

ELLIPTIC SELBERG INTEGRALS AND CONFORMAL BLOCKS

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ABSTRACT. We present an elliptic version of Selberg’s integral formula.

1. Introduction

The Selberg integral is the integral

$$B_p(\alpha, \beta, \gamma) = \int_{\Delta_p} \prod_{j=1}^p t_j^{\alpha-1} (1-t_j)^{\beta-1} \prod_{0 \leq j < k \leq p} (t_j - t_k)^{2\gamma} dt_1 \dots dt_p,$$

where $\Delta_p = \{t \in \mathbb{R}^p \mid 0 \leq t_p \leq \dots \leq t_1 \leq 1\}$. The Selberg integral is a generalization of the beta function. It can be calculated explicitly,

$$B_p(\alpha, \beta, \gamma) = \frac{1}{p!} \prod_{j=0}^{p-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(p+j-1)\gamma)}.$$

The Selberg integral has many applications, see [A1, A2, As, D, DF1, DF2, M, S]. In this paper, we present elliptic versions of the Selberg integral.

2. Conformal Blocks on the Torus

Let $\tau \in \mathbb{C}$ be such that $\text{Im } \tau > 0$. Let κ and p be non-negative integers satisfying $\kappa \geq 2p+2$. The KZB-heat equation is the partial differential equation

$$(1) \quad 2\pi i \kappa \frac{\partial u}{\partial \tau}(\lambda, \tau) = \frac{\partial^2 u}{\partial \lambda^2}(\lambda, \tau) + p(p+1) \rho'(\lambda, \tau) u(\lambda, \tau).$$

Here, the prime denotes the derivative with respect to the first argument, and ρ is defined in terms of the first Jacobi theta function,

$$\vartheta_1(\lambda, \tau) = 2q^{\frac{1}{8}} \sin(\pi\lambda) \prod_{j=1}^{\infty} (1 - q^j e^{2\pi i \lambda})(1 - q^j e^{-2\pi i \lambda})(1 - q^j),$$

$$\rho(\lambda, \tau) = \frac{\vartheta_1'(\lambda, \tau)}{\vartheta_1(\lambda, \tau)},$$

where $q = e^{2\pi i \tau}$. Holomorphic solutions of the KZB-heat equation with the properties,

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- (i) $u(\lambda + 2, \tau) = u(\lambda, \tau)$,
- (ii) $u(\lambda + 2\tau, \tau) = e^{-2\pi i \kappa(\lambda + \tau)} u(\lambda, \tau)$,
- (iii) $u(-\lambda, \tau) = (-1)^{p+1} u(\lambda, \tau)$,
- (iv) $u(\lambda, \tau) = \mathcal{O}((\lambda - m - n\tau)^{p+1})$ as $\lambda \rightarrow m + n\tau$ for any $m, n \in \mathbb{Z}$

are called conformal blocks (or elliptic hypergeometric functions) associated with the family of elliptic curves $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with the marked point $z = 0$ and the irreducible sl_2 representation of dimension $2p + 1$. It is known that the space of conformal blocks has dimension $\kappa - 2p - 1$.

3. Integral Representations of Conformal Blocks

Introduce special functions

$$\sigma_\lambda(t, \tau) = \frac{\vartheta_1(\lambda - t, \tau) \vartheta_1'(0, \tau)}{\vartheta_1(\lambda, \tau) \vartheta_1(t, \tau)}, \quad E(t, \tau) = \frac{\vartheta_1(t, \tau)}{\vartheta_1'(0, \tau)}.$$

Consider the theta functions

$$\theta_{\kappa, n}(\lambda, \tau) = \sum_{j \in \mathbb{Z}} e^{2\pi i \kappa(j + \frac{n}{2\kappa})^2 \tau + 2\pi i \kappa(j + \frac{n}{2\kappa}) \lambda}, \quad n \in \mathbb{Z}/2\kappa\mathbb{Z}.$$

They form a basis of the space of theta functions of level κ . They satisfy the equations

$$\theta_{\kappa, n}(\lambda + 1, \tau) = (-1)^n \theta_{\kappa, n}(\lambda, \tau), \quad \theta_{\kappa, n}(\lambda + \tau, \tau) = e^{-\pi i \kappa(\lambda + \frac{\tau}{2})} \theta_{\kappa, n+\kappa}(\lambda, \tau)$$

and have the modular properties

$$\theta_{\kappa, n}(\lambda, \tau + 1) = e^{\pi i \frac{n^2}{2\kappa}} \theta_{\kappa, n}(\lambda, \tau),$$

$$\theta_{\kappa, n} \left(\frac{\lambda}{\tau}, -\frac{1}{\tau} \right) = \sqrt{-\frac{i\tau}{2\kappa}} e^{\pi i \kappa \frac{\lambda^2}{2\tau}} \sum_{m=0}^{2\kappa-1} e^{-\pi i \frac{mn}{\kappa}} \theta_{\kappa, m}(\lambda, \tau),$$

where $|\arg(-i\tau)| < \pi/2$. Let $\theta_{\kappa, n}^s$ denote the symmetrization of $\theta_{\kappa, n}$ with respect to λ , $\theta_{\kappa, n}^s(\lambda, \tau) = \theta_{\kappa, n}(\lambda, \tau) + \theta_{\kappa, n}(-\lambda, \tau)$.

Define $u_{\kappa, n}$ by

$$u_{\kappa, n}(\lambda, \tau) = u_{p, \kappa, n}(\lambda, \tau) = J_{p, \kappa, n}(\lambda, \tau) + (-1)^{p+1} J_{p, \kappa, n}(-\lambda, \tau),$$

where

$$J_{p, \kappa, n}(\lambda, \tau) = \int_{\Delta_p} \prod_{j=1}^p E(t_j, \tau)^{-\frac{2p}{\kappa}} \prod_{1 \leq j < k \leq p} E(t_j - t_k, \tau)^{\frac{2}{\kappa}} \times \\ \prod_{j=1}^p \sigma_\lambda(t_j, \tau) \theta_{\kappa, n} \left(\lambda + \frac{2}{\kappa} \sum_{j=1}^p t_j, \tau \right) dt_1 \dots dt_p.$$

The branch of the logarithm is chosen in such a way that $\arg(E(t, \tau)) \rightarrow 0$ as $t \rightarrow 0^+$, and the integral is understood as the analytic continuation from the region where all of the exponents in the integrand have positive real parts.

Theorem 3.1. [FV1] *For all n , the integrals $u_{\kappa,n}(\lambda, \tau)$ are solutions of the KZB-heat equation having the properties (i)-(iv).*

Theorem 3.2. [FSV2] *We have*

- (a) $u_{\kappa,n} = u_{\kappa,n+2\kappa}$ and $u_{\kappa,n} = -e^{2\pi i p n / \kappa} u_{\kappa,-n}$.
- (b) *The set $\{u_{\kappa,n}(\lambda, \tau) \mid n = p+1, \dots, \kappa-p-1\}$ is a basis for the space of conformal blocks. The integrals $u_{\kappa,n}$ are identically zero for all other values of n in the interval from 0 to κ .*

4. Transformations acting on the space of conformal blocks

Introduce four transformations A , B , T , and S defined by

$$\begin{aligned} Au(\lambda, \tau) &= u(\lambda+1, \tau), & Bu(\lambda, \tau) &= e^{\pi i \kappa (\lambda + \frac{\tau}{2})} u(\lambda + \tau, \tau), \\ Tu(\lambda, \tau) &= u(\lambda, \tau+1), & Su(\lambda, \tau) &= e^{-\pi i \kappa \frac{\lambda^2}{2\tau} - \frac{1}{2} - \frac{p(p+1)}{\kappa}} u\left(\frac{\lambda}{\tau}, -\frac{1}{\tau}\right), \end{aligned}$$

where we fix $\arg \tau \in (0, \pi)$.

Proposition 4.1. *If $u(\lambda, \tau)$ is a solution of the KZB-heat equation, then $Au(\lambda, \tau)$, $Bu(\lambda, \tau)$, $Tu(\lambda, \tau)$, and $Su(\lambda, \tau)$ are solutions too. Moreover, the transformations A , B , T and S preserve the properties (i)-(iv).*

The proofs that T and S preserve the space of conformal blocks are given in [EK]. The proofs that A and B also preserve this space are straightforward and follow from the equations

$$\vartheta_1(\lambda+1, \tau) = -\vartheta_1(\lambda, \tau), \quad \vartheta_1(\lambda+\tau, \tau) = -e^{-\pi i (2\lambda+\tau)} \vartheta_1(\lambda, \tau).$$

Lemma 4.2. *Restricted to the space of conformal blocks, the transformations A , B , T , and S satisfy the relations*

$$\begin{aligned} A^2 &= I, & B^2 &= I, & S^2 &= (-1)^p i e^{-\pi i \frac{p(p+1)}{\kappa}} I, & (ST)^3 &= (-1)^p i e^{-\pi i \frac{p(p+1)}{\kappa}} I, \\ TA &= AT, & SAS^{-1} &= B, & AB &= (-1)^\kappa BA, & TB &= i^\kappa BAT, \end{aligned}$$

where I denotes the identity transformation. □

Lemma 4.3. *We have*

$$Au_{\kappa,n}(\lambda, \tau) = (-1)^n u_{\kappa,n}(\lambda, \tau), \quad Bu_{\kappa,n}(\lambda, \tau) = -e^{2\pi i \frac{pn}{\kappa}} u_{\kappa,\kappa-n}(\lambda, \tau).$$

□

Let $(t_{m,n})$ and $(s_{m,n})$ be the matrices of the transformations T and S , respectively, with respect to the basis $\{u_{\kappa,n}(\lambda, \tau) \mid p+1 \leq n \leq \kappa-p-1\}$, namely, $Tu_{\kappa,n} = \sum_{m=p+1}^{\kappa-p-1} t_{m,n} u_{\kappa,m}$, $Su_{\kappa,n} = \sum_{m=p+1}^{\kappa-p-1} s_{m,n} u_{\kappa,m}$. In Theorem 4.4, we give formulas for the matrices of T and S in terms of Macdonald polynomials of type A_1 .

The Macdonald polynomials [Ma] of type A_1 are x -even polynomials in terms of ϵ^{mx} , where $m \in \mathbb{Z}$. They depend on two parameters k and n , where k and n are non-negative integers. They are defined by the conditions:

1. $P_n^{(k)}(x) = \epsilon^{nx} + \epsilon^{-nx} + \text{lower order terms}$, except for $P_0^{(k)}(x) = 1$,
2. $\langle P_m^{(k)}, P_n^{(k)} \rangle = 0$ for $m \neq n$, where

$$\langle f, g \rangle = \frac{1}{2} \text{Const Term} \left(fg \prod_{j=0}^{k-1} (1 - \epsilon^{2(j+x)}) (1 - \epsilon^{2(j-x)}) \right).$$

Theorem 4.4. [FSV2] *Let $\epsilon = e^{\pi i/\kappa}$. For $p+1 \leq m, n \leq \kappa - p - 1$, we have*

$$t_{m,n} = \epsilon^{\frac{n^2}{2}} \delta_{mn},$$

$$s_{m,n} = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2\kappa}} \epsilon^{p(n-m) - \frac{p(p+1)}{2}} (\epsilon^{-m} - \epsilon^m) \left(\prod_{j=1}^p (\epsilon^{-n+j} - \epsilon^{n-j}) \right) P_{n-p-1}^{(p+1)}(m),$$

where $\delta_{mn} = 1$ for $m = n$ and 0 otherwise.

5. Integral identities

To formulate our main result, we need functions η , ϕ_1 , ϕ_2 , and ϕ_3 . The Dedekind η -function is the function $\eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j)$. We have

$$\vartheta'_1(0, \tau) = 2\pi\eta^3(\tau).$$

Consider the functions [W]

$$\phi_1(\tau) = \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)} = q^{-\frac{1}{48}} \prod_{j=1}^{\infty} (1 + q^{j-\frac{1}{2}}),$$

$$\phi_2(\tau) = \frac{\eta(\frac{\tau}{2})}{\eta(\tau)} = q^{-\frac{1}{48}} \prod_{j=1}^{\infty} (1 - q^{j-\frac{1}{2}}),$$

$$\phi_3(\tau) = \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)} = \sqrt{2} q^{\frac{1}{24}} \prod_{j=1}^{\infty} (1 + q^j).$$

We have

$$\phi_1\left(-\frac{1}{\tau}\right) = \phi_1(\tau), \quad \phi_2\left(-\frac{1}{\tau}\right) = \phi_3(\tau), \quad \phi_3\left(-\frac{1}{\tau}\right) = \phi_2(\tau),$$

$$\phi_1(\tau+1) = e^{-\frac{\pi i}{24}} \phi_2(\tau), \quad \phi_2(\tau+1) = e^{-\frac{\pi i}{24}} \phi_1(\tau), \quad \phi_3(\tau+1) = e^{\frac{\pi i}{12}} \phi_3(\tau).$$

Let

$$c_{\kappa,n} =$$

$$c_{p,\kappa,n} = (2\pi)^{\frac{p(p+1)}{\kappa}} e^{-\pi i \frac{p(3p-1)}{2\kappa}} e^{\pi i \frac{p+1}{2}} B_p\left(\frac{n+1}{\kappa}, -\frac{2p}{\kappa}, \frac{1}{\kappa}\right) \prod_{j=1}^p \left(1 - e^{2\pi i \frac{n+j}{\kappa}}\right).$$

Here, $B_p(\alpha, \beta, \gamma)$ is the Selberg integral.

Theorem 5.1. *We have ten series of identities,*

$$\begin{aligned}
 (2) \quad & u_{2p+2,p+1}(\lambda, \tau) = c_{2p+2,p+1} \vartheta_1(\lambda, \tau)^{p+1}, \\
 (3) \quad & u_{2p+3,p+1}(\lambda, \tau) = c_{2p+3,p+1} \eta(\tau)^{-\frac{3(p+1)}{2p+3}} \vartheta_1^{p+1}(\lambda, \tau) \theta_{1,0}(\lambda, \tau), \\
 (4) \quad & u_{2p+3,p+2}(\lambda, \tau) = c_{2p+3,p+2} \eta(\tau)^{-\frac{3(p+1)}{2p+3}} \vartheta_1^{p+1}(\lambda, \tau) \theta_{1,1}(\lambda, \tau), \\
 (5) \quad & u_{2p+4,p+2}(\lambda, \tau) = 2^{-\frac{2(p+1)}{2p+4}} c_{2p+4,p+2} (\phi_3(\tau) \eta(\tau)^{-1})^{\frac{4(p+1)}{2p+4}} \vartheta_1^{p+1}(\lambda, \tau) \theta_{2,1}^s(\lambda, \tau), \\
 (6) \quad & u_{2p+4,p+1}(\lambda, \tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+1)}{2p+4}} u_{2p+4,p+3}(\lambda, \tau) = \\
 & c_{2p+4,p+1} (\phi_2(\tau) \eta(\tau)^{-1})^{\frac{4(p+1)}{2p+4}} \vartheta_1^{p+1}(\lambda, \tau) (\theta_{2,0}(\lambda, \tau) - \theta_{2,2}(\lambda, \tau)), \\
 (7) \quad & u_{2p+4,p+1}(\lambda, \tau) + (-1)^p e^{2\pi i \frac{p(p+1)}{2p+4}} u_{2p+4,p+3}(\lambda, \tau) = \\
 & c_{2p+4,p+1} (\phi_1(\tau) \eta(\tau)^{-1})^{\frac{4(p+1)}{2p+4}} \vartheta_1^{p+1}(\lambda, \tau) (\theta_{2,0}(\lambda, \tau) + \theta_{2,2}(\lambda, \tau)), \\
 (8) \quad & u_{2p+6,p+1}(\lambda, \tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+1)}{2p+6}} u_{2p+6,p+5}(\lambda, \tau) = \\
 & 2^{\frac{3(p+1)}{2p+6}} c_{2p+6,p+1} (\phi_3(\tau) \eta(\tau))^{-\frac{6(p+1)}{2p+6}} \vartheta_1^{p+1}(\lambda, \tau) (\theta_{4,0}(\lambda, \tau) - \theta_{4,4}(\lambda, \tau)), \\
 (9) \quad & u_{2p+6,p+2}(\lambda, \tau) + (-1)^p e^{2\pi i \frac{p(p+2)}{2p+6}} u_{2p+6,p+4}(\lambda, \tau) = \\
 & c_{2p+6,p+2} (\phi_2(\tau) \eta(\tau))^{-\frac{6(p+1)}{2p+6}} \vartheta_1^{p+1}(\lambda, \tau) (\theta_{4,1}^s(\lambda, \tau) + \theta_{4,3}^s(\lambda, \tau)), \\
 (10) \quad & u_{2p+6,p+2}(\lambda, \tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+2)}{2p+6}} u_{2p+6,p+4}(\lambda, \tau) = \\
 & c_{2p+6,p+2} (\phi_1(\tau) \eta(\tau))^{-\frac{6(p+1)}{2p+6}} \vartheta_1^{p+1}(\lambda, \tau) (\theta_{4,1}^s(\lambda, \tau) - \theta_{4,3}^s(\lambda, \tau)), \\
 (11) \quad & u_{2p+8,p+2}(\lambda, \tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+2)}{2p+8}} u_{2p+8,p+6}(\lambda, \tau) = \\
 & c_{2p+8,p+2} \eta(\tau)^{-\frac{8(p+1)}{2p+8}} \vartheta_1^{p+1}(\lambda, \tau) (\theta_{6,1}^s(\lambda, \tau) - \theta_{6,5}^s(\lambda, \tau)).
 \end{aligned}$$

The integrals in Theorem 5.1 are appropriately called the elliptic Selberg integrals. The identity (2) appears in [FSV1] and in [FV1] for $p = 1$.

6. Differential equations

In Lemmas 6.1–6.3, let $'$ denote the derivative with respect to λ , let \cdot denote the derivative with respect to τ , and let

$$v(\lambda, \tau) = \vartheta_1^{p+1}(\lambda, \tau) \sum_{j=0}^{\kappa-2p-2} c_j(\tau) \theta_{\kappa-2p-2,j}^s(\lambda, \tau).$$

Lemma 6.1. *The function $v(\lambda, \tau)$ is a solution of the KZB-heat equation if and only if the differential equation*

$$(12) \quad \frac{\kappa}{p+1} \sum_{j=0}^{\kappa-2p-2} \left(\frac{d}{d\tau} c_j \right) \theta_{\kappa-2p-2,j}^s = (2p+2-\kappa) \frac{\dot{\vartheta}_1}{\vartheta_1} \sum_{j=0}^{\kappa-2p-2} c_j \theta_{\kappa-2p-2,j}^s \\ - 2 \sum_{j=0}^{\kappa-2p-2} c_j \dot{\theta}_{\kappa-2p-2,j}^s + \frac{1}{\pi i} \frac{\vartheta_1'}{\vartheta_1} \sum_{j=0}^{\kappa-2p-2} c_j (\theta_{\kappa-2p-2,j}^s)'$$

holds.

The proof of Lemma 6.1 uses the identities

$$(13) \quad 2\pi i(2p+2)(\vartheta_1^{p+1})(\lambda, \tau) = (\vartheta_1^{p+1})''(\lambda, \tau) + p(p+1)\rho'(\lambda, \tau)\vartheta_1^{p+1}(\lambda, \tau),$$

$$(14) \quad 2\pi i \kappa \dot{\theta}_{\kappa,m}^s(\lambda, \tau) = (\theta_{\kappa,m}^s)''(\lambda, \tau).$$

Proof of Lemma 6.1. Applying the differential operator $2\pi i \kappa \partial / \partial \tau$ to $v(\lambda, \tau)$ gives

$$2\pi i \kappa \left[(p+1) \dot{\vartheta}_1 \vartheta_1^p \sum_{j=0}^{\kappa-2p-2} c_j \theta_{\kappa-2p-2,j}^s + \vartheta_1^{p+1} \sum_{j=0}^{\kappa-2p-2} \left(\frac{d}{d\tau} c_j \right) \theta_{\kappa-2p-2,j}^s \right. \\ \left. + \vartheta_1^{p+1} \sum_{j=0}^{\kappa-2p-2} c_j \dot{\theta}_{\kappa-2p-2,j}^s \right].$$

Applying the differential operator $\partial^2 / \partial \lambda^2 + p(p+1)\rho'(\lambda, \tau)$ to $v(\lambda, \tau)$ gives

$$(\vartheta_1^{p+1})'' \sum_{j=0}^{\kappa-2p-2} c_j \theta_{\kappa-2p-2,j}^s + 2(p+1) \vartheta_1' \vartheta_1^p \sum_{j=0}^{\kappa-2p-2} c_j (\theta_{\kappa-2p-2,j}^s)' \\ + \vartheta_1^{p+1} \sum_{j=0}^{\kappa-2p-2} c_j (\theta_{\kappa-2p-2,j}^s)'' + p(p+1) \rho' \vartheta_1^{p+1} \sum_{j=0}^{\kappa-2p-2} c_j \theta_{\kappa-2p-2,j}^s.$$

Applying (13) and (14), we obtain the result. \square

Lemma 6.2. *If $v(\lambda, \tau)$ is a solution of the KZB-heat equation, then the functions $c_j(\tau)$ satisfy the differential equation*

$$\frac{\kappa}{p+1} \sum_{j=0}^{\kappa-2p-2} \left(\frac{d}{d\tau} c_j(\tau) \right) \theta_{\kappa-2p-2,j}(0, \tau) = \\ (2p+2-\kappa) \left(\frac{d}{d\tau} \ln \vartheta_1'(0, \tau) \right) \sum_{j=0}^{\kappa-2p-2} c_j(\tau) \theta_{\kappa-2p-2,j}(0, \tau) \\ + 2(\kappa-2p-3) \sum_{j=0}^{\kappa-2p-2} c_j(\tau) \frac{d}{d\tau} \theta_{\kappa-2p-2,j}(0, \tau).$$

Proof. For any fixed λ , equation (12) gives a differential equation for the functions $c_j(\tau)$. We take the limit of that equation as $\lambda \rightarrow 0$. In the ratio ϑ'_1/ϑ_1 , both the numerator and denominator tend to zero, so the limit of this term as $\lambda \rightarrow 0$ is equal to the limit of the ratio of the derivatives of the numerator and the denominator. The limit of the ratio $(\sum_{j=0}^{\kappa-2p-2} c_j(\theta_{\kappa-2p-2,j}^s)')/\vartheta_1$ is calculated in the same way, since each $\theta_{\kappa-2p-2,j}^s$ is a symmetric function and therefore $(\theta_{\kappa-2p-2,j}^s)'(0, \tau) = 0$. Then the result follows from (14). \square

Lemma 6.3. *If $v(\lambda, \tau)$ is a solution of the KZB-heat equation, then the functions $c_j(\tau)$ satisfy the differential equation*

$$\begin{aligned} \frac{\kappa}{p+1} \sum_{j=0}^{\kappa-2p-2} \left(\frac{d}{d\tau} c_j(\tau) \right) \theta_{\kappa-2p-2, \kappa-2p-2-j}(0, \tau) = \\ (2p+2-\kappa) \left(\frac{d}{d\tau} \ln \vartheta'_1(0, \tau) \right) \sum_{j=0}^{\kappa-2p-2} c_j(\tau) \theta_{\kappa-2p-2, \kappa-2p-2-j}(0, \tau) \\ + 2(\kappa-2p-3) \sum_{j=0}^{\kappa-2p-2} c_j(\tau) \frac{d}{d\tau} \theta_{\kappa-2p-2, \kappa-2p-2-j}(0, \tau). \end{aligned}$$

Proof. We take the limit of (12) as $\lambda \rightarrow \tau$. This limit is calculated in terms of the limit $\lambda \rightarrow 0$ using the formulas

$$\begin{aligned} \frac{\partial}{\partial z} \theta_1(z, \tau)|_{z=\lambda+\tau} &= e^{-2\pi i \lambda - \pi i \tau} (2\pi i \theta_1(\lambda, \tau) - \theta'_1(\lambda, \tau)), \\ \frac{\partial}{\partial z} \theta_1(\lambda + \tau, z)|_{z=\tau} &= e^{-2\pi i \lambda - \pi i \tau} \left(-\pi i \theta_1(\lambda, \tau) + \theta'_1(\lambda, \tau) - \dot{\theta}_1(\lambda, \tau) \right), \\ \frac{\partial}{\partial z} \theta_{\kappa, m}^s(z, \tau)|_{z=\lambda+\tau} &= e^{-\pi i \kappa \lambda - \pi i \kappa \frac{\tau}{2}} \left(-\pi i \kappa \theta_{\kappa, \kappa-m}^s(\lambda, \tau) + (\theta_{\kappa, \kappa-m}^s)'(\lambda, \tau) \right), \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial z} \theta_{\kappa, m}^s(\lambda + \tau, z)|_{z=\tau} = \\ e^{-\pi i \kappa \lambda - \pi i \kappa \frac{\tau}{2}} \left(\pi i \frac{\kappa}{2} \theta_{\kappa, \kappa-m}^s(\lambda, \tau) - (\theta_{\kappa, \kappa-m}^s)'(\lambda, \tau) + \dot{\theta}_{\kappa, \kappa-m}^s(\lambda, \tau) \right). \end{aligned}$$

It is straightforward to calculate the limit of the left hand side. Using the above formulas, the limit as $\lambda \rightarrow \tau$ of the right hand side is equal to the limit as $\lambda \rightarrow 0$ of the expression

$$\begin{aligned} e^{\pi i (2p+2-\kappa)(\lambda+\frac{\tau}{2})} \left((2p+2-\kappa) \frac{\vartheta'_1}{\vartheta_1} \sum_{j=0}^{\kappa-2p-2} c_j \theta_{\kappa-2p-2, \kappa-2p-2-j}^s \right. \\ \left. - 2 \sum_{j=0}^{\kappa-2p-2} c_j \dot{\theta}_{\kappa-2p-2, \kappa-2p-2-j}^s + \frac{1}{\pi i} \frac{\vartheta'_1}{\vartheta_1} \sum_{j=0}^{\kappa-2p-2} c_j (\theta_{\kappa-2p-2, \kappa-2p-2-j}^s)' \right). \end{aligned}$$

This limit is calculated using L'Hôpital's rule. \square

7. Identities for theta functions

In the next section, we give the proofs of the integral identities in Theorem 5.1. We will use the following results.

Lemma 7.1. *We have $\theta_{2,1}^s(\lambda) = \vartheta_1(\lambda + 1/2)$.*

Lemma 7.1 is proved by comparing the Fourier series expansions of the functions.

Corollary 7.2. *We have $2\theta_{2,1}(0) = \eta(\tau)\phi_3(\tau)^2$.*

Lemma 7.3. *Let*

$$f_1(\tau) = \frac{\theta_{4,1}(0) - \theta_{4,3}(0)}{\eta(\tau)}, \quad f_2(\tau) = \frac{\theta_{4,1}(0) + \theta_{4,3}(0)}{\eta(\tau)}, \quad f_3(\tau) = \frac{\theta_{4,0}(0) - \theta_{4,4}(0)}{\sqrt{2}\eta(\tau)}.$$

Then $f_1(\tau) = \phi_1(\tau)^{-1}$, $f_2(\tau) = \phi_2(\tau)^{-1}$, $f_3(\tau) = \phi_3(\tau)^{-1}$.

The proof of Lemma 7.3 is based on the following result.

Lemma 7.4. [W] *Suppose $g_1(\tau)$, $g_2(\tau)$, and $g_3(\tau)$ are holomorphic functions on the upper half plane \mathbb{C}_+ satisfying the following conditions.*

P1. *The functions g_1 , g_2 , and g_3 can be written in the forms*

$$g_1(\tau) = q^{-\frac{a}{48}} \sum_{j=0}^{\infty} a_j q^{\frac{j}{2}}, \quad g_2(\tau) = q^{-\frac{a}{48}} \sum_{j=0}^{\infty} (-1)^j a_j q^{\frac{j}{2}}, \quad g_3(\tau) = q^{\frac{a}{24}} \sum_{j=0}^{\infty} b_j q^j,$$

where a is an integer, $a_j, b_j \in \mathbb{C}$, and $a_0 = 1$.

P2. *The functions g_1 , g_2 , and g_3 have the modular properties*

$$g_1\left(-\frac{1}{\tau}\right) = g_1(\tau), \quad g_2\left(-\frac{1}{\tau}\right) = g_3(\tau), \quad g_3\left(-\frac{1}{\tau}\right) = g_2(\tau).$$

Then $g_i(\tau) = \phi_i(\tau)^a$, for $i = 1, 2, 3$.

Proof of Lemma 7.3. We have

$$\begin{aligned} \theta_{4,1}(0) &= \sum_{j \in \mathbb{Z}} q^{4(j+\frac{1}{8})^2} = q^{\frac{1}{16}} \sum_{j \in \mathbb{Z}} q^{\frac{1}{2}(8j^2+2j)}, \\ \theta_{4,3}(0) &= \sum_{j \in \mathbb{Z}} q^{4(j+\frac{3}{8})^2} = q^{\frac{1}{16}} \sum_{j \in \mathbb{Z}} q^{\frac{1}{2}(8j^2+6j+1)}, \\ \theta_{4,0}(0) &= \sum_{j \in \mathbb{Z}} q^{4j^2} = \sum_{j \in \mathbb{Z}} q^{(2j)^2}, \\ \theta_{4,4}(0) &= \sum_{j \in \mathbb{Z}} q^{4(j+\frac{1}{2})^2} = \sum_{j \in \mathbb{Z}} q^{(2j+1)^2}. \end{aligned}$$

Hence, the functions

$$\begin{aligned} \tilde{f}_1(\tau) &= q^{-\frac{1}{24}} (\theta_{4,1}(0) - \theta_{4,3}(0)), \\ \tilde{f}_2(\tau) &= q^{-\frac{1}{24}} (\theta_{4,1}(0) + \theta_{4,3}(0)), \\ \tilde{f}_3(\tau) &= 2^{-\frac{1}{2}} q^{-\frac{1}{24}} (\theta_{4,0}(0) - \theta_{4,4}(0)) \end{aligned}$$

are holomorphic functions on \mathbb{C}_+ with the property P1 for $a = -1$ and $g_i = \tilde{f}_i$. Let $y(q) = \sum_{j=1}^{\infty} c_j q^j$ be defined by the condition $1 + y(q) = q^{-1/24} \eta(\tau)$. Then for $i = 1, 2, 3$,

$$f_i(\tau) = \frac{\tilde{f}_i(\tau)}{1 + y(q)} = \tilde{f}_i(\tau)(1 - y(q) + y(q)^2 + \dots)$$

are holomorphic functions on \mathbb{C}_+ with the property P1 for $a = -1$ and $g_i = f_i$. One checks that f_1, f_2 and f_3 have the property P2 using the modular properties of $\theta_{4,n}(0)$ and $\eta(\tau)$. \square

Lemma 7.5. *We have $\theta_{6,1}(0) - \theta_{6,5}(0) = \eta(\tau)$.*

Lemma 7.5 is proved by comparing the infinite series expansions of the functions.

8. The proof of Theorem 5.1

Proof of (2). For $\kappa = 2p + 2$, the space of conformal blocks is one-dimensional. The right hand side of (2) is a solution of (1) with the properties (i)-(iv) [FV1]. According to Theorem 3.1, the left hand side also has these properties. Thus the two functions are proportional. The coefficient of proportionality is calculated by comparing the leading terms of ϑ_1^{p+1} and $u_{\kappa,p+1}$ in the limit as $\tau \rightarrow i\infty$. The leading term of ϑ_1^{p+1} is $(-i)^{p+1} q^{(p+1)/8} (e^{\pi i \lambda} - e^{-\pi i \lambda})^{p+1}$. Let $dt = dt_1 \dots dt_p$. The leading term of $u_{\kappa,p+1}$ is

$$\begin{aligned} & \int_{\Delta_p} \prod_{j=1}^p \left(\frac{e^{\pi i t_j} - e^{-\pi i t_j}}{2\pi e^{\frac{\pi i}{2}}} \right)^{-\frac{2p}{2p+2}-1} \prod_{1 \leq j < k \leq p} \left(\frac{e^{\pi i(t_j - t_k)} - e^{-\pi i(t_j - t_k)}}{2\pi e^{\frac{\pi i}{2}}} \right)^{\frac{2}{2p+2}} \\ & \left(\left(\prod_{j=1}^p \frac{e^{\pi i(\lambda - t_j)} - e^{-\pi i(\lambda - t_j)}}{e^{\pi i \lambda} - e^{-\pi i \lambda}} \right) q^{\frac{(p+1)^2}{4(2p+2)}} e^{\pi i(p+1)(\lambda + \frac{2}{2p+2} \sum_{j=1}^p t_j)} \right. \\ & \left. + (-1)^{p+1} \left(\prod_{j=1}^p \frac{e^{\pi i(\lambda + t_j)} - e^{-\pi i(\lambda + t_j)}}{e^{\pi i \lambda} - e^{-\pi i \lambda}} \right) q^{\frac{(p+1)^2}{4(2p+2)}} e^{\pi i(p+1)(-\lambda + \frac{2}{2p+2} \sum_{j=1}^p t_j)} \right) dt. \end{aligned}$$

The above expression is equal to

$$(2\pi e^{\frac{\pi i}{2}})^{\frac{p(p+1)}{2p+2}+p} e^{-\pi i(\frac{2p^2}{2p+2}+p)} q^{\frac{p+1}{8}} (e^{\pi i \lambda} - e^{-\pi i \lambda})^{-p} \sum_{l=0}^p I_l (e^{\pi i(2l+1)\lambda} - e^{-\pi i(2l+1)\lambda}),$$

where

$$\begin{aligned} I_l = & \int_{\Delta_p} f_l(t_1, \dots, t_p) \prod_{j=1}^p e^{2\pi i t_j (\frac{p+2}{2p+2} + \frac{1}{2})} (1 - e^{2\pi i t_j})^{-\frac{2p}{2p+2}-1} \times \\ & \prod_{1 \leq j < k \leq p} (e^{2\pi i t_j} - e^{2\pi i t_k})^{\frac{2}{2p+2}} dt, \end{aligned}$$

for some function f_l , symmetric in the variables t_1, \dots, t_p . Comparing the coefficients of $e^{\pi i(p+1)\lambda}$ in the leading terms, we find that

$$u_{\kappa, p+1} = i^{p+1} (2\pi e^{\frac{\pi i}{2}})^{\frac{p(p+1)}{2p+2} + p} e^{-\pi i(\frac{2p^2}{2p+2} + p)} I_p \vartheta_1^{p+1}.$$

To complete the proof, it remains to compute I_p . It is not difficult to show that $f_p(t_1, \dots, t_p) = \prod_{j=1}^p e^{-\pi i t_j}$. Let $x_j = e^{2\pi i t_j}$. Let $\tilde{\Delta}_p$ be the image of Δ_p under the map $t_j \mapsto x_j$. We have

$$I_p = (2\pi i)^{-p} \int_{\tilde{\Delta}_p} \prod_{j=1}^p x_j^{\frac{p+2}{2p+2}-1} (1-x_j)^{-\frac{2p}{2p+2}-1} \prod_{1 \leq j < k \leq p} (x_j - x_k)^{\frac{2}{2p+2}} dx.$$

Applying the Stokes theorem, we deform the contour $\tilde{\Delta}_p$ to get

$$I_p = (2\pi i)^{-p} \prod_{j=1}^p (e^{2\pi i \frac{j+p+1}{2p+2}} - 1) \times \\ \int_{\Delta_p} \prod_{j=1}^p x_j^{\frac{p+2}{2p+2}-1} (1-x_j)^{-\frac{2p}{2p+2}-1} \prod_{1 \leq j < k \leq p} (x_j - x_k)^{\frac{2}{2p+2}} dx.$$

Observe that

$$\int_{\Delta_p} \prod_{j=1}^p x_j^{\frac{p+2}{2p+2}-1} (1-x_j)^{-\frac{2p}{2p+2}-1} \prod_{1 \leq j < k \leq p} (x_j - x_k)^{\frac{2}{2p+2}} dx$$

is the Selberg integral $B_p((p+2)/(2p+2), -2p/(2p+2), 1/(2p+2))$. This completes the proof. \square

Proof of (3) and (4). Let $\kappa = 2p+3$. Then any solution of (1) with the properties (i)-(iv) has the form

$$v(\lambda, \tau) = \vartheta_1^{p+1}(\lambda, \tau) (c_0(\tau) \theta_{1,0}(\lambda, \tau) + c_1(\tau) \theta_{1,1}(\lambda, \tau)).$$

Let A be the transformation introduced in section 4. By Proposition 4.1, $Av(\lambda, \tau) = (-1)^{p+1} \vartheta_1^{p+1}(\lambda, \tau) (c_0(\tau) \theta_{1,0}(\lambda, \tau) - c_1(\tau) \theta_{1,1}(\lambda, \tau))$ is also a solution. Hence, for $j = 0$ or 1 , the function $v_j(\lambda, \tau) = c_j(\tau) \vartheta_1^{p+1}(\lambda, \tau) \theta_{1,j}(\lambda, \tau)$ gives a solution too. Moreover, $Av_j = (-1)^{p+1+j} v_j$. By Theorem 3.2, the integrals $u_{\kappa, p+1}$ and $u_{\kappa, p+2}$ span the space of conformal blocks. By Lemma 4.3, $Au_{\kappa, p+1} = (-1)^{p+1} u_{\kappa, p+1}$ and $Au_{\kappa, p+2} = (-1)^p u_{\kappa, p+2}$. So for $j = 0$ or 1 , the integral $u_{\kappa, p+1+j}$ is proportional to v_j . By Lemma 6.2, $c_j(\tau)$ must satisfy the differential equation

$$\frac{\kappa}{p+1} \left(\frac{d}{d\tau} c_j(\tau) \right) \theta_{1,j}(0, \tau) = - \left(\frac{d}{d\tau} \ln \vartheta_1'(0, \tau) \right) c_j(\tau) \theta_{1,j}(0, \tau).$$

The function $c_j(\tau) = ((2\pi)^{-1} \vartheta_1'(0, \tau))^{-(p+1)/\kappa} = \eta(\tau)^{-3(p+1)/\kappa}$ is a solution of this equation for $j = 0$ and $j = 1$. The coefficients of proportionality are computed in the limit as $\tau \rightarrow i\infty$, cf. the proof of (2). This completes the proof. \square

Proof of (5), (6), and (7). Let $\kappa = 2p + 4$. Let

$$v_j(\lambda, \tau) = \vartheta_1^{p+1}(\lambda, \tau) \theta_{2,j}^s(\lambda, \tau), \quad 0 \leq j \leq 2.$$

Any solution of (1) with the properties (i)-(iv) has the form $v(\lambda, \tau) = \sum_{j=0}^2 c_j(\tau) v_j(\lambda, \tau)$. Let A be the transformation in section 4. By Proposition 4.1, $Av = \sum_{j=0}^2 (-1)^{p+j+1} c_j v_j$ is also a solution. Hence the function $c_1 v_1$ gives a solution too. Moreover, it is an eigenvector of A with eigenvalue $(-1)^p$. By Theorem 3.2, the space of conformal blocks is three-dimensional with spanning set $\{u_{\kappa,n} \mid p+1 \leq n \leq p+3\}$. According to Lemma 4.3, the eigenspace of A corresponding to the eigenvalue $(-1)^p$ is one-dimensional and is spanned by $u_{\kappa,p+2}$. It follows that $u_{\kappa,p+2}$ is proportional to $c_1 v_1$. By Lemma 6.2, $c_1(\tau)$ must satisfy the differential equation

$$\begin{aligned} \frac{\kappa}{p+1} \left(\frac{d}{d\tau} c_1(\tau) \right) \theta_{2,1}(0, \tau) = \\ - 2 \left(\frac{d}{d\tau} \ln \vartheta_1'(0, \tau) \right) c_1(\tau) \theta_{2,1}(0, \tau) + 2c_1(\tau) \frac{d}{d\tau} \theta_{2,1}(0, \tau). \end{aligned}$$

The function $c_1(\tau) = (4\pi\theta_{2,1}(0, \tau)\vartheta_1'(0, \tau)^{-1})^{2(p+1)/\kappa}$ is a solution of the above equation. By Corollary 7.2, $c_1(\tau) = (\phi_3(\tau)\eta(\tau)^{-1})^{4(p+1)/\kappa}$. The coefficient of proportionality is computed in the limit $\tau \rightarrow i\infty$, cf. the proof of (2). This proves (5). To prove (6), we apply the transformation S to both sides of (5). To prove (7), we apply the transformation T to both sides of (6). This completes the proof. \square

Proof of (8), (9), and (10). Let $\kappa = 2p + 6$. Let

$$v_j(\lambda, \tau) = \vartheta_1^{p+1}(\lambda, \tau) \theta_{4,j}^s(\lambda, \tau), \quad 0 \leq j \leq 4.$$

Any solution of (1) with the properties (i)-(iv) has the form $v(\lambda, \tau) = \sum_{j=0}^4 c_j(\tau) v_j(\lambda, \tau)$. Let A and B be the transformations in section 4. By Proposition 4.1, $Av = \sum_{j=0}^4 (-1)^{p+j+1} c_j v_j$ is also a solution. Hence the function $c_0 v_0 + c_2 v_2 + c_4 v_4$ gives a solution too. Moreover, $B(c_0 v_0 + c_2 v_2 + c_4 v_4) = (-1)^{p+1} (c_4 v_0 + c_2 v_2 + c_0 v_4)$ is also a solution. So there exists a solution of the form $c(\tau)(v_0 - v_4)$. It is an eigenvector of A with eigenvalue $(-1)^{p+1}$ and an eigenvector of B with eigenvalue $(-1)^p$. We show that the subspace of conformal blocks with this property is one-dimensional. By Theorem 3.2, the space of conformal blocks is five-dimensional with spanning set $\{u_{\kappa,n} \mid p+1 \leq n \leq p+5\}$. By Lemma 4.3, the eigenspace of A corresponding to the eigenvalue $(-1)^{p+1}$ is three-dimensional and is spanned by $u_{\kappa,p+1}$, $u_{\kappa,p+3}$, and $u_{\kappa,p+5}$. By Lemma 4.3, the transformation B preserves the subspace $\langle u_{\kappa,p+1}, u_{\kappa,p+3}, u_{\kappa,p+5} \rangle$. The

matrix of B restricted to this subspace is

$$\begin{pmatrix} 0 & 0 & -e^{2\pi i \frac{p(p+5)}{\kappa}} \\ 0 & (-1)^{p+1} & 0 \\ -e^{2\pi i \frac{p(p+1)}{\kappa}} & 0 & 0 \end{pmatrix}.$$

Thus the restriction of B to $\langle u_{\kappa,p+1}, u_{\kappa,p+3}, u_{\kappa,p+5} \rangle$ has eigenvalues $(-1)^p$ and $(-1)^{p+1}$ of multiplicities 1 and 2, respectively. The eigenspace corresponding to the eigenvalue $(-1)^p$ is spanned by the vector $u_{\kappa,p+1} + (-1)^{p+1} e^{2\pi i p(p+1)/\kappa} u_{\kappa,p+5}$. It follows that this vector is proportional to $c(\tau)(v_0 - v_4)$. By Lemma 6.2, $c(\tau)$ must satisfy the differential equation

$$\begin{aligned} & \frac{\kappa}{p+1} \left(\frac{d}{d\tau} c(\tau) \right) (\theta_{4,0}(0, \tau) - \theta_{4,4}(0, \tau)) = \\ & -4 \left(\frac{d}{d\tau} \ln \vartheta'_1(0, \tau) \right) c(\tau) (\theta_{4,0}(0, \tau) - \theta_{4,4}(0, \tau)) + 6c(\tau) \frac{d}{d\tau} (\theta_{4,0}(0, \tau) - \theta_{4,4}(0, \tau)). \end{aligned}$$

The function $c(\tau) = ((2\pi)^2(\theta_{4,0}(0, \tau) - \theta_{4,4}(0, \tau))^3 \vartheta'_1(0, \tau)^{-2})^{2(p+1)/\kappa}$ is a solution of the above equation. By Lemma 7.3, we have

$$c(\tau) = 2^{3(p+1)/\kappa} (\phi_3(\tau) \eta(\tau))^{-6(p+1)/\kappa}.$$

The coefficient of proportionality is computed as in the proof of (2). This proves (8). To prove (9) and (10), apply the transformations S and TS , respectively, to both sides of (8). This completes the proof. \square

Proof of (11). Let $\kappa = 2p + 8$. Let $v_j(\lambda, \tau) = \vartheta_1^{p+1}(\lambda, \tau) \theta_{6,j}^s(\lambda, \tau)$, $0 \leq j \leq 6$. Any solution of (1) with the properties (i)-(iv) has the form

$$v(\lambda, \tau) = \sum_{j=0}^6 c_j(\tau) v_j(\lambda, \tau).$$

Let A and B be as in section 4. By Proposition 4.1, $Av = \sum_{j=0}^6 (-1)^{p+j+1} c_j v_j$ is also a solution. Hence the function $c_1 v_1 + c_3 v_3 + c_5 v_5$ gives a solution too. The function $B(c_1 v_1 + c_3 v_3 + c_5 v_5) = (-1)^{p+1} (c_5 v_1 + c_3 v_3 + c_1 v_5)$ also gives a solution. So there is a solution of the form $c(\tau)(v_1 - v_5)$ which is an eigenvector of A and B with eigenvalue $(-1)^p$ under both transformations. We show that the subspace of conformal blocks with this property is one-dimensional. By Theorem 3.2, the space of conformal blocks is seven-dimensional with spanning set $\{u_{\kappa,n} \mid p+1 \leq n \leq p+7\}$. By Lemma 4.3, the three-dimensional eigenspace of A corresponding to the eigenvalue $(-1)^p$ is spanned by $u_{\kappa,p+2}$, $u_{\kappa,p+4}$, and $u_{\kappa,p+6}$. The matrix of B restricted to $\langle u_{\kappa,p+2}, u_{\kappa,p+4}, u_{\kappa,p+6} \rangle$ is

$$\begin{pmatrix} 0 & 0 & -e^{2\pi i \frac{p(p+6)}{\kappa}} \\ 0 & (-1)^{p+1} & 0 \\ -e^{2\pi i \frac{p(p+2)}{\kappa}} & 0 & 0 \end{pmatrix}.$$

Thus, B has eigenvalues $(-1)^p$ and $(-1)^{p+1}$ of multiplicities 1 and 2, respectively. The eigenspace corresponding to the eigenvalue $(-1)^p$ is spanned by the vector $u_{\kappa,p+2} + (-1)^{p+1} e^{2\pi i p(p+2)/\kappa} u_{\kappa,p+6}$. So this vector is proportional to $c(\tau)(v_1 - v_5)$. By Lemma 6.2, $c(\tau)$ must be a solution of the differential equation

$$\begin{aligned} & \frac{\kappa}{p+1} \left(\frac{d}{d\tau} c(\tau) \right) (\theta_{6,1}(0, \tau) - \theta_{6,5}(0, \tau)) = \\ & -6 \left(\frac{d}{d\tau} \ln \vartheta_1'(0, \tau) \right) c(\tau) (\theta_{6,1}(0, \tau) - \theta_{6,5}(0, \tau)) + 10c(\tau) \frac{d}{d\tau} (\theta_{6,1}(0, \tau) - \theta_{6,5}(0, \tau)). \end{aligned}$$

The function $c(\tau) = ((2\pi)^3(\theta_{6,1}(0, \tau) - \theta_{6,5}(0, \tau))^5 \vartheta_1'(0, \tau)^{-3})^{2(p+1)/\kappa}$ is a solution of the preceding equation. By Lemma 7.5, $c(\tau) = (\eta(\tau))^{-8(p+1)/\kappa}$. The coefficient of proportionality is computed as in the proof of (2). Notice that the solution in (11) is invariant with respect to the action of the modular group. \square

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