{aligned} & \mathrm{Ext}^i_{\mathrm{Sh}(|\Delta|)}(\, j^*(M^\bullet)^+,\, \mathcal{D}^\bullet_{|\Delta|}\,) \\ \cong\; & \mathrm{Ext}^i_{\mathrm{Sh}(B)}(\,(M^\bullet)^+,\, \mathcal{D}^\bullet_B\,) \qquad \text{(by [I, VII. Theorem 3.1])} \\ \cong\; & \mathrm{Ext}^i_{\mathrm{Sh}(B)}(\,(M^\bullet)^+,\, or_B[n-1]\,) \\ \cong\; & H^i(\,\mathrm{Hom}_{\mathrm{Sh}(B)}(\,(P^\bullet)^+,\, (K_{k[Q]})^+[n-1]\,)\,) \\ \cong\; & H^i(\,[\,\mathrm{Hom}_{k[Q]}(\,P^\bullet,\, K_{k[Q]}[n-1]\,)\,]_0\,) \qquad (\text{since } P^\bullet \in D^b(\mathrm{Sq}_+)) \\ \cong\; & [\,\mathrm{Ext}^i_{k[Q]}(\,M^\bullet,\, K_{k[Q]}[n-1]\,)\,]_0 \\ \cong\; & [\,\mathrm{Ext}^i_{k[\Delta]}(\,M^\bullet,\, \omega^\bullet_{k[\Delta]}\,)\,]_0. \end{aligned}$$

Finally, we study the Cohen-Macaulay property of $k[\Delta]$ and $I_{\Delta/\Sigma}$. If $\dim k[\Delta] \leq 1$, then $k[\Delta]$ is always Cohen-Macaulay. So we may assume that $\dim k[\Delta] \geq 2$. The same thing is true for $I_{\Delta/\Sigma}$. When $k[Q]$ is a polynomial ring, the next result is a well-known theorem of Munkres.

Theorem 4.9 (c.f. [M, Y3]). *Let $\Delta \subset L$ be an order ideal with $d := \dim|\Delta| \geq 1$ (i.e., $\dim k[\Delta] = d+1 \geq 2$). Then the following are equivalent.*

(a) *$k[\Delta]$ is a Cohen-Macaulay ring of dimension $d+1$,*
(b) *$\tilde{H}_i(|\Delta|; k) = H_i(|\Delta|, |\Delta| - \{p\}; k) = 0$ for all $i < d$ and all $p \in |\Delta|$,*
(c) *$\mathcal{H}^i(\mathcal{D}^\bullet_{|\Delta|}) = 0$ for all $i \neq -d$, $H^i\Gamma(|\Delta|, \mathcal{D}^\bullet_{|\Delta|}) = 0$ for all $i \neq -d, 0$, and $H^0\Gamma(|\Delta|, \mathcal{D}^\bullet_{|\Delta|}) \cong k$.*

In particular, the Cohen-Macaulay property of $k[\Delta]$ is a topological property of $|\Delta|$.

Proof. The equivalence between (a) and (b) has been proved in [M, Y3]. Recall that $H_i(|\Delta|, |\Delta| - \{p\}; k) = 0$ for all $i < d$ and all $p \in |\Delta|$ if and only if $\mathcal{H}^i(\mathcal{D}^\bullet_{|\Delta|}) = 0$ for all $i \neq -d$. Since $H^{-i}\Gamma(|\Delta|, \mathcal{D}^\bullet_{|\Delta|}) \cong H^i(|\Delta|; k)^\vee$, (b) and (c) are equivalent. □

Proposition 4.10. *Let $\Delta, \Sigma \subset L$ be order ideals with $\Delta \supset \Sigma \neq \emptyset$, and h the embedding map from $Z := |\Delta| - |\Sigma|$ to $|\Delta|$.*

(a) *$I_{\Delta/\Sigma}$ is a Cohen-Macaulay module of dimension $d+1$,*
(b) *$R^i h_* \mathcal{D}^\bullet_Z = H^i\Gamma(Z, \mathcal{D}^\bullet_Z) = 0$ for all $i \neq -d$.*

In particular, the Cohen-Macaulay property of $I_{\Delta/\Sigma}$ is a topological property of the pair $(|\Delta|, |\Sigma|)$.

Proof. $I_{\Delta/\Sigma}$ is Cohen-Macaulay if and only if it is Buchsbaum and $[H^i_{\mathfrak{m}}(I_{\Delta/\Sigma})]_0 = 0$ for all $i \neq d+1$. As we have seen in Remark 4.8, $I_{\Delta/\Sigma}$ is Buchsbaum if and only if $R^i h_* \mathcal{D}_Z^\bullet = 0$ for all $i \neq -d$. Since $I_{\Delta/\Sigma} \in \mathrm{Sq}_+$, we have $[H_\mathfrak{m}^{i+1}(I_{\Delta/\Sigma})]_0 \cong H^i(B, (I_{\Delta/\Sigma})^+) \cong H_c^i(Z; k) \cong H^{-i}\Gamma(Z, \mathcal{D}_Z^\bullet)^\vee$ for all i. So we are done. □

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References

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings, revised edition*, Cambridge University Press, 1998.

[GW] S. Goto and K. Watanabe, *On graded rings. II. ($\mathbf{Z}^n$-graded rings)*. Tokyo J. Math. **1** (1978), 237–261.

[H] R. Hartshorne, *Residues and duality*, Lectures Notes in Math. **20**, Springer-Verlag, 1966.

[I] B. Iversen, *Cohomology of sheaves*. Springer-Verlag, 1986.

[M] E. Miller, *Cohen-Macaulay quotients of normal affine semigroup rings via irreducible resolutions*, Math. Research Letters **9** (2002), 117–128.

[RWY] V. Reiner, V. Welker and K. Yanagawa. *Local cohomology of Stanley-Reisner rings with supports in general monomial ideals*, J. Algebra **244** (2001), 706–736.

[SS] U. Schäfer and P. Schenzel, *Dualizing complexes of affine semigroup rings*, Trans. Amer. Math. Soc. **322** (1990), 561–582.

[Sta] R. Stanley, *Combinatorics and commutative algebra, 2nd ed*, Progress in Mathematics, **41**, (1996).

[SV] S. Stuckrad and W. Vogel, *Buchsbaum rings and applications*, Springer-Verlag, 1986.

[Y1] K. Yanagawa, *Alexander duality for Stanley-Reisner rings and squarefree $\mathbb{N}^n$-graded modules*, J. Algebra **225** (2000), 630–645.

[Y2] ———, *Sheaves on finite posets and modules over normal semigroup rings*, J. Pure and Appl. Algebra **161** (2001), 341–366.

[Y3] ———, *Squarefree modules and local cohomology modules at monomial ideals*, in "Local Cohomology and Its Applications (G. Lyubeznik, ed.)," pp. 207–231, Dekker, 2002.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: yanagawa@math.sci.osaka-u.ac.jp