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# LOWER ESTIMATE OF MULTIPLICITY OF ISOLATED COMPLETE INTERSECTION SINGULARITIES WITH APPLICATIONS IN WEAKLY ELLIPTIC SINGULARITIES

Yi-Jing Xu and Stephen S.-T. Yau

*Dedicated to Yum-Tong Siu on the occasion of his 60th birthday.*

## 1. Introduction

Let $p$ be a normal singularity of the 2-dimensional Stein space $V$. Let $\pi : M \to V$ be a resolution of $V$ such that the irreducible components $A_i$, $1 \le i \le n$, of $A = \pi^{-1}(p)$ are nonsingular and have only normal crossings. Associated to $A$ is a weighted dual graph $\Gamma$ (e.g. see [Hi] or [La1]) which, along with the genera of the $A_i$, fully describes the topology and differentiable structure of $A$ and the topological and differentiable nature of the embedding of $A$ in $M$.

One of the important questions in normal two dimensional singularities is the classification of all weighted dual graphs for complete intersection singularities. Alternatively, we may ask: what condition does the existence of complete intersection representative $(V,p)$ put on a weighted dual graph $\Gamma$? In other words, we would like to identify the image of the mapping

$$\begin{array}{ccc} \{\text{isolated complete intersection singularities}\} & \longrightarrow & \{\text{weighted dual graphs}\} \\ (V,p) & \longrightarrow & \Gamma \end{array}$$

A complete intersection singularity $(V,p)$ is Gorenstein [Ba]. So there exists an integral cycle $K'$ on $\Gamma$ which satisfies the adjunction formula [Se]. M. Artin has studied the rational singularities (those for which $R^1\pi_*(\mathcal{O}) = 0$). It is well known that rational complete intersection singularities are hypersurfaces. Artin has shown that all hypersurface rational singularities have multiplicities two and the graphs associated to these singularities are one of the graphs $A_k$, $k \ge 1$; $D_k, k \ge 4$; $E_6, E_7, E_8$ which arise in the classification of simple Lie groups. In [Wa] Wagreich introduces a definition for $p$ to be weakly elliptic. In the seventies, weakly elliptic singularities were studied extensively by Laufer [La2], and Yau [Ya1] [Ya2] [Ya3] [Ya4] [Ya5] [Ya6] [Ya7]. In the eighties, Tomari has made some important contributions to the theory of weakly elliptic singularities [To1] [To2]. Recently, there is a renewed interest of studying weakly elliptic singularities. Némethi [Ne] has proved several beautiful theorems about weakly elliptic singularities. He has completed Yau's program [Ya7] on understanding

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the Hilbert-Samuel functions of maximally elliptic singularities. He also proves that weakly elliptic Gorenstein singularity with $H^1(A, \mathbb{Z}) = 0$ is automatically maximally elliptic which generalizes a result of Yau in [Ya5]. After the topological classification of complete intersection rational singularities by Artin, it is a natural question to ask whether we can do the topological classification for complete intersection weakly elliptic singularities. In [Ya3], Yau gives a complete list (131 cases) of all weighted dual graphs for weakly elliptic double points. Moreover, for each of those weighted dual graphs, a typical defining equation is given. The main purpose of this paper is to give the essential step in topological classification for complete intersection weakly elliptic singularities.

**Theorem 1.1.** *Let $(V, p)$ be a two-dimensional complete intersection weakly elliptic singularity. Let* $\mathrm{mult}(V, p)$ *be the multiplicity of $(V, p)$. Then*

1. *$(V, p)$ is a hypersurface singularity if and only if* $\mathrm{mult}(V, p)$ *is either* 2 *or* 3
2. *$(V, p)$ is a complete intersection but not hypersurface if and only if* $\mathrm{mult}(V, p) = 4$. *In this case $(V, p)$ is defined by two functions in $\mathbb{C}^4$, each of which has multiplicity two.*

Theorem 1.1 will enable us to completely classify all the weighted dual graphs arising from weakly elliptic complete intersection singularities. Let $Z$ be the fundamental cycle (cf. Definition 1.1) of a weakly elliptic complete intersection singularities. Theorem 1.1 implies $Z^2$ is either $-1$, $-2$, $-3$ or $-4$. By using the theory of elliptic sequence (cf. Definition 1.10) developed in [Ya7], a complete list of weighted dual graphs of weakly elliptic complete intersection singularities can be found. Details can be found in our forthcoming paper.

There are two ingredients in the proof of the above theorem. They are of independent interest. The first one is the following theorem which gives lower estimate of the multiplicity of isolated complete intersection singularity in terms of the codimension.

**Theorem 1.2.** *Let $(V, p)$ be an isolated complete intersection of dimension $n$. Suppose that the minimal embedding dimension of $(V, p)$ is* $\mathrm{embed\,dim}(V, p) = n + k$. *Let $f_1, \dots, f_k$ be the defining equations of $(V, p)$. Then*

$$\mathrm{mult}(V, p) \geq \mathrm{mult}(f_1, p) \cdots \mathrm{mult}(f_k, p).$$

*In particular*

$$\mathrm{mult}(V, p) \geq 2^k.$$

Theorem 1.2 above generalizes a classical result of Abhyanker [Ab] which states that for Cohen-Macaulay singularity $(V, p)$,
$\mathrm{mult}(V, p) \geq 2 + \mathrm{codim}(V, p)$ if $\mathrm{mult}(V, p) \geq 3$.

**Definition 1.1** Let $(V, p)$ be an isolated Cohen-Macaulay singularity $(V, p)$. Then $(V, p)$ is said to have the maximal embedding dimension if either $\mathrm{mult}(V, p) = 2$ or $\mathrm{mult}(V, p) = \mathrm{codim}(V, p) + 2$, i.e. embed dim $(V, p)$ reaches to Abhyanker bound $\mathrm{mult}(V, p) + \dim(V, p) - 2$.

The second ingredient of the proof of Theorem 1.1 is the following theorem.

**Theorem 1.3.** *Let $(V, p)$ be a germ of normal two-dimensional complete intersection weakly elliptic singularity. Then $(V, p)$ has maximal embedding dimension. In particular,* $\operatorname{mult}(V, p) = \operatorname{embed}\dim(V, p)$ *if* $\operatorname{mult}(V, p) \geq 3$.

We should remark that the above Theorem 1.3 is a trivial Corollary of the celebrated work of Némethi [Ne] in case $H^1(A, \mathbb{Z}) = 0$. However his theory cannot be generalized to cover Theorem 1.3 in case $H^1(A, \mathbb{Z}) \neq 0$. For more detail, please see section 2. We should also remark that in the Remark 6.4(c) of [Ne], Némethi stated that Theorem 1.3 can be verified by using similar argument in section 6 of [Ne]. Theorem 1.3 was part of the 1990 Ph,D. thesis at UIC of the first author under the guidance of the second author.

In section 2, we shall recall some notations and known results. In section 3, we give the lower estimate of multiplicity in terms of codimension for isolated complete intersection singularities. In section 4, we prove that weakly elliptic complete intersection singularity has maximal embedding dimension and the multiplicity is at most 4.

## 2. Preliminaries

Let $\pi : M \to V$ be a resolution of normal two dimensional Stein space $V$. We assume that $p$ is the only singularity of $V$. Let $\pi^{-1}(p) = A = \cup A_i$, $1 \leq i \leq n$, be the decomposition of the exceptional set $A$ into irreducible components. Suppose $\pi$ is the minimal good resolution. Let $\Gamma$ be the associated weighted dual graph. The vertices of $\Gamma$ correspond to the $A_i$. The edge of $\Gamma$ connecting the vertices corresponding to $A_i$ and $A_j$, $i \neq j$, corresponds to the points of $A_i \cap A_j$. Finally, associated to each $A_i$ is its genus, $g_i$, as a Riemann surface, and its weight, $A_i\, A_i$, the topological self-intersection number. $\Gamma$ will denote the graph along with the genera and the weights.

A cycle $D$ on $A$ is an integral combination of the $A_i$, $D = \Sigma d_i\, A_i$, $1 \leq i \leq n$ with $d_i$ an integer. There is a natural partial ordering, denoted by $<$, between cycles defined by comparing the coefficients. We shall only be considering cycles $D \geq 0$. The support of $D$ is supp $D = \cup A_i, d_i \neq 0$. Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on $M$. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on $M$ which vanish to order $d_i$ on $A_i$. Let $\mathcal{O}_D$ denote $\mathcal{O}/\mathcal{O}(-D)$. Then

$$\chi(D) := \dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D). \tag{2.1}$$

Arithmetic genus of $D$ is $p_a(D) := 1 - \chi(D)$. The Riemann-Roch Theorem [Se, p. 75] says

$$\chi(D) = -\frac{1}{2}(D \cdot D + D \cdot K) \tag{2.2}$$

where $K$ is the canonical divisor on $M$. By adjunction formula

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2. \tag{2.3}$$

It follows immediately from (2.2) that if $B$ and $C$ are cycles, then

$$\chi(B+C) = \chi(B) + \chi(C) - B \cdot C. \tag{2.4}$$

**Definition 2.1** (Artin) Associated to $\pi$ is a unique fundamental cycle $Z$ which is the minimal element among the set $\{D : D > 0 \text{ and } D \cdot A_i \leq 0 \text{ for all } 1 \leq i \leq n\}$.

**Definition 2.2** (Wagreich) The geometric genus $p_g(V,p)$ is defined to be $\dim H^1(M,\mathcal{O})$. The arithmetic genus $p_a(V,p)$ is defined to be $\sup\{p_a(D) : D \text{ is a positive cycle}\}$.

**Definition 2.3** $(V,p)$ is called rational singularity if $p_g(V,p) = 0$. $(V,p)$ is called weakly elliptic singularity if $p_a(V,p) = 1$.

The following theorem is well known [Ar] [La2].

**Theorem 2.4.** (i) $(V,p)$ *is rational* $\iff \chi(Z) = 1 \iff \min\limits_{D>0} \chi(D) \geq 1$

(ii) $(V,p)$ *is weakly elliptic* $\iff \chi(Z) = 0 \iff \min\limits_{D>0} \chi(D) = 0$

The class of weakly elliptic singularities contains all the singularities with $p_g = 1$ [La2] and all the Gorenstein singularities with $p_g = 2$. A weakly elliptic singularity can have arbitrarily high geometric genus [Ya7].

**Definition 2.5** (Laufer) A cycle $E > 0$ is minimally elliptic if $\chi(E) = 0$ and $\chi(D) > 0$ for all cycles $D$ such that $0 < D < E$.

**Definition 2.6** (Laufer) Let $\pi : M \to V$ be the minimal resolution of $V$ with $A$ as the exceptional set. $p$ is called minimally elliptic if it is weakly elliptic and every connected proper subvariety of $A$ is the exceptional set for a rational singularity.

**Definition 2.7** Let $K$ be the canonical divisor on $M$. The negative cycle $K' = \Sigma k_i A_i$ on $A$ with $k_i \in \mathbb{Z}$ is the unique cycle such that $A_i \cdot K' = A_i \cdot K$ for $A_i \subset A$.

**Remark 2.8** $K'$ does not always exist. If $(V,p)$ is a Gorenstein singularity or complete intersection singularity, then $K'$ exists.

In [Ya2] [Ya7], Yau first introduce the following definition.

**Definition 2.9** Let $\mathcal{O}_{V,p}$ be the local ring of $V$ at $p$ and $m_{V,p}$ be its maximal ideal. Let $v_i$ be the valuation on $\mathcal{O}_{V,p}$ defined by $v_i(f)$ = vanishing order of

$\pi^*(f)$ on $A_i$ for $f$ in $\mathcal{O}_{V,p}$, $1 \leq i \leq n$. Then the maximal ideal cycle $Y_\pi$ on $M$ is defined by

$$Y_\pi = \sum_{i=1}^{n} (\min_{f \in m_{V,p}} v_i(f)) A_i. \tag{2.5}$$

The following notion of elliptic sequence introduced by Yau [Ya7] is very useful in studying weakly elliptic singularities.

**Definition 2.10** Let $A$ be the exceptional set of the minimal good resolution $\pi : M \to V$, where $V$ is a normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. By Laufer [La2], there is a unique minimal elliptic cycle $E$ such that $\chi(E) = 0$ and $\chi(D) > 0$ for all cycles $D$ with $0 < D < E$. Let $Z$ be the fundamental cycle on $A$. If $E \cdot Z < 0$, we say that the elliptic sequence is $\{Z\}$ and the length of elliptic sequence is equal to one. Suppose $E \cdot Z = 0$. Let $B_1$ be the maximal connected subvariety of $A$ such that $B_1 \supset \mathrm{Supp}E$ and $A_i \cdot Z = 0$ for all $A_i \subset B_1$. Since $A$ is an exceptional set, $Z \cdot Z < 0$. So $B_1$ is properly contained in $A$. Let $Z_{B_1}$ be the fundamental cycle on $B_1$. Suppose $Z_{B_1} \cdot E = 0$. Let $B_2$ be the maximal connected subvariety of $B_1$ such that $B_2 \supset \mathrm{Supp}E$ and $A_i \cdot Z_{B_i} = 0$ for all $A_i \supset B_2$. By the same argument as above, $B_2$ is properly contained in $B_1$. Continuing this process, we finally obtain $B_m$ with $Z_{B_m} \cdot E < 0$. We call $\{Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_m}\}$ the elliptic sequence, and the length of the elliptic sequence is $m + 1$.

Elliptic sequence defined above is topological in nature. It does give a lot of information about the weakly elliptic singularity $(V, p)$.

**Theorem 2.11.** [Ya7] *Let $\pi : M \to V$ be the minimal good resolution of normal two dimensional Stein space with $p$ as its only weakly elliptic singularity. Suppose $p$ is not a minimally elliptic singularity. If $E \cdot Z < 0$ and* $\mathrm{supp}E \neq A$, *then $K'$ does not exist. If $K'$ exists, then the elliptic sequence is of the following form:*

$$Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_\ell}, \quad Z_{B_{\ell+1}} = Z_E, \qquad \ell \geq 0.$$

*Moreover* $-K' = \sum_{i=0}^{\ell} Z_{B_i} + E$.

**Theorem 2.12.** [Ya7] *Let $\pi : M \to V$ be the minimal good resolution of a normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. If $K'$ exists, then the geometric genus is less than or equal to the length of the elliptic sequence.*

In view of Theorem 2.12, it is natural to introduce the following definition.

**Definition 2.13** Let $V$ be a normal 2-dimensional Stein space with $p$ as its only weakly elliptic singularity. Suppose $K'$ exists. If the geometric genus is equal to the length of the elliptic sequence, then $p$ is called a maximally elliptic singularity.

Recently Némethi [Ne] has made very important contributions to weakly elliptic singularities. He has succeeded in computing the Hilbert-Samuel functions of maximally elliptic singularities.

**Theorem 2.14.** (Némethi) *Assume that $(V,p)$ is a maximally elliptic singularity. Let $Z$ be the fundamental cycle in the minimal resolution $\pi : (M,A) \to (V,p)$. Let $Y$ be the maximal ideal cycle and $m$ be the maximal ideal of $\mathcal{O}_{V,p}$. Then*

(a) *If $Z^2 \le -2$, then $m\mathcal{O}_M = \mathcal{O}(-Z)$, hence* $\operatorname{mult}(V,p) = -Z^2$*;*
(b) *If $Z^2 = -1$, then $m\mathcal{O}_M = m_q\mathcal{O}(-Z)$ for some smooth point $q$ of $A$ where $m_q$ is the maximal ideal of $\mathcal{O}_{M,q}$, and* $\operatorname{mult}(V,p) = 2$ *(hence, in any case, $Z = Y$);*
(c) embed $\dim(V,p) = \max(3, -Z^2)$*;*
(d) *$H^0(M, \mathcal{O}(-kZ)) = m^k$ for any $k \ge 0$ provided that $Z^2 \le -3$ (i.e. the $\mathcal{O}_V$-algebra $R := \bigoplus_{k\ge 0} H^0(\mathcal{O}(-kZ))$ is generated by $m$);*
(e) $\dim \mathcal{O}_{V,p}/m^k = \chi(\mathcal{O}_{kZ}) + 1$, *and* $\dim m^k/m^{k+1} = -kZ^2$ *for any $k \ge 1$ provided that $Z^2 \le -3$;*
(f) *One can write a generator set of $R := \bigoplus_{k\ge 0} H^0(\mathcal{O}(-kZ))$ in the case $d := -Z^2 \le 2$ as well. If $d = 2$, then $R$ is generated by $m$ in degree 1 and an element $y \in m\backslash m^2$ in degree two. If $d = 1$, then $R$ is generated by $m$ in degree 1, by an element $y \in m\backslash m^2$ in degree* 2, *and $z \in m\backslash(m^2, y)$ in degree* 3.

The other beautiful theorem proved by Némethi is the following theorem.

**Theorem 2.15.** (Némethi) *Assume that $(V,p)$ is an elliptic Gorenstein singularity with $H^1(A,\mathbb{Z}) = 0$. Then $(V,p)$ is a maximally elliptic singularity.*

If $H^1(A,\mathbb{Z}) = 0$, then Theorem 1.3 in the introduction follows immediately from Theorem 2.14 and Theorem 2.15. The following example shows that Némethi's theorem above is the best possible theorem in some sense.

**Example 2.16** Let $V$ be the locus in $\mathbb{C}^3$ of $z^2 = y(x^4 + y^6)$. Then the weighted dual graph is

$$\underset{\text{torus}}{\overset{-1}{\bullet}} \;\text{———}\; \overset{-2}{\bullet} \;\text{———}\; \overset{-2}{\bullet}$$

This is a weakly elliptic singularity and the length of the elliptic sequence is equal to three. It can be calculated Milnor number $\mu = 22$. By Theorem 2.11

$$-K' = Z_{B_0} + Z_{B_1} + E, \qquad K' \cdot K' = Z_{B_0}^2 + Z_{B_1}^2 + E^2 = -3.$$

By Laufer's formula

$$\mu = n + K' \cdot K' - \dim H^1(A,\mathbb{C}) + 12\, p_g$$

where $n$ is the number of irreducible components of $A$, we know $p_g = 2$. Therefore $(V, 0)$ is not a maximally elliptic singularity. Notice that Theorem 2.15 does not apply here because $H^1(A, \mathbb{C}) \neq 0$.

## 3. Lower estimate of multiplicity of isolated complete intersection singularities

In [Ab] Abhyanker first observed that for Cohen-Macaulay singularity, one can get lower bound of its multiplicity in terms of linear function of the codimension of the singularity. In this section we shall improve the lower bound of the multiplicity if the singularity is a complete intersection. Specifically, we shall prove that multiplicity is bounded below by an exponential function of the codimension of the singularity.

**Theorem 3.1.** *Let $(V, 0)$ be a germ of irreducible analytic space in $\mathbb{C}^N$ and let $H$ be a hypersurface passing through 0 defined by an holomorphic function $f$. If $\dim V \cap H = \dim V - 1$, then* $\operatorname{mult}(V \cap H, 0) \geq \operatorname{mult}(V, 0)\operatorname{mult}(H, 0)$.

*Proof.* Let $\mathcal{O}_{V,0}$ be the local ring of $V$ at 0. Let $m$ be the maximal ideal of $\mathcal{O}_{V,0}$. Let $\overline{f}$ be the image of $f$ in $\mathcal{O}_{V,0}$. Suppose $\overline{f} \in m^r - m^{r+1}$. Then we have $r \geq \operatorname{mult}(H, 0)$. Consider the filtration of $(\overline{f})$ given by $(\overline{f})_n = (\overline{f}) \cap m^n$ and the associated $gr(\overline{f}) = \bigoplus_{n \geq 0} (\overline{f})_n / (\overline{f})_{n+1}$. It is clear that $gr(\overline{f})$ is generated by the $r$-th homogeneous element represented by $\overline{f}$. Therefore the inclusion $m(\overline{f})_i \longrightarrow (\overline{f})_{i+1}$ is surjective $\operatorname{mod}(\overline{f})_{i+2}$ for all $i \geq r$. On the other hand, the Artin-Rees theorem (e.g. [At-Ma]) guarantees the existence of an integer $s$ so that

$$m(\overline{f})_i = (\overline{f})_{i+1}, \qquad \text{for all} \;\; i \geq s.$$

An easy descending induction starting at $\max(r, s)$ shows

$$m(\overline{f})_i = (\overline{f})_{i+1}, \qquad \text{for all} \;\; i \geq r.$$

Thus

$$m^n(\overline{f})_r = (\overline{f})_{n+r} \text{ for all } \;\; n \geq 0$$

i.e.,

$$m^{n+r} \cap (\overline{f}) = m^n(\overline{f}) \text{ for all } \;\; n \geq 0. \tag{3.1}$$

Let $W = V \cap H$ and $m_{W,0}$ be the maximal ideal of the local ring $\mathcal{O}_{W,0}$. We have $m^n_{W,0} \cong (m^n + (\overline{f}))/(\overline{f})$ and hence

$$m^n_{W,0}/m^{n+1}_{W,0} \cong (m^n + (\overline{f}))/(m^{n+1} + (\overline{f})) \cong m^n/(m^{n+1} + m^n \cap (\overline{f})). \tag{3.2}$$

Now let us consider

$$m^n/m^{n+1} \xrightarrow{\phi} m^{n+r}/m^{n+r+1} \xrightarrow{\rho} m^{n+r}_{W,0}/m^{n+r+1}_{W,0} \longrightarrow 0 \tag{3.3}$$

where $\phi$ is given by the multiplication of $\overline{f}$ and $\rho$ is the natural map. It is clear that $\rho$ is surjective and $\rho \circ \phi = 0$. We are going to show $\ker \rho \subseteq \operatorname{Im}\phi$. By (3.2), it is enough to show that $m^n \cdot (\overline{f}) + m^{n+r+1} = m^{n+r+1} + m^{n+r} \cap (\overline{f})$. But this follows from (3.1). By the exactness of the sequence (3.2), we have

$$\dim m_{W,0}^{n+r}/m_{W,0}^{n+r+1} \geq \dim m^{n+r}/m^{n+r+1} - \dim m^n/m^{n+1}. \tag{3.4}$$

Let $H_W(n) = \dim m_W^n/m_W^{n+1}$ and $H_V(n) = \dim m^n/m^{n+1}$ be the Hilbert-Samuel function of $(W,0)$ and $(V,0)$ respectively. Define $H_W^1(n) = \sum_{i=0}^{n} H_W(i)$ and $H_V^1(n) = \sum_{i=0}^{n} H_V(n)$. Then by (3.4),

$$\begin{aligned}
H_W^1(n) &= \sum_{i=0}^{r} H_W(i) + \sum_{i=1}^{n} H_W(i+r) \\
&\geq \sum_{i=0}^{r} H_W(i) + \sum_{i=1}^{n} (H_V(r+i) - H_V(i)) \\
&= \sum_{i=0}^{r} H_W(i) - \sum_{i=0}^{r} H_V(i) + \sum_{i=0}^{n+r} H_V(i) - \sum_{i=0}^{n} H_V(i) \\
&= H_V^1(n+r) - H_V^1(n) + K_r
\end{aligned} \tag{3.5}$$

where $K_r = \sum_{i=0}^{r} H_W(i) - \sum_{i=0}^{r} H_V(i)$ is a constant depending on $r$. Recall hat for $n$ sufficiently large, $H_W^1(n)$ is a polynomial in $n$ of degree $\dim W = \dim V - 1$ with leading coefficient $\mathrm{mult}(W,0)/(d-1)!$, where $d = \dim V$. On the other hand, the right hand side of (3.5) is also a polynomial in $n$ of degree $d-1$ with leading coefficient $r\ \mathrm{mult}(V,0)/(d-1)!$. In view of (3.5), it follows that $\mathrm{mult}(W,0) \geq r\mathrm{mult}(V,0) \geq \mathrm{mult}(H,0)\mathrm{mult}(V,0)$. □

It is clear that Theorem 1.2 in section 1 follows from Theorem 3.1 above.

**Corollary 3.1.** *Theorem* 1.2 *in section* 1 *holds.*

## 4. Upper estimate of multiplicity of elliptic Gorenstein singularity

The purpose of this section is to prove that elliptic complete intersection singularities have maximal embedding dimensions. We begin with a lemma proved by Tomari (Proposition 1.3 of [To2]).

**Lemma 4.1.** (Tomari) *Let $C$ be a projective curve and $p \in C$ is a singular point with multiplicity* $\mathrm{mult}(C,p)$. *Let $\pi : C_1 \to C$ be the blowing up of $C$ at $p$. Then*

$$\chi(C_1) - \chi(C) = \sum_{k \geq 0} (\mathrm{mult}(C,p) - \dim m^k/m^{k+1}) \tag{4.1}$$

*where $m$ is the maximal ideal of the local ring $\mathcal{O}_{C,p}$ and $\chi(Y)$ is the analytic Euler characteristic of a scheme $Y$.*

**Theorem 4.1.** *Let $(V,p)$ be a germ of a two-dimensional elliptic complete intersection singularity. Then $(V,p)$ has maximal embedding dimension. In particular,* $\operatorname{mult}(V,p) = \operatorname{embed}\dim(V,p)$ *if* $\operatorname{mult}(V,p) \geq 3$.

PROOF: Since $(V,p)$ is an isolated singularity, we may assume that $(V,0)$ is defined by algebraic equations in $\mathbb{C}^N$. By taking the closure of $V$ in $\mathbb{C}P^N$, we may assume that $V$ is projective.

We are going to show that if $\operatorname{mult}(V,p) \geq 3$, then $\operatorname{mult}(V,p)=\operatorname{embed}\dim(V,p)$. Let $\pi : M \to V$ resolve the singularity of $V$ at $p$ such that $\pi^{-1}(m)$ is locally principal, where $m$ is the maximal ideal of the local ring $\mathcal{O}_{V,p}$. (Notice that $\pi : \pi^{-1}(V-p) \to V-p$ is biholomorphic). Let $H$ be a hyperplane in $\mathbb{C}P^N$ and $C = H \cap V$. By taking $H$ general enough, we may assume that $C$ has the following properties:

(i) $\pi^{-1}(C) = C' + Y_\pi$ where $C'$ is the proper transform of $C$, i.e. the closure of $\pi^{-1}(C-p)$ and $Y_\pi$ is the maximal ideal cycle supported on $A = \pi^{-1}(p)$.
(ii) $\operatorname{mult}(C,p) = \operatorname{mult}(V,p)$ and $\operatorname{embed}\dim(C,p) = \operatorname{embed}\dim(V,p) - 1$.
(iii) $(C,p)$ is defined by $f$ which is not a zero divisor of $\mathcal{O}_{V,p}$.

**Claim:** $\chi(\pi^{-1}(C)) = \chi(C)$.

*Proof.* Consider the exact sequences

$$0 \longrightarrow I_C \to \mathcal{O}_V \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$$0 \longrightarrow I_{\pi^{-1}(C)} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_{\pi^{-1}(C)} \longrightarrow 0.$$

Thus $\chi(C) = \chi(\mathcal{O}_V) - \chi(I_C)$ and $\chi(\pi^{-1}(C)) = \chi(\mathcal{O}_M) - \chi(I_{\pi^{-1}(C)})$. By Leray spectral sequence, we have

$$\chi(\mathcal{O}_M) = \sum_{k\geq 0}(-1)^k \chi(R^k\pi_*\mathcal{O}_M) \tag{4.2}$$

$$\chi(I_{\pi^{-1}(C)}) = \sum_{k\geq 0}(-1)^k \chi(R^k\pi_*(I_{\pi^{-1}(C)})). \tag{4.3}$$

At $p \in V$, the projection formula gives

$$\begin{aligned} R^k\pi_*(\mathcal{O}_M)_p &\cong (I_C \otimes R^k\pi_*(\mathcal{O}_M))_p \cong (R^k\pi_*(\pi^*(I_C)))_p \\ &\cong (R^*\pi_*(I_{\pi^{-1}(C)}))_p. \end{aligned} \tag{4.4}$$

The first isomorphism in (4.4) comes from the fact that the local equation $f$ of $C$ at $p$ is not a zero divisor in $\mathcal{O}_{V,p}$ and hence $(I_C)_p \cong \mathcal{O}_{V,p}$ as $\mathcal{O}_{V,p}$-module. The supports of $R^k\pi_*(\mathcal{O}_M)$ and $R^k\pi_*(\pi^*(I_C))$ are contained in $p$ for $k \geq 1$. Thus (4.3) gives

$$R^k\pi_*(\mathcal{O})_M \cong R^k\pi_*(\pi^*(I_C)) \cong R^k\pi_*(I_{\pi^{-1}(C)}) \qquad \text{for } k \geq 1. \tag{4.5}$$

For $R^0\pi_*(\mathcal{O}_M) = \pi_*(\mathcal{O}_M)$, we have a map $\psi : \mathcal{O}_V \to \pi_*(\mathcal{O}_M)$ and the exact sequence

$$0 \longrightarrow \ker\psi \longrightarrow \mathcal{O}_V \longrightarrow \pi_*(\mathcal{O}_M) \longrightarrow \operatorname{coker}\psi \longrightarrow 0. \tag{4.6}$$

Observe that $\ker\psi$ and $\operatorname{coker}\psi$ are supported at $p$. Since $I_{C,p} \cong \mathcal{O}_{V,p}$, by projection formula, we have

$$0 \longrightarrow \ker\psi \longrightarrow I_C \longrightarrow \pi_*(\pi^* I_C) \longrightarrow \operatorname{coker}\psi \longrightarrow 0. \tag{4.7}$$

Combining (4.6) and (4.7), we get

$$\chi(\pi_*(\pi^* I_C)) - \chi(I_C) = \chi(\pi_*(\mathcal{O}_M)) - \chi(\mathcal{O}_V). \tag{4.8}$$

By (4.2), (4.3), (4.5) and (4.8), we have

$$\begin{aligned} \chi(\mathcal{O}_M) - \chi(I_{\pi^{-1}(C)}) &= \chi(\pi_*\mathcal{O}_M) - \chi(\pi_* I_{\pi^{-1}(C)}) \\ &= \chi(\mathcal{O}_V) - \chi(I_C) \end{aligned} \tag{4.9}$$

(4.9) is equivalent to $\chi(\pi^{-1}(C)) = \chi(C)$. This complete the proof of our claim.

Recall that $\pi^{-1}(C) = C' + Y_\pi$. Since $\pi^{-1}(C)$ is defined by the pull back of a generic element in $m$, we have

$$Y_\pi \cdot (C' + Y_\pi) = 0. \tag{4.10}$$

It follows that

$$\begin{aligned} \chi(C) &= \chi(C' + Y_\pi) = \chi(C') + \chi(Y_\pi) - Y_\pi \cdot C' \\ &= \chi(C') + \chi(Y_\pi) + Y_\pi^2. \end{aligned} \tag{4.11}$$

Since $\pi^{-1}(m)$ is locally principal, by Theorem 2.17 of [Ya7], $Y_\pi^2 = -\mathrm{mult}(V, p)$. So we have

$$\mathrm{mult}(V, p) = \chi(C') - \chi(C) + \chi(Y_\pi). \tag{4.12}$$

Consider the map $\pi\Big|_{C'} : C' \longrightarrow C$. Since $\pi^{-1}(m)$ is locally principal, its restriction on $C'$ is also locally principal. Thus the map $\pi\Big|_{C'}$ factors through a blowing-up $\rho : C_1 \longrightarrow C$ and we have the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\pi} & V \\ \cup\Big| & & \cup\Big| \\ C' & \xrightarrow{\pi|_{C'}} & C \\ & \searrow^{\mu} \quad C_1 \quad {}^{\rho}\nearrow & \end{array}$$

Now we have

$$\chi(C') - \chi(C) = (\chi(C') - \chi(C_1)) + (\chi(C_1) - \chi(C)).$$

In view of (4.1), we can rewrite the above equality as

$$\chi(C') - \chi(C) = (\chi(C') - \chi(C_1)) + \sum_{k\geq 0}(\operatorname{mult}(C,p) - \dim m^k_{C,p}/m^{k+1}_{C,p}). \tag{4.13}$$

Applying (4.12) to (4.13), we have

$$\begin{aligned}\operatorname{mult}(V,p) = (\chi(C') - \chi(C_1)) + \operatorname{mult}(C,p) - 1 + \\ \sum_{k\geq 1}(\operatorname{mult}(C,p) - \dim m^k_{C,p}/m^{m+1}_{C,p}) + \chi(Y_\pi).\end{aligned}$$

Since $\operatorname{mult}(V,p) = \operatorname{mult}(C,p)$, we have

$$\begin{aligned}0 = (\chi(C') - \chi(C_1)) + (\operatorname{mult}(C,p) - \text{embed}\dim(C,p) - 1) + \chi(Y_\pi) \\ + \sum_{k\geq 2}(\operatorname{mult}(C,p) - \dim m^k_{C,p}/m^{k+1}_{C,p}).\end{aligned} \tag{4.14}$$

By Leray spectral sequence and the birationality of the map $\mu$, we have $\chi(C') - \chi(C_1) = \chi(\mu_*\mathcal{O}_{C'}/\mathcal{O}_{C_1})$. Since $\mu_*\mathcal{O}_{C'}/\mathcal{O}_{C_1}$ is of dimension zero, $\chi(\mu_*\mathcal{O}_{C'}/\mathcal{O}_{C_1}) \geq 0$ i.e. $\chi(C') - \chi(C_1) \geq 0$.

By construction $(C,p)$ is a complete intersection singularity of dimension one and $\operatorname{mult}(C,p) = \operatorname{mult}(V,p)$, embed $\dim(C,p) =$ embed $\dim(V,p) - 1$. On sees that $\operatorname{mult}(C,p) -$ embed $\dim(C,p) - 1 = \operatorname{mult}(V,p) -$ embed $\dim(V,p) \geq 0$ by Abhyanker's inequality.

On the other hand, we always have $\operatorname{mult}(C,p) \geq \dim m^k_{C,p}/m^{k+1}_{C,p}$ for $k \geq 0$ in view of [Sa] (see p. 40 and p. 44 of [Sa]). We also observe that $\chi(Y_\pi) \geq 0$ by a theorem of Laufer [La2] (cf. Theorem 2.4 (ii) above).

The above observations together with (4.14) imply that every term on the right hand side of (4.14) must be zero. In particular we have $\operatorname{mult}(V,p) =$ embed $\dim(V,p)$. □

**Theorem 4.2.** *Let* $(V,p)$ *be a two-dimensional weakly elliptic complete intersection singularity. Let* $\operatorname{mult}(V,p)$ *be the multiplicity of* $(V,p)$. *Then*

(1) $(V,p)$ *is a hypersurface singularity if and only if* $\operatorname{mult}(V,p)$ *is either* 2 *or* 3
(2) $(V,p)$ *is a complete intersection but not hypersurface if and only if* $\operatorname{mult}(V,p) = 4$. *In this case* $(V,p)$ *is defined by two functions in* $\mathbb{C}^4$, *each of which has multiplicity two.*

*Proof.* If $\operatorname{mult}(V,p) = 2$, then it is well known that $(V,p)$ is necessarily a hypersurface singularity [Ab]. In particular the embed $\dim(V,p) = 3$.

From now on, we shall assume that $\operatorname{mult}(V,p) \geq 3$. Then Theorem 4.1 and Corollary 3.1 imply

$$2 + k = \text{embed}\dim(V,p) = \operatorname{mult}(V,p) \geq 2^k \tag{4.15}$$

where $k = \operatorname{codim}(V,p) =$ number of defining equations of $(V,p)$. From (4.15), we know that $k \leq 2$. $k = 1$ if and only if $(V,p)$ is a hypersurface singularity in $\mathbb{C}^3$ with multiplicity 3. $k = 2$, if and only if $(V,p)$ is a complete intersection singularity with embedding dimension 4 and multiplicity 4.

To finish the proof, we shall assume that $\mathrm{mult}(V,p) = 4$. Let $f_1$, $f_2$ be the holomorphic functions which define $(V,p)$. Let $H_1, H_2$ be the hypersurface defined by $f_1$ and $f_2$ respectively. We claim that $\mathrm{mult}(H_i,p) \geq 2$ for $i = 1,2$. Suppose $\mathrm{mult}(H_1,p) = 1$. Then $p$ is a smooth point of $H_1$ and hence we may embed$(V,p)$ into $\mathbb{C}^3$, i.e. $\mathrm{mult}(V,p) = \mathrm{embed}\dim(V,p) = 3$. It is absurd. By Corollary 3.1, we have $4 \geq \mathrm{mult}(H_1,p)\mathrm{mult}(H_2,p) \geq 4$. This implies that $\mathrm{mult}(H_1,p) = \mathrm{mult}(H_2,p) = 2$. □

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John Tyler Community College, 13101 Jefferson Davis Highway, Chester, VA 23831-5316

Department of MSCS (M/C 249), University of Illinois at Chicago, 851 South Morgan Street, Chicago, Illinois 60607-7045
*E-mail address*: yau@uic.edu