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ON A QUESTION OF LOUIS NIRENBERG

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Abstract. This note proves that if A, B are $\mathcal{C}^\infty$ real vector fields in an open set $\Omega \subset \mathbb{R}^3$ such that A, B and $[A, B]$ are linearly independent then, given any $\mathcal{C}^\infty$ real vector field C in Ω and any function $\varphi \in \mathcal{C}^\infty(\Omega)$, the second order operator $L = AB + C + \varphi$ is locally solvable at every point of Ω. The result can be extended to first-order real pseudodifferential operators with simple real characteristics.

1. Statement and Proof of Theorem 1

Theorem 1. *Let A, B be $\mathcal{C}^\infty$ real vector fields in an open set $\Omega \subset \mathbb{R}^3$ such that A, B and $[A, B]$ are linearly independent. Let the $\mathcal{C}^\infty$ real vector field C in Ω and the function $\varphi \in \mathcal{C}^\infty(\Omega)$ be arbitrary and call L the second order operator $AB + C + \varphi$. Given any point $x^\circ \in \Omega$ and any number $\varepsilon > 0$ there is an open neighborhood $U_{x^\circ,\varepsilon} \subset \Omega$ of x° with the following property: there is a bounded linear operator $G_{x^\circ,\varepsilon} : H^{-1}(U_{x^\circ,\varepsilon}) \longrightarrow H^{-1}(U_{x^\circ,\varepsilon})$ with norm $\leq \varepsilon$ and such that $LG_{x^\circ,\varepsilon} f = f$ in $U_{x^\circ,\varepsilon}$ for every $f \in H^{-1}(U_{x^\circ,\varepsilon})$.*

In the statement $H^{-1}(U_{x^\circ})$ denotes the standard Sobolev space. We indicate in the second section of this note how right-inverses of L acting from the Sobolev space $H^s(U_{x^\circ})$ to itself can be found, for each $s \in \mathbb{R}$ (after some contraction of U_{x° about x°). The proof will also make clear under which hypotheses one can get right-inverses acting from $H^s(U_{x^\circ})$ to $H^{s+1}(U_{x^\circ})$ (cf. Corollary 1).

Theorem 1 answers a question of Louis Nirenberg originating in joint work, currently in progress, with I. Ekeland. At the microlocal level Theorem 1 is closely related to the works [Ha1], [Ha2].

Below we use systematically the notation $\|\cdot\|$ and $(\cdot,\cdot)$ for the L^2 norm and the L^2 inner product, respectively; but we shall use the notation $\|\cdot\|_s$ for the norm in the Sobolev space H^s, $s \neq 0$. The letters $K, K_1, \ldots$ will denote various constants that depend solely on the (pseudo)differential operators being considered.

Proof. Call L^* the adjoint of L. Let $x^\circ \in \Omega$ be arbitrary; actually we take it to be the origin in the coordinates x_i, $i = 1, 2, 3$. We select a number $\delta > 0$ such that the closure of the ball $\mathfrak{B}_\delta = \{x \in \mathbb{R}^3;\ |x| < \delta\}$ is contained in Ω.

By our hypothesis we can write

$$C = \alpha(x)\,A + \beta(x)\,B + \gamma(x)\,T \tag{1.1}$$

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where $T = [A, B]$ and $\alpha, \beta, \gamma \in \mathcal{C}^\infty(\Omega)$. For most vector fields C the claim in Theorem 1 is a consequence of the following

Lemma 1. *Under the hypotheses of Theorem 1, if $\gamma(0) \neq -\frac{1}{2}$ then there are constants $K, \delta > 0$ such that*

$$\|ABu\| + \|BAu\| + \|Tu\| \leq K \|L^* u\| \tag{1.2}$$

for all $u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta)$.

Proof. Below, given two quadratic functionals $Q_1(u)$ and $Q_2(u)$, we shall write $Q_1(u) \cong Q_2(u)$ if to each $\varepsilon > 0$ there is $\delta > 0$ such that

$$|Q_1(u) - Q_2(u)| \leq \varepsilon \left(\|ABu\|^2 + \|BAu\|^2 + \|Tu\|^2 \right)$$

for all $u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta)$.

Since the origin is not a critical point of the vector fields A and B the following is true:

- $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, for all $u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta)$,

$$\|u\| \leq \varepsilon \|Bu\| \leq \varepsilon^2 \|ABu\|, \ \|u\| \leq \varepsilon \|Au\| \leq \varepsilon^2 \|BAu\|. \tag{1.3}$$

We have

$$L^* = BA - \gamma(x) T + p(x) A + q(x) B + r(x) + \overline{\varphi}(x)$$

with $p, q, r \in \mathcal{C}^\infty(\Omega)$. It follows at once from (1.3) that

$$\|L^* u\|^2 \cong \|(BA - \gamma T) u\|^2. \tag{1.4}$$

We use the fact that

$$\|(BA - \gamma T) u\|^2 = \|BAu\|^2 + \|\gamma Tu\|^2 - 2\Re e (BAu, \gamma Tu).$$

and

$$\|(BA - \gamma T) u\|^2 = \|(AB - (1 + \gamma) T) u\|^2 =$$
$$\|ABu\|^2 + \|((1 + \gamma) T) u\|^2 - 2\Re e (ABu, (1 + \gamma) Tu)$$

to derive

$$\begin{gathered} \|(BA - \gamma T) u\|^2 = \\ \frac{1}{2} \|ABu\|^2 + \frac{1}{2} \|BAu\|^2 + \frac{1}{2} \|\gamma Tu\|^2 + \frac{1}{2} \|(1 + \gamma) Tu\|^2 \\ -\Re e ((BAu, \gamma Tu) + (ABu, \gamma Tu)) - \Re e (ABu, Tu). \end{gathered} \tag{1.5}$$

We claim that

$$(BAu, \gamma Tu) + (\gamma Tu, ABu) \cong 0. \tag{1.6}$$

Indeed,

$$\begin{gathered}(BAu, \gamma Tu) = (Au, (B^* + B)(\gamma Tu)) - (Au, \gamma TBu) - (Au, [B, \gamma T] u) = \\ (Au, (B^* + B)(\gamma Tu)) - (u, (A^* + A)(\gamma TBu)) - (Au, [B, \gamma T] u) + (u, A(\gamma TBu)) = \\ (Au, (B^* + B)(\gamma Tu)) - (T^* (\gamma (A^* + A) u), Bu) - (Au, [B, \gamma T] u) \\ + \left([A, \gamma T]^* u, Bu\right) + \left((\gamma T)^* u, ABu\right) = \\ (Au, (B^* + B)(\gamma Tu)) - (T^* (\gamma (A^* + A) u), Bu) - (Au, [B, \gamma T] u) \\ + \left([A, \gamma T]^* u, Bu\right) + \left(\left((\gamma T)^* + \gamma T\right) u, ABu\right) - (\gamma Tu, ABu).\end{gathered}$$

Putting (1.6) into (1.5) yields

$$\begin{aligned}2\|(BA - \gamma T) u\|^2 \cong \|ABu\|^2 + \|BAu\|^2 + \|\gamma Tu\|^2 \\ + \|(1 + \gamma) Tu\|^2 - 2\Re e (ABu, Tu).\end{aligned} \tag{1.7}$$

We have

$$(ABu, [A, B] u) \cong -(Bu, [A, B] Au) \cong (B [A, B] u, Au) \cong -([A, B] u, BAu)$$

and therefore

$$2\Re e (ABu, [A, B] u) \cong \|[A, B] u\|^2. \tag{1.8}$$

Combining (1.7) and (1.8) yields

$$2\|(BA - \gamma T) u\|^2 \cong \|ABu\|^2 + \|BAu\|^2 + 2\int \gamma (1 + \gamma) |Tu|^2 \, dx.$$

But for any $0 < \theta < 1$,

$$\frac{1}{2}(1 - \theta)\|(AB - BA) u\|^2 \leq (1 - \theta)\|ABu\|^2 + (1 - \theta)\|BAu\|^2$$

whence

$$\begin{gathered}\|ABu\|^2 + \|BAu\|^2 + 2\int \gamma (1 + \gamma) |Tu|^2 \, dx \geq \\ \theta\left(\|ABu\|^2 + \|BAu\|^2\right) + 2\int \left(\gamma (1 + \gamma) + \frac{1}{4}(1 - \theta)\right) |Tu|^2 \, dx\end{gathered}$$

The hypothesis $\gamma(0) \neq -\frac{1}{2}$ is equivalent to

$$\gamma(0)(1 + \gamma(0)) + \frac{1}{4} > 0.$$

We can find θ and $\delta > 0$ such that

$$\forall x \in \mathfrak{B}_\delta,\ \gamma(x)(1 + \gamma(x)) + \frac{1}{4}(1 - \theta) \geq \theta$$

and therefore such that

$$\begin{gathered}\|ABu\|^2 + \|BAu\|^2 + 2\int \gamma (1 + \gamma) |Tu|^2 \, dx \geq \\ \theta\left(\|ABu\|^2 + \|BAu\|^2 + \|Tu\|^2\right).\end{gathered}$$

Combining this with (1.4) and possibly further reducing δ yields (1.2). □

Since A, B, C are linearly independent we see that (1.2) has the following consequence

$$\|u\|_1 \leq K_1 \|L^* u\|, \; u \in \mathcal{C}_c^\infty(\Gamma_\delta), \tag{1.9}$$

where $\|\cdot\|_1$ is the norm in the Sobolev space $H^1(\mathfrak{B}_\delta)$. We may state:

Corollary 1. *Suppose the hypotheses of Theorem 1 satisfied and $\gamma(0) \neq -\frac{1}{2}$. Then, if the number $\delta > 0$ is sufficiently small there is a bounded linear operator $G_\delta : H^{-1}(\mathfrak{B}_\delta) \longrightarrow L^2(\mathfrak{B}_\delta)$ such that $LG_\delta f = f$ for every $f \in H^{-1}(\mathfrak{B}_\delta)$.*

It remains to prove Theorem 1 when $\gamma(0) = -\frac{1}{2}$. To simplify notation it is convenient to assume $A = -A^*$; to achieve this it suffices to choose the local coordinates x_i $(i = 1, 2, 3)$ in such a way that $A = \frac{\partial}{\partial x_1}$.

Still under the hypothesis that $\gamma(0) \neq -\frac{1}{2}$ we apply (1.9) with Au substituted for u, thus obtaining, for all $u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta)$,

$$\|u\|_1 \leq 2\delta \|Au\|_1 \leq 4\delta K \|L^* A^* u\|.$$

Let D be any first-order linear differential operator in Ω with smooth coefficients and let D^* denote its formal adjoint. We derive from the preceding inequality:

$$\|u\|_1 \leq 4\delta K_1 \|L^* A^* u + D^* u\| + \delta K_2 \|u\|_1$$

whence, provided $\delta K_2 \leq \frac{1}{2}$,

$$\|u\|_1 \leq 8\delta K_1 \|L^* A^* u + D^* u\|, \; u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta). \tag{1.10}$$

This last inequality has the following implication:

Corollary 2. *Suppose the hypotheses of Theorem 1 satisfied and $\gamma(0) \neq -\frac{1}{2}$. Let D be any first-order linear differential operator in Ω with smooth coefficients. Then, to each number $\varepsilon > 0$ there is a number $\delta > 0$ and a bounded linear operator $G_{\varepsilon,D} : H^{-1}(\mathfrak{B}_\delta) \longrightarrow L^2(\mathfrak{B}_\delta)$ whose norm does not exceed ε and which is such that $(AL + D) G_{\varepsilon,D} f = f$ for every $f \in H^{-1}(\mathfrak{B}_\delta)$.*

At this juncture we assume $\gamma(0) = -\frac{1}{2}$. We form

$$(AB + C + \varphi) A = A (AB - [A, B] + C + \varphi) - [A, C + \varphi].$$

and we apply Corollary 2 with $L = AB - [A, B] + C + \varphi$ and $D = -[A, C + \varphi]$. This is permitted since, at the origin,

$$-[A, B] + C = -\frac{3}{2} T \mod (A, B).$$

Corollary 2 states that if $\delta > 0$ is sufficiently small then

$$(AB + C + \varphi) A G_{\varepsilon,D} f = f$$

for every $f \in H^{-1}(\mathfrak{B}_\delta)$. To complete the proof of Theorem 1 it suffices to observe that $AG_{\varepsilon,D}$ is a bounded linear operator $H^{-1}(\mathfrak{B}_\delta) \longrightarrow H^{-1}(\mathfrak{B}_\delta)$ whose norm does not exceed $\varepsilon \|A\|$ where $\|A\|$ is the norm of the operator $A : L^2(\mathfrak{B}_\delta) \longrightarrow H^{-1}(\mathfrak{B}_\delta)$.

Remark 1. *Inspection of the proof of Theorem 1 shows that the requirement that C be real can be slightly weakened: for instance the coefficients α and β in (1.1) need not be real. It is also clear that the regularity requirements on all the coefficients can be weakened, to $\mathcal{C}^3$ and possibly further.*

2. Further Remarks

2.1. Meaning of the condition on $\gamma(0)$. The meaning of the value $\gamma(0) = -\frac{1}{2}$ (cf. Corollaries 1, 2) becomes clearer if we write $AB + C = \frac{1}{2}(AB + BA) + C + \frac{1}{2}T$. The best way to understand this meaning is through the subprincipal symbol of the operator $L = AB + C + \varphi$. Call $A(x,\xi)$ the symbol of A; $A(x,\xi)$ is purely imaginary; likewise for B and C. The symbol of $AB + C$ is

$$A(x,\xi)\, B(x,\xi) - i\nabla_\xi A(x,\xi) \cdot \nabla_x B(x,\xi) + C(x,\xi).$$

The subprincipal symbol of L is

$$\sigma_{\text{sub}}(x,\xi) = C(x,\xi) - i\nabla_\xi A(x,\xi) \cdot \nabla_x B(x,\xi) - \frac{1}{2i}(\nabla_x \cdot \nabla_\xi)(A(x,\xi)\, B(x,\xi)).$$

Using the notation $\{,\}$ for the Poisson bracket we see that

$$\sigma_{\text{sub}}(x,\xi) \cong C(x,\xi) + \frac{1}{2i}\{A(x,\xi), B(x,\xi)\}$$

mod $(A(x,\xi), B(x,\xi))$. The right-hand side is the principal symbol of $C + \frac{1}{2}T$. The hypothesis that $\gamma(0) \neq -\frac{1}{2}$ is equivalent to the **ellipticity** of $C + \frac{1}{2}T$ on the double characteristics of L. For those values we get the best possible estimates, ie, the estimates (1.2), yielding solutions $u \in L^2$ of the equation $Lu = f \in H^{-1}$. When the ellipticity of $C + \frac{1}{2}T$ fails, ie, when $\gamma(0) = -\frac{1}{2}$, solvability still holds but we have only obtained solutions in H^{-1}. Considering that L is a second-order differential operator and comparing to the elliptic case, one could say that there is local solvability with loss of one derivative when $\gamma(0) \neq -\frac{1}{2}$ and loss of two derivatives when $\gamma(0) = -\frac{1}{2}$.

2.2. The pseudodifferential case and solvability in H^s. It remains to prove the local solvability of $Lu = f$ in the sense of the Sobolev space H^s for an arbitrary real number s. We shall do this through the extension of Theorem 1 to classical pseudodifferential operators of principal type in $\Omega \subset \mathbb{R}^n$ ($n \geq 2$ arbitrary). Inspection of the proof of Theorem 1 shows that the extension is valid:

Theorem 2. *If P_1 and P_2 are two first-order classical pseudodifferential operators of principal type in $\Omega \subset \mathbb{R}^n$, with real principal symbols and such that*

$$P_1^2 + P_2^2 + \{P_1, P_2\}^2$$

is elliptic, then $L = P_1P_2 + \sqrt{-1}Q$ *is locally solvable whatever the first-order classical pseudodifferential operator* Q *in* Ω *having a real principal symbol.*

More precisely, given any point $x^\circ \in \Omega$ *and any number* $\varepsilon > 0$ *there is an open neighborhood* $U_{x^\circ,\varepsilon} \subset \Omega$ *of* x° *with the following property: there is a bounded linear operator* $G_{x^\circ,\varepsilon} : H^{-1}(U_{x^\circ,\varepsilon}) \longrightarrow H^{-1}(U_{x^\circ,\varepsilon})$ *with norm* $\leq \varepsilon$ *such that* $LG_{x^\circ,\varepsilon}f = f$ *in* $U_{x^\circ,\varepsilon}$ *for every* $f \in H^{-1}(U_{x^\circ,\varepsilon})$*. If moreover the subprincipal symbol of* L *does not vanish on the common characteristics of* P_1 *and* P_2 *then* $G_{x^\circ,\varepsilon}$ *can be taken to be a continuous linear operator* $H^{-1}(U_{x^\circ,\varepsilon}) \longrightarrow L^2(U_{x^\circ,\varepsilon})$ *with norm* $\leq \varepsilon$.

The proof duplicates that of Theorem 1 replacing A by $\sqrt{-1}P_1$, B by $\sqrt{-1}P_2$ and C by $\sqrt{-1}Q$. Its "pivot" is the analogue of Estimate (1.10), valid when the subprincipal symbol of L does not vanish on the double characteristics of L:

$$\|u\|_1 \leq \delta K_2 \|L^*P_1^*u + D^*u\|, \; u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta), \tag{2.1}$$

where now D is an arbitrary first-order pseudodifferential operator with real principal symbol (the positive constant K_2 depends on L and D but not on δ nor, of course, on u).

Now let E be a properly supported, classical, elliptic, self-adjoint pseudodifferential operator in Ω of order $s \in \mathbb{R}$. If we write $L = L_2 + L_1$ modulo pseudodifferential operators of order zero and use the notation $\sigma(\cdot)$ for the principal symbol, the subprincipal symbol of $E^{-1}LE = L + E^{-1}[L, E]$ is equal to

$$\sigma(L_1) - \frac{1}{2i}(\nabla_x \cdot \nabla_\xi)\sigma(L) - \sigma(E)^{-1}\{\sigma(L), \sigma(E)\}.$$

But $\{\sigma(L), \sigma(E)\} \equiv 0$ on the double characteristics of L. This allows us to apply (2.1) with $E^{-1}LE$ in the place of L:

$$\|u\|_1 \leq \delta K_2 \left\|EL^*E^{-1}P_1^*u + D^*u\right\|, \; u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta). \tag{2.2}$$

Let $\chi \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta)$, $\chi \equiv 1$ in $\mathfrak{B}_{\delta/2}$. Take $u \in \mathcal{C}_c^\infty(\mathfrak{B}_{\delta/2})$ and apply (2.2) with χEu in the place of u. We obtain

$$\begin{gathered}\|u\|_{s+1} \leq \|\chi Eu\|_1 + \|u\|_s \leq \\ \delta K_3 \|E(L^*P_1^*u + D^*u)\| + \delta K_3 \left\|EL^*E^{-1}[P_1^*, E]u\right\| \\ +\delta K_3 \|[D^*, E]u\| + \delta K_3 \left\|\left(EL^*E^{-1}P_1^* + D^*\right)[E, \chi]u\right\|.\end{gathered}$$

On the one hand the order of $[D^*, E]$ is $\leq s$ while the operator $(L^*P_1^* + D^*)[E, \chi]$ acting on compactly supported distributions in $\mathfrak{B}_{\delta/2}$ is regularizing. On the other hand the order of $L^*E^{-1}[P_1^*, E]$ is ≤ 1. By taking $\delta > 0$ suitably small we conclude that

$$\|u\|_{s+1} \leq \delta K_4 \|L^*P_1^*u + D^*u\|_s + \|u\|_s, \; , u \in \mathcal{C}_c^\infty(\mathfrak{B}_{\delta/2}). \tag{2.3}$$

From (2.3) we proceed as we did in the proof of Theorem 1 starting from Corollary 2. When the subprincipal symbol of L vanishes on the double characteristics of L we obtain an inequality of the following kind (after a redefinition of s and δ):

$$\|u\|_s \leq \delta K_5 \|L^* u\|_s + \|u\|_{s-1}\,, \;, \; u \in \mathcal{C}_c^\infty(\mathfrak{B}_\delta)\,. \tag{2.4}$$

Such an inequality implies (by standard arguments) the solvability of $Lu = f$ in $H^s(\mathfrak{B}_\delta)$ after further decreasing of δ. We can state

Corollary 3. *Let L be as in Theorem 2 and let $s \in \mathbb{R}$ be arbitrarily given. Given any point $x^\circ \in \Omega$ and any real number s there is an open neighborhood $U_{x^\circ,s} \subset \Omega$ of x° with the following property: there is a bounded linear operator $G_{x^\circ,s} : H^s(U_{x^\circ,s}) \longrightarrow H^s(U_{x^\circ,s})$ such that $LG_{x^\circ,s} f = f$ in $U_{x^\circ,s}$ for every $f \in H^s(U_{x^\circ,s})$. If moreover the subprincipal symbol of L does not vanish on the common characteristics of P_1 and P_2 then $G_{x^\circ,\varepsilon}$ can be taken to be a continuous linear operator $H^s(U_{x^\circ,\varepsilon}) \longrightarrow H^{s+1}(U_{x^\circ,\varepsilon})$.*

2.3. Open questions. Questions related to Theorems 1 and 2 that come to mind are the following.

1. Is there a convenient symbolic calculus specifically adapted to the construction of a parametrix for operators L like those in Theorem 2 (cf. [Ha1])?
2. What is the geometry of the null bicharacteristics of the symbols p_1, p_2 or of the "double" half bicharacteristics of $p_1 p_2$ defined by the sign of $\{p_1, p_2\}$ ensuring semiglobal solvability of the operator L in Theorem 2?
3. What are the generalizations of Theorem 1 to real, smooth vector fields $X_1, ..., X_r$ with $r \geq 3$ satisfying Hörmander's condition? This is of course related to the local solvability of left-invariant differential operators on a nilpotent Lie group (see e.g. [MüR]).

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References

[Ha1] N, Hanges, *Parametrices and local solvability for a class of singular hyperbolic operators,* Comm. **3** (1978), 105–152.

[Ha2] ———, *Parametrices and propagation of singularities for operators with noninvolutive characteristics*, Indiana U. Math. Journal **28** (1979), 87–97.

[MüR] D. Müller, and F. Ricci, *Solvability for a class of doubly characteristic differential operators on 2-step nilpotent groups,* Ann. of Math. **143**, (1996), 1–49.

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