Mathematical Research Letters **11**, 187–196 (2004)

ARITHMETIC PROPERTIES OF PERIODIC MAPS

Zhi-Wei Sun

Abstract. Let $\psi_1,\dots,\psi_k$ be periodic maps from $\mathbb{Z}$ to a field of characteristic p (where p is zero or a prime). Assume that positive integers $n_1,\dots,n_k$ not divisible by p are their periods respectively. We show that $\psi_1+\dots+\psi_k$ is constant if $\psi_1(x)+\dots+\psi_k(x)$ equals a constant for $|S|$ consecutive integers x where $S=\bigcup_{s=1}^k\{r/n_s:\ r=0,\dots,n_s-1\}$. We also present some new results on finite systems of arithmetic sequences.

1. Introduction

For $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+=\{1,2,3,\dots\}$ we call

$$a(n)=a+n\mathbb{Z}=\{a+nx:\ x\in\mathbb{Z}\}$$

an *arithmetic sequence* with modulus n. For a finite system

$$A=\{a_s(n_s)\}_{s=1}^k \tag{1.1}$$

of such sequences, the *covering function* $w_A\colon\mathbb{Z}\to\mathbb{Z}$ given by

$$w_A(x)=|\{1\leqslant s\leqslant k:\ x\in a_s(n_s)\}| \tag{1.2}$$

is obviously periodic modulo the least common multiple $[n_1,\dots,n_k]$ of all the moduli $n_1,\dots,n_k$. If $w_A(x)\leqslant 1$ for all $x\in\mathbb{Z}$ (i.e., $a_i(n_i)\cap a_j(n_j)=\emptyset$ if $1\leqslant i<j\leqslant k$), then we say that (1.1) is *disjoint*. When $w_A(x)\geqslant 1$ for all $x\in\mathbb{Z}$ (i.e., $\bigcup_{s=1}^k a_s(n_s)=\mathbb{Z}$), (1.1) is called a *cover* of $\mathbb{Z}$.

A famous result of H. Davenport, L. Mirsky, D. Newman and R. Radó (cf. [NZ]) states that if (1.1) is a disjoint cover of $\mathbb{Z}$ with $1<n_1\leqslant\dots\leqslant n_{k-1}\leqslant n_k$ then we must have $n_{k-1}=n_k$. In 1958 S. K. Stein [St] conjectured that if (1.1) is disjoint with $1<n_1<\dots<n_k$ then there exists an integer $x\notin\bigcup_{s=1}^k a_s(n_s)$ with $1\leqslant x\leqslant 2^k$. In 1965 P. Erdős [E2] offered a prize for a proof of his

Received November 17, 2002, Revised March 2, 2004.

2000 *Mathematics Subject Classification.* Primary 11B25; Secondary 11A07, 11A25, 11Y16, 68Q25.

Supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, and the Key Program of the National Natural Science Foundation of P. R. China.

following stronger conjecture (see [E1]): (1.1) forms a cover of $\mathbb{Z}$ if it covers those integers from 1 to 2^k. (The above 2^k is best possible because $\{2^{s-1}(2^s)\}_{s=1}^k$ covers $1,\dots,2^k-1$ but does not cover any multiple of 2^k.) In 1969–1970 R. B. Crittenden and C. L. Vanden Eynden [CV1, CV2] supplied a long and awkward proof of the Erdős conjecture for $k \geqslant 20$.

Let m be a positive integer. In [Su4, Su5] the author called (1.1) an *m-cover* of $\mathbb{Z}$ if $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$, and an *exact m-cover* of $\mathbb{Z}$ if $w_A(x) = m$ for all $x \in \mathbb{Z}$. Recently the author [Su10] found that m-covers of $\mathbb{Z}$ are closely related to subset sums in a field and zero-sum problems on abelian groups.

Here is a result of [Su4, Su5] stronger than Erdős' conjecture: (1.1) *forms an m-cover of $\mathbb{Z}$ if it covers $|\{\{\sum_{s\in I} m_s/n_s\} : I \subseteq \{1,\dots,k\}\}|$ consecutive integers at least m times, where the given $m_1,\dots,m_k \in \mathbb{Z}^+$ are relatively prime to $n_1,\dots,n_k$ respectively.* (As usual the fractional part of a real number x is denoted by $\{x\}$.) In [Su5] the author asked whether we have a similar result for exact m-covers of $\mathbb{Z}$. The answer is actually negative, moreover there is no constant $c(k,m) \in \mathbb{Z}^+$ such that (1.1) forms an exact m-cover of $\mathbb{Z}$ whenever it covers $c(k,m)$ consecutive integers exactly m times. In fact, if (1.1) is an exact m-cover of $\mathbb{Z}$, then for any integer $N > 1$ the system $\{a_1(n_1),\dots,a_k(n_k),0(N)\}$ covers $1,\dots,N-1$ exactly m times but it covers 0 exactly $m+1$ times! (This observation is due to the author's student H. Pan.)

For an assertion P we adopt Iverson's notation

$$[\![P]\!] = \begin{cases} 1 & \text{if } P \text{ holds,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Observe that $w_A(x) = \sum_{s=1}^k \psi_s(x)$ where $\psi_s(x) = [\![n_s \mid x - a_s]\!]$ is periodic modulo n_s.

Our first result is completely new!

Theorem 1.1. *Let F be a field of characteristic p where p is zero or a prime. Let $n_1,\dots,n_k$ be positive integers not divisible by p, and let $\psi_1,\dots,\psi_k$ be maps from $\mathbb{Z}$ to F with periods $n_1,\dots,n_k$ respectively. Then $\psi_1+\dots+\psi_k = 0$ if $\psi_1(x)+\dots+\psi_k(x) = 0$ for $\sum_{d\in D}\varphi(d)$ consecutive integers x, where φ is Euler's totient function, $D = \bigcup_{s=1}^k D(n_s)$, and $D(n)$ denotes the set of positive divisors of $n \in \mathbb{Z}^+$.*

Remark 1.1. Clearly $\sum_{d\in D}\varphi(d)$ in Theorem 1.1 equals the cardinality of the set

$$\bigcup_{d\in D}\left\{\frac{c}{d} : \ 0 \leqslant c < d \text{ and } (c,d) = 1\right\} = \bigcup_{s=1}^k \left\{\frac{r}{n_s} : \ r = 0,1,\dots,n_s-1\right\},$$

where (c,d) is the greatest common divisor of c and d. The result stated in the abstract is equivalent to Theorem 1.1 since a constant can be viewed as a function on $\mathbb{Z}$ periodic mod 1.

Corollary 1.1. *Let $w(x)$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$ with period $n_0 \in \mathbb{Z}^+$. Then $w(x)$ is the covering function of* (1.1) *if $w_A(x) = w(x)$ for*

$$\Big|\bigcup_{s=0}^{k}\Big\{0, \frac{1}{n_s}, \dots, \frac{n_s-1}{n_s}\Big\}\Big| \leqslant n_0 + n_1 + \dots + n_k - k$$

consecutive integers x. In particular, (1.1) *forms an exact m-cover of $\mathbb{Z}$ if it covers $|\bigcup_{s=1}^{k}\{r/n_s :\ r = 0, \dots, n_s-1\}|$ consecutive integers exactly m times.*

Proof. Let $D = \bigcup_{s=0}^{k} D(n_s)$. As

$$\psi(x) := w_A(x) - w(x) = -w(x) + \sum_{s=1}^{k}[\![n_s \mid x - a_s]\!]$$

vanishes for $|\bigcup_{s=0}^{k}\{r/n_s :\ r = 0, \dots, n_s-1\}| = \sum_{d\in D}\varphi(d)$ consecutive integers x, we have $\psi(x) = 0$ for all $x \in \mathbb{Z}$ by Theorem 1.1. When $n_0 = 1$ and $w(x) = m \in \mathbb{Z}^+$, this yields the latter result in Corollary 1.1. □

Remark 1.2. The problem whether a given $A = \{a_s(n_s)\}_{s=1}^{k}$ forms a cover of $\mathbb{Z}$ is known to be co-NP-complete. (See, e.g. [GJ] and [T].) However, Corollary 1.1 indicates that we can check whether system A has a given covering function in polynomial time! In 1997 the author [Su6] showed that if (1.1) covers all the integers the same number of times then

$$\Big\{\sum_{s\in I}\frac{1}{n_s} :\ I \subseteq \{1, \dots, k\}\Big\} \supseteq \bigcup_{s=1}^{k}\Big\{\frac{r}{n_s} :\ r = 0, \dots, n_s-1\Big\}.$$

Example 1.1. Let (1.1) be an exact m-cover of $\mathbb{Z}$, and let n be an integer greater than n_k. Then the system

$$A' = \{a_1(n_1), \dots, a_{k-1}(n_{k-1}), a_k + n_k(n)\}$$

covers each of the consecutive integers $a_k + 1, \dots, a_k + 2n_k - 1$ exactly m times but it does not cover a_k or $a_k + 2n_k$ exactly m times. For example, $B = \{1(2), 2(4), 0(4)\}$ is a disjoint cover of $\mathbb{Z}$, thus $B' = \{1(2), 2(4), 4(6)\}$ covers $1, \dots, 7$ exactly once but it is not a disjoint cover of $\mathbb{Z}$. Note that the set $\bigcup_{n\in\{2,4,6\}}\{r/n :\ r = 0, \dots, n-1\}$ just has 8 elements.

Corollary 1.2. *Let* (1.1) *be a system of arithmetic sequences, and let m be any integer greater than $k - f([n_1, \dots, n_k])$.* (*The function f is given by $f(1) = 0$ and $f(\prod_{i=1}^{r} p_i) = \sum_{i=1}^{r}(p_i - 1)$ where $p_1, \dots, p_r$ are primes.*) *Then there is an $x \in \{0, 1, \dots, |S|-1\}$ such that $w_A(x) \neq m$ where $S = \bigcup_{s=1}^{k}\{r/n_s :\ r = 0, 1, \dots, n_s-1\}$.*

Proof. If (1.1) is an exact m-cover of $\mathbb{Z}$, then $k \geqslant m + f([n_1, \dots, n_k])$ by Corollary 4.5 of [Su7]. Thus, in view of the condition, (1.1) does not form an exact m-cover of $\mathbb{Z}$ and hence the desired result follows from Corollary 1.1. □

Our next theorem extends some earlier work in [Su4, Su5].

Theorem 1.2. *Let $n_1, \dots, n_k$ be positive integers, and let $R_1, \dots, R_k$ be finite subsets of $\mathbb{Z}$. For $s = 1, \dots, k$, let c_{st} lie in the complex field $\mathbb{C}$ for each $t \in R_s$, and set*

$$X_s = \Big\{x \in \mathbb{Z} : \sum_{t \in R_s} c_{st} e^{2\pi i \frac{t}{n_s} x} = 0\Big\}. \tag{1.4}$$

If the system $\{X_s\}_{s=1}^k$ covers W consecutive integers at least m times where $1 \leqslant m \leqslant k$ and

$$W = \max_{\substack{I \subseteq \{1,\dots,k\} \\ |I| = k-m+1}} \left| \left\{ \left\{ \sum_{s \in I} \frac{r_s}{n_s} \right\} : \ r_s \in R_s \right\} \right| \leqslant \max_{\substack{I \subseteq \{1,\dots,k\} \\ |I| = k-m+1}} \prod_{s \in I} |R_s|, \tag{1.5}$$

then it covers every integer at least m times.

Corollary 1.3. *Let (1.1) be a system of arithmetic sequences, and let $m_1, \dots, m_k$ be integers relatively prime to $n_1, \dots, n_k$ respectively. Let l be any nonnegative integer with $w_A(x) \geqslant l$ for all $x \in \mathbb{Z}$, and set*

$$W_l = \max_{\substack{I \subseteq \{1,\dots,k\} \\ |I| = k-l}} \left| \left\{ \left\{ \sum_{s \in J} \frac{m_s}{n_s} \right\} : \ J \subseteq I \right\} \right| \leqslant 2^{k-l}. \tag{1.6}$$

Then the covering function $w_A(x)$ takes its minimum on every set of W_l consecutive integers.

Proof. Without loss of generality we may assume that $1 \leqslant m_s \leqslant n_s$ for all $s = 1, \dots, k$. As $m(A) = \min_{x \in \mathbb{Z}} w_A(x) \geqslant l$ and $W_l \geqslant W_{m(A)}$, it suffices to work with $l = m(A)$ below.

The case $l = k$ is trivial, so we let $l < k$. Set $c_{s0} = 1$ and $c_{sm_s} = -e^{-2\pi i a_s m_s / n_s}$ for $s = 1, \dots, k$. Since m_s and n_s are relatively prime,

$$X_s := \Big\{x \in \mathbb{Z} : \ c_{s0} e^{2\pi i \frac{0}{n_s} x} + c_{sm_s} e^{2\pi i \frac{m_s}{n_s} x} = 0\Big\} = a_s(n_s).$$

Applying Theorem 1.2 with $m = l + 1$ and $R_s = \{0, m_s\}$ $(1 \leqslant s \leqslant k)$, we immediately get the desired result. □

Remark 1.3. (a) [Su9] contains some other interesting results on the covering function of (1.1). (b) W_l in (1.6) might be smaller than its value in the case $m_1 = \dots = m_k = 1$. Let $n_1 = 3,\ n_2 = 5$ and $n_3 = 15$. Set

$$W_0(m_1, m_2, m_3) = \left| \left\{ \left\{ \sum_{s \in J} \frac{m_s}{n_s} \right\} : \ J \subseteq \{1, 2, 3\} \right\} \right|$$

for $m_1, m_2, m_3 \in \mathbb{Z}$. Then $W_0(1, 1, 2) = 7 < W_0(1, 1, 1) = 8$.

Our third theorem characterizes the least period of a covering function.

Theorem 1.3. *Let $\lambda_s \in \mathbb{C}$, $a_s \in \mathbb{Z}$ and $n_s \in \mathbb{Z}^+$ for $s = 1, \ldots, k$. Then the smallest positive period n_0 of the (weighted) covering function*

$$w(x) = \sum_{s=1}^{k} \lambda_s [\![n_s \mid x - a_s]\!]$$

is the least $n \in \mathbb{Z}^+$ such that $\alpha n \in \mathbb{Z}$ for all those $\alpha \in [0, 1)$ with

$$\sum_{\substack{1 \leqslant s \leqslant k \\ \alpha n_s \in \mathbb{Z}}} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \neq 0.$$

Remark 1.4. Under the condition of Theorem 1.3, it can be easily checked that $\sum_{x=0}^{N-1} w(x)/N = \sum_{s=1}^{k} \lambda_s/n_s$ where $N = [n_1, \ldots, n_k]$. If $w(x) = 0$ for all $x \in \mathbb{Z}$, then $n_0 = 1$ and hence

$$\sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} = 0 \quad \text{for all } \alpha \in [0, 1). \tag{1.7}$$

This was first obtained by the author [Su2] in 1991 via an analytic method, and the converse was proved in [Su3]. In [Su8] the author determined those functions f: $\bigcup_{n \in \mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ such that $\sum_{s=1}^{k} \lambda_s f(a_s + n_s\mathbb{Z})$ only depends on the covering function $w(x)$, this was announced by the author [Su1] in 1989.

Let l be a positive integer, and let

$$\mathbb{Z}^l = \{\vec{x} = \langle x_1, \ldots, x_l \rangle : \ x_1, \ldots, x_l \in \mathbb{Z}\}$$

be the direct sum of l copies of the ring $\mathbb{Z}$. For $\vec{x}, \vec{y} \in \mathbb{Z}^l$, we use $\vec{x} \mid \vec{y}$ to mean that $\vec{y} = \vec{q}\vec{x} = \langle q_1 x_1, \ldots, q_l x_l \rangle$ for some $\vec{q} \in \mathbb{Z}^l$. A function $\Psi : \mathbb{Z}^l \to \mathbb{C}$ is said to be *periodic modulo* $\vec{n} \in \mathbb{Z}^l$ if $\Psi(\vec{x}) = \Psi(\vec{y})$ whenever $\vec{x} - \vec{y} = \langle x_1 - y_1, \ldots, x_l - y_l \rangle$ is divisible by $\vec{n}$. For $x_1, \ldots, x_l \in \mathbb{Z}$, we also use $[x_t]_{1 \leqslant t \leqslant l}$ to denote the least common multiple of $x_1, \ldots, x_l$.

Theorem 1.4. *Let $\lambda_s \in \mathbb{C}$, $\vec{a}_s \in \mathbb{Z}^l$ and $\vec{n}_s \in (\mathbb{Z}^+)^l$ for $s = 1, \ldots, k$ where $l \in \mathbb{Z}^+$. Suppose that the function*

$$w(\vec{x}) = \sum_{s=1}^{k} \lambda_s [\![\vec{n}_s \mid \vec{x} - \vec{a}_s]\!] \tag{1.8}$$

is periodic modulo $\vec{n}_0 \in (\mathbb{Z}^+)^l$. Let $\vec{d} \in (\mathbb{Z}^+)^l$, $\vec{d} \nmid \vec{n}_0$ and

$$I(\vec{d}) = \{1 \leqslant s \leqslant k : \ \vec{d} \mid \vec{n}_s\} \neq \emptyset.$$

If $\sum_{s\in I(\vec{d}\,)}\lambda_s/(n_{s1}\cdots n_{sl})\neq 0$, *then*

$$|I(\vec{d}\,)|\geqslant\left|\left\{\left\{\sum_{t=1}^{l}\frac{a_{st}}{d_t}\right\}: s\in I(\vec{d}\,)\right\}\right|\geqslant\min_{\substack{0\leqslant s\leqslant k\\ \vec{d}\nmid\vec{n}_s}}\left[\frac{d_t}{(d_t,n_{st})}\right]_{1\leqslant t\leqslant l}\geqslant p(d_1\cdots d_l)$$

where we use $p(m)$ *to denote the least prime divisor of an integer* $m>1$.

Remark 1.5. Theorem 1.4 is a generalization of the main result of [Su2] which corresponds to the case $l=1$ and improves the Znám–Newman result [N].

Corollary 1.4. *Let* $\lambda_s\in\mathbb{C}\setminus\{0\}$, $\vec{a}_s\in\mathbb{Z}^l$ *and* $\vec{n}_s\in(\mathbb{Z}^+)^l$ *for* $s=1,\dots,k$ *where* $l\in\mathbb{Z}^+$. *Suppose that all those moduli* $\vec{n}_s$ *which are maximal with respect to divisibility are distinct. Then the function* $w(\vec{x})$ *given by* (1.8) *is periodic modulo* $\vec{n}_0\in(\mathbb{Z}^+)^l$ *if and only if* $\vec{n}_0$ *is divisible by all the moduli* $\vec{n}_1,\dots,\vec{n}_k$.

Proof. If $\vec{n}_s\mid\vec{n}_0$ for all $s=1,\dots,k$, then the function $w(\vec{x})$ is obviously periodic mod $\vec{n}_0$.

Now suppose that $w(\vec{x})$ is periodic modulo $\vec{n}_0$ but not all the moduli divide $\vec{n}_0$. Then there exists a maximal modulus $\vec{n}_r$ with respect to divisibility such that $\vec{n}_r\nmid\vec{n}_0$. By the condition,

$$I(\vec{n}_r):=\{1\leqslant s\leqslant k:\ \vec{n}_r\mid\vec{n}_s\}=\{1\leqslant s\leqslant k:\ \vec{n}_s=\vec{n}_r\}=\{r\}.$$

On the other hand, by Theorem 1.4 we should have $|I(\vec{n}_r)|\geqslant p(n_{r1}\cdots n_{rl})$. The contradiction ends our proof. □

Remark 1.6. Corollary 1.4 in the case $l=1$ was essentially established by Š. Porubský [P].

2. Proofs of Theorems 1.1–1.4

Lemma 2.1. *Let* $c_1,\dots,c_n$ *lie in a field* F, *and let* $z_1,\dots,z_n$ *be distinct elements of* $F\setminus\{0\}$. *If* $\sum_{j=1}^n c_jz_j^x$ *vanishes for* n *consecutive integers* x, *then it vanishes for all* $x\in\mathbb{Z}$.

Proof. Suppose that $\sum_{j=1}^n c_jz_j^{h+i-1}=0$ for every $i=1,\dots,n$ where $h\in\mathbb{Z}$. Since the Vandermonde determinant

$$\|z_j^{i-1}\|_{1\leqslant i,j\leqslant n}=\begin{vmatrix}1&1&\dots&1\\ z_1&z_2&\dots&z_n\\ \vdots&\vdots&\ddots&\vdots\\ z_1^{n-1}&z_2^{n-1}&\dots&z_n^{n-1}\end{vmatrix}=\prod_{1\leqslant i<j\leqslant n}(z_j-z_i)$$

does not vanish, by Cramer's rule we have $c_jz_j^h=0$ and hence $c_j=0$ for all $j=1,\dots,n$. Therefore $\sum_{j=1}^n c_jz_j^x=0$ for any $x\in\mathbb{Z}$. □

Proof of Theorem 1.1. As p does not divide $N = [n_1, \dots, n_k]$, the equation $x^N - 1 = 0$ has N distinct roots in the algebraic closure E of the field F. The multiplicative group $\{\zeta \in E : \ \zeta^N = 1\}$ of order N is cyclic, so E contains an element ζ of multiplicative order N. For $a \in \mathbb{Z}$ and $1 \leqslant s \leqslant k$, we have the geometric series

$$\frac{1}{n_s} \sum_{r=0}^{n_s - 1} \zeta^{\frac{N}{n_s} ar} = [\![n_s \mid a]\!]. \tag{2.1}$$

Therefore

$$\begin{aligned}
\sum_{s=1}^{k} \psi_s(x) &= \sum_{s=1}^{k} \sum_{a=0}^{n_s - 1} [\![n_s \mid a - x]\!] \psi_s(a) \\
&= \sum_{s=1}^{k} \sum_{a=0}^{n_s - 1} \frac{1}{n_s} \sum_{r=0}^{n_s - 1} \zeta^{\frac{N}{n_s}(a-x)r} \psi_s(a) \\
&= \sum_{s=1}^{k} \frac{1}{n_s} \sum_{a=0}^{n_s - 1} \psi_s(a) \sum_{\substack{0 \leqslant \alpha < 1 \\ \alpha n_s \in \mathbb{Z}}} \zeta^{\alpha N (a - x)} \\
&= \sum_{\alpha \in S} \left(\zeta^{-\alpha N} \right)^x \left(\sum_{s=1}^{k} \frac{[\![\alpha n_s \in \mathbb{Z}]\!]}{n_s} \sum_{a=0}^{n_s - 1} \psi_s(a) \zeta^{\alpha N a} \right),
\end{aligned}$$

where S is the set

$$\{\alpha \in [0,1) : \ \alpha n_s \in \mathbb{Z} \text{ for some } 1 \leqslant s \leqslant k\} = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : \ r = 0, \dots, n_s - 1 \right\}.$$

As those $\zeta^{-\alpha N}$ with $\alpha \in S$ are distinct, applying Lemma 2.1 we find that $\sum_{s=1}^{k} \psi_s(x) = 0$ for $|S|$ consecutive integers x if and only if $\sum_{s=1}^{k} \psi_s(x) = 0$ for all $x \in \mathbb{Z}$. By Remark 1.1, $|S| = \sum_{d \in D} \varphi(d)$. This concludes the proof. □

Proof of Theorem 1.2. Clearly an integer x is covered by $\{X_s\}_{s=1}^{k}$ at least m times if and only if x is covered by $\{X_s\}_{s \in I}$ for all $I \subseteq \{1, \dots, k\}$ with $|I| = k - m + 1$.

Now let $I \subseteq \{1, \dots, k\}$ and $|I| = k - m + 1$. For any $x \in \mathbb{Z}$, we have

$$\begin{aligned}
\prod_{s \in I} \sum_{t \in R_s} c_{st} e^{2\pi i \frac{t}{n_s} x} &= \sum_{r_s \in R_s \text{ for } s \in I} \left(\prod_{s \in I} c_{s r_s} \right) e^{2\pi i x \sum_{s \in I} r_s / n_s} \\
&= \sum_{\theta \in R(I)} C_{I,\theta} e^{2\pi i \theta x}
\end{aligned}$$

where

$$R(I) = \left\{ \left\{ \sum_{s \in I} \frac{r_s}{n_s} \right\} : \ r_s \in R_s \right\} \ \text{ and } \ C_{I,\theta} = \sum_{\substack{r_s \in R_s \text{ for } s \in I \\ \{\sum_{s \in I} r_s / n_s\} = \theta}} \prod_{s \in I} c_{s r_s}.$$

Since those $e^{2\pi i\theta}$ with $\theta \in R(I)$ are distinct, by Lemma 2.1 the system $\{X_s\}_{s\in I}$ covers $|R(I)|$ consecutive integers x if and only if it covers all $x \in \mathbb{Z}$.

In view of the above, we immediately obtain the desired result. $\square$

Proof of Theorem 1.3. Let $S = \{0 \leqslant \alpha < 1 :\ \alpha n_s \in \mathbb{Z} \text{ for some } 1 \leqslant s \leqslant k\}$ and

$$T = \Big\{0 \leqslant \alpha < 1 :\ c_\alpha = \sum_{\substack{1\leqslant s\leqslant k \\ \alpha n_s \in \mathbb{Z}}} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \neq 0\Big\}.$$

For each $s = 1, \ldots, k$ the arithmetical function $\psi_s(x) = \lambda_s[\![n_s \mid x - a_s]\!]$ is periodic modulo n_s. By the proof of Theorem 1.1, for any $x \in \mathbb{Z}$ we have

$$w(x) = \sum_{s=1}^{k} \lambda_s [\![n_s \mid x - a_s]\!] = \sum_{\alpha \in S} e^{-2\pi i \alpha x} c_\alpha = \sum_{\alpha \in T} e^{-2\pi i \alpha x} c_\alpha.$$

Let n be the least positive integer such that $\alpha n \in \mathbb{Z}$ for all $\alpha \in T$. By the above, $w(x) = w(x + n)$ for all $x \in \mathbb{Z}$. Thus $n_0 \mid n$.

If $T = \emptyset$, then $n = 1$ and hence $n_0 = n$. In the case $T \neq \emptyset$, we have

$$0 = w(x) - w(x + n_0) = \sum_{\alpha \in T} e^{-2\pi i \alpha x}(1 - e^{-2\pi i \alpha n_0}) c_\alpha$$

for every $x = 0, \ldots, |T| - 1$, and hence $(1 - e^{-2\pi i \alpha n_0}) c_\alpha = 0$ for any $\alpha \in T$ (Vandermonde). Now that $\alpha n_0 \in \mathbb{Z}$ (i.e., $e^{-2\pi i \alpha n_0} = 1$) for all $\alpha \in T$, we have $n_0 \geqslant n$ and thus $n_0 = n$.

The proof of Theorem 1.3 is now complete. $\square$

Proof of Theorem 1.4. Let $\vec{c}$ be any vector in $\mathbb{Z}^l$ with $\vec{d} \nmid \vec{c}\vec{n}_0$. Then, for some $1 \leqslant r \leqslant l$ we have $d_r \nmid c_r n_{0r}$. Note that $\vec{n}_0$ divides the vector $\langle 0, \ldots, 0, n_{0r}, 0, \ldots, 0\rangle$. For any $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_l \in \mathbb{Z}$, since

$$\sum_{s=1}^{k} \Big(\lambda_s \prod_{\substack{t=1 \\ t\neq r}}^{l} [\![n_{st} \mid x_t - a_{st}]\!]\Big) [\![n_{sr} \mid x_r - a_{sr}]\!] = w(\vec{x})$$

is periodic mod n_{0r} as a function of x_r, by Theorem 1.3 we must have

$$\sum_{\substack{s=1 \\ d_r \mid c_r n_{sr}}}^{k} \Big(\lambda_s \prod_{\substack{t=1 \\ t\neq r}}^{l} [\![n_{st} \mid x_t - a_{st}]\!]\Big) \frac{e^{2\pi i (c_r/d_r) a_{sr}}}{n_{sr}} = 0.$$

(Recall that $(c_r/d_r) n_{0r} \notin \mathbb{Z}$.)

Let $J = \{1 \leqslant s \leqslant k : \ d_r \mid c_r n_{sr}\}$ and $\lambda'_s = \lambda_s n_{sr}^{-1} e^{2\pi i a_{sr} c_r / d_r}$ for $s \in J$. Given $r' \in \{1, \ldots, l\} \setminus \{r\}$ and $x_t \in \mathbb{Z}$ with $t \neq r, r'$, we have

$$\begin{aligned}\sum_{s \in J} \Bigg(\lambda'_s \prod_{\substack{t=1 \\ t \neq r, r'}}^{l} [\![n_{st} \mid x_t - a_{st}]\!] \Bigg) [\![n_{sr'} \mid x_{r'} - a_{sr'}]\!] \\ = \sum_{s \in J} \lambda'_s \prod_{\substack{t=1 \\ t \neq r}}^{l} [\![n_{st} \mid x_t - a_{st}]\!] = 0\end{aligned}$$

for all $x_{r'} \in \mathbb{Z}$. By applying Remark 1.4 $l-1$ times we finally obtain that

$$\sum_{\substack{s=1 \\ \vec{d} \mid \vec{c}\vec{n}_s}}^{k} \frac{\lambda_s}{n_{s1} \cdots n_{sl}} e^{2\pi i \sum_{t=1}^{l} a_{st} c_t / d_t} = 0. \tag{2.2}$$

Set $m = \min_{0 \leqslant s \leqslant k, \, \vec{d} \nmid \vec{n}_s} [d_t / (d_t, n_{st})]_{1 \leqslant t \leqslant l}$. Clearly $m \geqslant p(d_1 \cdots d_l)$. Let c be any positive integer less than m. For $s = 0, 1 \cdots, k$ we have

$$\vec{d} \mid c\vec{n}_s \Leftrightarrow d_t \mid c n_{st} \text{ for all } t = 1, \ldots, l \Leftrightarrow \left[\frac{d_t}{(d_t, n_{st})} \right]_{1 \leqslant t \leqslant l} \Bigg| \, c \Leftrightarrow \vec{d} \mid \vec{n}_s.$$

In other words, $\vec{d} \mid c\vec{n}_s$ if and only if $s \in I(\vec{d})$. (2.2) in the case $\vec{c} = \langle c, \ldots, c \rangle$ yields that

$$\sum_{s \in I(\vec{d})} \frac{\lambda_s}{n_{s1} \cdots n_{sl}} e^{2\pi i c \sum_{t=1}^{l} a_{st} / d_t} = 0.$$

Let $\Theta = \{\{\sum_{t=1}^{l} a_{st}/d_t\} : \ s \in I(\vec{d})\}$. Suppose that $|\Theta| < m$. Then for each $c = 1, \ldots, |\Theta|$ we have

$$\begin{aligned}\sum_{\theta \in \Theta} e^{2\pi i c \theta} \sum_{\substack{s \in I(\vec{d}) \\ \{\sum_{t=1}^{l} a_{st}/d_t\} = \theta}} \frac{\lambda_s}{n_{s1} \cdots n_{sl}} \\ = \sum_{s \in I(\vec{d})} \frac{\lambda_s}{n_{s1} \cdots n_{sl}} e^{2\pi i c \sum_{t=1}^{l} a_{st} / d_t} = 0.\end{aligned}$$

By Lemma 2.1 this holds for all integers c, in particular $c = 0$:

$$\sum_{s \in I(\vec{d})} \frac{\lambda_s}{n_{s1} \cdots n_{sl}} = 0.$$

This directly contradicts one of the hypotheses, whence $|\Theta| \geqslant m$. □

Acknowledgment

The author is indebted to the referee for his valuable suggestions to improve the presentation.

References

[CV1] R. B. Crittenden and C. L. Vanden Eynden, *A proof of a conjecture of Erdős*, Bull. Amer. Math. Soc. **75** (1969), 1326–1329.

[CV2] ———, *Any n arithmetic progressions covering the first* 2^n *integers cover all integers*, Proc. Amer. Math. Soc. **24** (1970), 475–481.

[E1] P. Erdős, *Remarks on number theory IV: Extremal problems in number theory I*, Mat. Lapok **13** (1962), 228–255.

[E2] ———, *Extremal problems in number theory*, Proc. Sympos. Pure Math. **8** (1965), 181–189, Amer. Math. Soc., Providence, R. I..

[GJ] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, W.H. Freeman, New York, 1983.

[N] M. Newman, *Roots of unity and covering sets*, Math. Ann. **191** (1971), 279–282.

[NZ] B. Novák, and Š. Znám, *Disjoint covering systems*, Amer. Math. Monthly **81** (1974), 42–45.

[P] Š. Porubský, *Covering systems and generating functions*, Acta Arith. **26** (1975), 223–231.

[St] S. K. Stein, *Unions of arithmetic sequences*, Math. Ann. **134** (1958), 289–294.

[Su1] Z. W. Sun, *Several results on systems of residue classes*, Adv. Math. (China) **18** (1989), no. 2, 251–252.

[Su2] ———, *An improvement to the Znám–Newman result*, Chinese Quart. J. Math. **6** (1991), no. 3, 90–96.

[Su3] ———, *On a generalization of a conjecture of Erdős*, Nanjing Univ. J. Natur. Sci. **27** (1991), no. 1, 8–15.

[Su4] ———, *Covering the integers by arithmetic sequences*, Acta Arith. **72** (1995), 109–129.

[Su5] ———, *Covering the integers by arithmetic sequences* II, Trans. Amer. Math. Soc. **348** (1996), 4279–4320.

[Su6] ———, *Exact m-covers and the linear form* $\sum_{s=1}^k x_s/n_s$, Acta Arith. **81** (1997), 175–198.

[Su7] ———, *Exact m-covers of groups by cosets*, European J. Combin. **22** (2001), 415–429.

[Su8] ———, *Algebraic approaches to periodic arithmetical maps*, J. Algebra **240** (2001), 723–743.

[Su9] ———, *On the function* $w(x) = |\{1 \leqslant s \leqslant k\colon x \equiv a_s \pmod{n_s}\}|$, Combinatorica **23** (2003), 681–691.

[Su10] ———, *Unification of zero-sum problems, subset sums and covers of* $\mathbb{Z}$, Electron. Res. Announc. Amer. Math. Soc. **9** (2003), 51–60.

[T] S. P. Tung, *Complexity of sentences over number rings*, SIAM J. Comp. **20** (1991), 126–143.

Department of Mathematics (and Institute of Mathematical Science), Nanjing University, Nanjing 210093, The People's Republic of China

E-mail address: zwsun@nju.edu.cn *Homepage*: http://pweb.nju.edu.cn/zwsun