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THE RELATION BETWEEN LOCAL AND GLOBAL DUAL PAIRS

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Abstract. In this note we clarify the relationship between the local and global definitions of dual pairs in Poisson geometry. It turns out that these are not equivalent. For the passage from local to global one needs a connected fiber hypothesis (this is well known), while the converse requires a dimension condition (which appears not to be known). We also provide examples illustrating the necessity of the extra conditions.

1. Regular dual pairs

The set-up we consider is the following. Let (M, ω) be a symplectic manifold (we assume our manifolds to be paracompact), $(P_1, \{\cdot,\cdot\}_1)$ and $(P_2, \{\cdot,\cdot\}_2)$ two Poisson manifolds, and $\pi_1 : M \to P_1$ and $\pi_2 : M \to P_2$ two surjective submersive Poisson maps. In Remark 7 we describe the effect of weakening the condition "submersion" to "open".

Let $\mathcal{F}_j$ denote the algebra of the pull-backs of smooth functions on P_j, that is,

$$\mathcal{F}_j = \pi_j^*(C^\infty(P_j)).$$

Since π_1 and π_2 are Poisson it follows that $\mathcal{F}_1$ and $\mathcal{F}_2$ are Poisson subalgebras of $C^\infty(M)$. If $U \subset M$ is open, we write $\mathcal{F}_j(U)$ for the algebra

$$\mathcal{F}_j(U) = \pi_j^*(C^\infty(\pi_j(U))).$$

This is a Poisson subalgebra of $C^\infty(U)$.

For a subset $A \subset C^\infty(U)$ we write A^c for the centralizer of A with respect to the Poisson structure on $(U, \omega|_U)$, that is,

$$A^c := \{f \in C^\infty(U) \mid \{f, g\}^U = 0 \text{ for all } g \in A\},$$

where $\{\cdot,\cdot\}^U$ is the restriction of the Poisson bracket to U.

Note that in the following two definitions π_1 and π_2 are assumed to be Poisson maps but not necessarily submersions.

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Definition 1. *Consider the diagram*

$$\begin{tikzcd}[column sep=small] & (M,\omega) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (P_1,\{\cdot,\cdot\}_1) & & (P_2,\{\cdot,\cdot\}_2) \end{tikzcd}$$

- *The diagram forms a* ***Howe (H) dual pair*** *if the Poisson subalgebras* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *centralize each other:*

$$\mathcal{F}_1^c = \mathcal{F}_2 \quad \textit{and} \quad \mathcal{F}_2^c = \mathcal{F}_1. \tag{1}$$

- *The diagram forms a* ***Lie-Weinstein (LW) dual pair*** *when* $\ker T\pi_1$ *and* $\ker T\pi_2$ *are symplectically orthogonal distributions. That is, for each* $m \in M$*,*

$$(\ker T_m\pi_1)^\omega = \ker T_m\pi_2. \tag{2}$$

In each case, the dual pair is ***regular*** *when the maps* π_j *are assumed to be surjective submersions; otherwise it is* ***singular****.*

This notion of singular dual pair is less general than that in [11]. Notice that for a regular Lie-Weinstein dual pair, the dimensions of P_1 and P_2 sum to the dimension of M. Actually, we note that if the manifold M is Lindelöf or paracompact as a topological space then the Lie-Weinstein condition cannot hold unless the dual pair is regular. This is because the LW condition implies that the two maps π_1 and π_2 are of complementary rank (the ranks sum to $\dim(M)$) and, by the lower semicontinuity of the rank of a smooth map, the maps must both be of constant rank. Since they are surjective they must be submersions (by Sard's theorem).

We emphasize that the definition of a Lie-Weinstein dual pair is local, while that of a Howe dual pair is global. However, the latter definition can be localized as follows.

Definition 2. *The diagram above forms a* ***local Howe (LH) dual pair*** *if for each* $m \in M$ *and each neighbourhood* V *of* m *there is a neighbourhood* U *of* m *with* $U \subset V$ *such that the algebras* $\mathcal{F}_1(U)$ *and* $\mathcal{F}_2(U)$ *centralize each other in* $C^\infty(U)$*. If in addition,* π_1 *and* π_2 *are surjective submersions, then the local Howe dual pair is said to be* ***regular****; otherwise it is* ***singular****.*

The notion of Howe dual pair has its origins in the study of group representations arising in quantum mechanics (see for instance [3, 5, 13, 4], and references therein) and it appears for the first time in the context of Poisson geometry in [14]. The definition of Lie-Weinstein dual pair can be traced back to [7] and, in its modern formulation, is due to [14]. Examples of dual pairs arising in classical mechanics can be found in [8, 9], and references therein. Further details on dual pairs can also be found in [12].

The relationships between the three notions of regular dual pair can be summed up in the following two results.

Proposition 3. *The two local notions of regular dual pair, that is, Lie-Weinstein and local Howe, are equivalent.*

In Remark 7 we provide an example showing that this result no longer holds if the regularity is dropped.

Theorem 4.

1. *If a regular Howe dual pair is such that the Poisson manifolds P_1 and P_2 are of complementary dimension, that is,* $\dim P_1 + \dim P_2 = \dim M$, *then it forms a regular local Howe dual pair.*
2. *If a regular local Howe dual pair is such that the fibers of π_1 and π_2 are connected then it is a regular Howe dual pair.*

Before giving the proofs (in Section 2), we give two examples showing the necessity of the hypotheses.

Example 5. This example (suggested to us by Andrea Giacobbe) shows that the hypothesis of connected fibers in the passage from local to global in the theorem above is necessary. Let

$$\mathbb{T}^2 = \{(\theta_1, \theta_2) \mid \theta_j \in \mathbb{R}/2\pi\mathbb{Z}\}$$

be the 2-torus considered as a symplectic manifold with the area form $\omega := \mathbf{d}\theta_1 \wedge \mathbf{d}\theta_2$. Consider the diagram $S^1 \overset{\pi_1}{\leftarrow} \mathbb{T}^2 \overset{\pi_2}{\rightarrow} S^1$ with $\pi_j(\theta_1, \theta_2) := j\theta_1$. The fibers of π_2 have two connected components. It is easy to see that this forms a Lie-Weinstein dual pair (and hence a local Howe dual pair) but not a Howe dual pair. Indeed, the function $\cos(\theta_1)$ belongs to $\mathcal{F}_1^c$ but not to $\mathcal{F}_2$.

In the example below and in subsequent proofs, we use the following notation. On the symplectic manifold (M, ω), we write the Poisson tensor as $B \in \Lambda^2(M)$. If $h \in C^\infty(M)$ then the Hamiltonian vector field X_h is defined by $\mathbf{d}h = \mathbf{i}_{X_h}\omega := \omega(X_h, \cdot)$. The Poisson tensor, defined by $B(\mathbf{d}g, \mathbf{d}h) := X_h[g] = \langle \mathbf{d}g, X_h\rangle = \omega(X_g, X_h)$, induces the vector bundle morphism over the identity $B^\sharp : T^*M \to TM$ given by $B^\sharp(\mathbf{d}h) = X_h$.

Example 6. This example shows that without the dimension hypothesis, a regular Howe dual pair need not be locally Howe (nor Lie-Weinstein). Let $M := \mathbb{T}^3 \times \mathbb{R}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Define on M the symplectic structure ω whose Poisson tensor $B \in \Lambda^2(M)$ is given by

$$B^\sharp = \begin{pmatrix} 0 & 1 & 0 & \lambda_1 \\ -1 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & \lambda_3 \\ -\lambda_1 & -\lambda_2 & -\lambda_3 & 0 \end{pmatrix}.$$

Let $\pi : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}$ be the projection onto the $\mathbb{R}$ factor. The Hamiltonian vector field associated to the function π is

$$X_\pi = \lambda_1 \frac{\partial}{\partial \theta_1} + \lambda_2 \frac{\partial}{\partial \theta_2} + \lambda_3 \frac{\partial}{\partial \theta_3}.$$

Then the diagram $\mathbb{R} \xleftarrow{\pi} M \xrightarrow{\pi} \mathbb{R}$ is a regular Howe pair but clearly not a Lie-Weinstein dual pair and hence not a regular local Howe dual pair. In order to see that it is a regular Howe dual pair let $g \in (\pi^* C^\infty(\mathbb{R}))^c$. The trajectories of the vector field X_π on M are irrational windings which are dense in the fibers of π. Since g is invariant under this Hamiltonian flow it must be constant on the fibers of π and hence $g \in \pi^* C^\infty(\mathbb{R})$, as required.

Remarks 7.

1. If one merely assumes the maps π_j to be open rather than submersions, then one can still pass from local Howe to global, provided of course the fibers are connected (see the proof in Section 2).

2. As was already observed, if the Lie-Weinstein condition is satisfied then the two Poisson maps π_j are of constant rank. Any example of a local Howe dual pair which is not Lie-Weinstein must of course be singular (the maps cannot be submersions) and not of constant rank. A simple example is obtained by putting $M = \mathbb{R}^4$ with coordinates (x_1, x_2, y_1, y_2) and its usual symplectic form $\omega = \sum_j \mathbf{d}x_j \wedge \mathbf{d}y_j$. Let $\pi_1 : M \to \mathbb{R}$ be given by $\pi_1(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ (inner product) and $\pi_2 : M \to \mathbb{R}^4$ be given by $\pi_2(\mathbf{x}, \mathbf{y}) = \mathbf{x} \otimes \mathbf{y}$ (outer product!). In coordinates,

$$\pi_2(x_1, x_2, y_1, y_2) = (x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2).$$

The fibers of both π_1 and π_2 are connected and the image $\pi_2(M)$ is a cone in $\mathbb{R}^4$.

We claim that $\mathbb{R} \xleftarrow{\pi_1} M \xrightarrow{\pi_2} \mathbb{R}^4$ forms a singular Howe dual pair. It is clearly not Lie-Weinstein at the origin, as $T\pi_1$ and $T\pi_2$ both vanish at that point. That it is a Howe pair follows from the paper of Karshon and Lerman [6], since π_1 is the orbit map for the $U(2)$ action on $\mathbb{C}^2$ and π_2 its momentum map. Note that in [10] it is shown that momentum maps of representations are G-open, although we do not know whether G-openness is sufficient to be able to pass from local to global in the singular case.

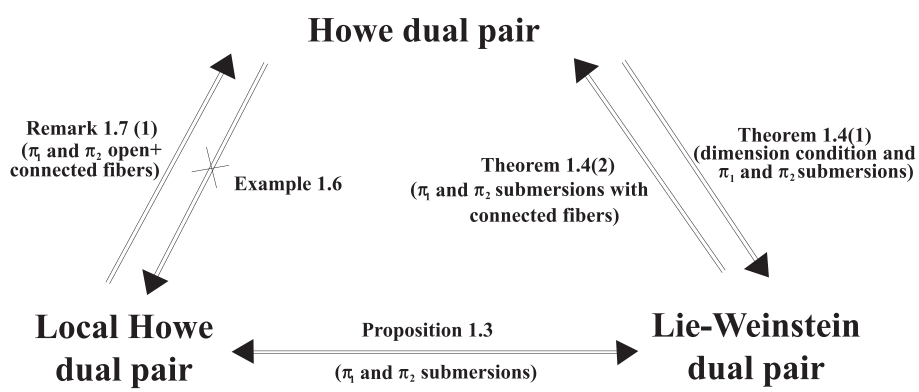
2. Proofs

Define $K_j := \ker T\pi_j$ (for $j = 1, 2$), which are two subbundles of TM. Since the maps π_j are submersions, we have for each $m \in M$

$$K_j(m)^\circ = \{\mathbf{d}f(m) \mid f \in \mathcal{F}_j\}. \tag{3}$$

Furthermore, since $K_j(m)^\omega = B^\sharp(K_j(m)^\circ)$ it follows that

$$K_j(m)^\omega = \{X_f(m) \mid f \in \mathcal{F}_j\}. \tag{4}$$



Proof of Proposition 3. We establish the equivalence of the two local notions of regular dual pair. Recall that π_1 and π_2 are assumed to be surjective submersions.

LH $\Rightarrow$ LW: We wish to show that $(K_1)^\omega = K_2$, which we do by double inclusion.

First we show $(K_1)^\omega \subset K_2$. Let z_j be coordinates on P_2. Then by the local Howe condition, $f \in \mathcal{F}_1$ implies

$$0 = \{f,\, z_j \circ \pi_2\} = \omega(X_f, X_{z_j \circ \pi_2}) \quad \text{for all} \quad j$$

so that $X_f \in \{X_g \mid g \in \mathcal{F}_2\}^\omega = K_2$, by (4).

For the converse inclusion, $K_2 \subset (K_1)^\omega$, let $m \in M$ and $v \in K_2(m)$. Then there is a function f such that $v = X_f(m)$. Then for $g \in \mathcal{F}_2$ we have $\langle \mathbf{d}f(m),\, X_g(m)\rangle = -\mathbf{d}g(m)(v) = 0$, so that $\mathbf{d}f(m) \in (K_2(m)^\omega)^\circ$. Since $(K_2{}^\omega)^\circ$ is a subbundle of T^*M and hence locally trivial, we can choose f on a neighbourhood V of m so that at each $x \in V$, $\mathbf{d}f(x) \in (K_2(x)^\omega)^\circ$. Thus

$$0 = \langle \mathbf{d}(z_j \circ \pi_2)(x), X_f(x)\rangle = \{z_j \circ \pi_2, f\}(x).$$

By the local Howe condition, this implies there is a sub-neighbourhood $U \subset V$ of m such that $f \in \mathcal{F}_1(U)$. Consequently, X_f is a section of $(K_1)^\omega$ over U and, since $v = X_f(m)$, it follows that $K_2 \subset (K_1)^\omega$.

LW $\Rightarrow$ LH: We choose U such that both submersions $\pi_j|_U$ have connected fibers (this can be done using Lemma 8, by choosing any function h on M with a non-degenerate local minimum at m). We prove the equality $\mathcal{F}_2(U) = \mathcal{F}_1(U)^c$, again by double inclusion.

Let $f \in \mathcal{F}_2(U)$. Since $\mathbf{d}f(m) \in K_2(m)^\circ$, for any $m \in U$ and, by hypothesis, $(K_1)^\omega = K_2$, we have

$$X_f(m) \in B^\sharp(m)(K_2(m)^\circ) = K_2(m)^\omega = K_1(m) = (\{\mathbf{d}g(m) \mid g \in \mathcal{F}_1(U)\})^\circ\,,$$

where the last equality follows from (3). Consequently, for an arbitrary $g \in \mathcal{F}_1(U)$, we conclude

$$\{g, f\} = \mathbf{d}g\,(X_f) = 0,$$

which implies that $f \in \mathcal{F}_1(U)^c$.

Conversely, let $f \in \mathcal{F}_1(U)^c$ for some U. In order to prove that $f \in \mathcal{F}_2(U)$ we start by showing that it is locally constant on the fibers of π_2. Indeed, since π_2 is a surjective submersion and the diagram $(P_1, \{\cdot,\cdot\}_1) \stackrel{\pi_1}{\leftarrow} (M, \omega) \stackrel{\pi_2}{\rightarrow} (P_2, \{\cdot,\cdot\}_2)$ forms a Lie-Weinstein dual pair, for any $m \in M$ we have

$$T_m\left(\pi_2^{-1}(\pi_2(m))\right) = K_2(m) = (K_1(m))^{\omega} = B^{\sharp}(m)\left((K_1(m))^{\circ}\right).$$

This equality, together with (3), guarantees that any vector v tangent at m to the fiber $\pi_2^{-1}(\pi_2(m))$ can be written as $v = X_g(m)$, for some $g \in \mathcal{F}_1(U)$. Hence,

$$\mathbf{d}f(m)(v) = \mathbf{d}f(m)\,(X_g(m)) = \{f, g\}(m) = 0.$$

Since both $m \in M$ and $v \in T_m\left(\pi_2^{-1}(\pi_2(m))\right)$ are arbitrary and, by the choice of U, the fibers of $\pi_2|_U$ are connected, this equality guarantees that the function $f \in \mathcal{F}_1(U)^c$ is constant on the fibers of π_2. This implies that there exists a unique function $\overline{f} : \pi_2(U) \to \mathbb{R}$ that satisfies the equality $f = \overline{f} \circ \pi_2|_U$. Since f is smooth and $\pi_2|_U$ a submersion it follows that $\overline{f}$ is smooth and hence $f \in \mathcal{F}_2(U)$.

The equality $\mathcal{F}_1(U) = \mathcal{F}_2(U)^c$ is proved analogously. □

Proof of Theorem 4.

Global ⇒ Local: Let $m \in M$ be an arbitrary point. We will now show that the hypotheses in the statement imply that $(K_1(m))^{\omega} = K_2(m)$. The inclusion $(K_2(m))^{\omega} \subset K_1(m)$ follows from the global Howe hypothesis and the equalities (4) and (3). Indeed,

$$\begin{aligned}(K_2(m))^{\omega} = \{X_f(m) \mid f \in \mathcal{F}_2\} = \{X_g(m) \mid g \in \mathcal{F}_1^c\} \subset (\{\mathbf{d}f(m) \mid f \in \mathcal{F}_1\})^{\circ} \\ = K_1(m).\end{aligned}$$

The converse inclusion follows immediately from the hypothesis on the dimensions and the submersiveness of π_1 and π_2. Indeed,

$$\dim(K_2(m))^{\omega} = \dim P_2 = \dim M - \dim P_1 = \dim K_1(m),$$

and hence $(K_2(m))^{\omega} = K_1(m)$, as required.

Local ⇒ Global: Using the symmetry of the statement with respect to the exchange of π_1 with π_2 it suffices to show that, for instance, $\mathcal{F}_1^c = \mathcal{F}_2$. Notice that the proof below only requires the Poisson maps π_j to be open onto P_j, rather than submersions, and that the P_j are not even required to be manifolds (in which case the proof would hold with an appropriate algebraic definition of smooth function on P_j).

First, it is easy to show that given $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_2$ then $\{f, g\} = 0$. Indeed, let $m \in M$ and let U be a neighbourhood of m on which the local Howe condition holds. Then, $f|_U \in \mathcal{F}_1(U)$ and similarly $g|_U \in \mathcal{F}_2(U)$ and hence,

$$\{f, g\}(m) = \{f|_U, g|_U\}^U(m) = 0, \tag{5}$$

where $\{\cdot,\cdot\}^U$ is the restriction of the Poisson bracket to U. Since $m \in M$, $f \in \mathcal{F}_1$, and $g \in \mathcal{F}_2$ are arbitrary, it follows that $\mathcal{F}_2 \subset \mathcal{F}_1^c$.

Second, let $g \in \mathcal{F}_1^c$; one needs to show that $g \in \mathcal{F}_2$. For each $m \in M$ there is a neighbourhood U_m on which the local Howe hypothesis is valid. This provides a cover of M, from which we can extract a locally finite subcover $\{U_a\}_{a\in A}$.

For each U_a, we claim that $g|_{U_a} \in \mathcal{F}_1(U_a)^c$. It then follows by hypothesis that $g|_{U_a} \in \mathcal{F}_2(U_a)$, which allows us to write

$$g|_{U_a} = \overline{g}_a \circ \pi_2|_{U_a} \tag{6}$$

for some $\overline{g}_a \in C^\infty(\pi_2(U_a))$. Our second claim is that there is a function $\overline{g} \in C^\infty(P_2)$ such that $\overline{g}_a = \overline{g}|_{\pi_2(U_a)}$. The result then follows as $g = \overline{g} \circ \pi_2$. We now establish the two claims.

The first claim is proved by contradiction. Suppose that there exists a function $f \in C^\infty(\pi_1(U_a))$ and $x \in U_a$ such that $\{g|_{U_a}, f \circ \pi_1|_{U_a}\}^{U_a}(x) \neq 0$. Let V_x be an open neighbourhood of x such that $\overline{V_x} \subset \pi_1(U_a)$. Then there is an extension $F \in C^\infty(P_1)$ of $f|_{V_x}$. Since $g \in \mathcal{F}_1^c$ it follows that

$$0 = \{g, F \circ \pi_1\}(x) = \{g|_{U_a}, f \circ \pi_1|_{U_a}\}^{U_a}(x) \neq 0,$$

contradicting (5).

As to the second claim, notice first that (6) implies that g is locally constant along the fibers of π_2. Since by hypothesis these fibers are connected, g is constant on the fibers of π_2 and $\overline{g}$ is therefore well defined. Moreover, $\overline{g}$ coincides with $\overline{g}_a$ on the open sets of the form $\pi_2(U_a)$ and so is smooth. □

Lemma 8. *Let U be a manifold and h be a smooth real-valued function with a non-degenerate local minimum at $u_0 \in U$, and suppose $h(u_0) = 0$. Let $\pi : U \to P$ be a submersion in a neighbourhood of u_0. Then for ε sufficiently small the level sets of π restricted to B_ε are diffeomorphic to an open ball, where B_ε is the connected component of $\{u \in U \mid h(u) < \varepsilon\}$ containing u_0.*

Proof. Write $F_{y,\varepsilon} = B_\varepsilon \cap \pi^{-1}(y)$. We wish to show that when it is nonempty, $F_{y,\varepsilon}$ is diffeomorphic to an open ball. First, choose coordinates near u_0 and $\pi(u_0)$ such that $\pi(x,y) = y$. Since the restriction of h to $\{y = 0\}$ has a non-degenerate minimum at $x = 0$, by the splitting lemma (or Morse lemma with parameters, see e.g. [2, p. 97]) there is a neighbourhood U_1 of u_0 on which one can change coordinates by $(x,y) \mapsto (X,y) = (X(x,y),y)$ in such a way that $h(X,y) = Q(X) + g(y)$, where Q is a positive definite quadratic form and g is a smooth function. Choose ε_1 sufficiently small so that B_{ε_1} is contained in this neighbourhood. In these coordinates, for $\varepsilon \leq \varepsilon_1$ and for each $y \in \pi(U_1)$,

$$F_{y,\varepsilon} = \{(X,y) \in B_\varepsilon \mid Q(X) < \varepsilon - g(y)\}.$$

It is clear that for each y,ε this set is either empty or diffeomorphic to an open ball. □

3. Final remarks

There are a number of theorems in the literature which are stated with the hypothesis that a given set-up is a Howe dual pair, while the proof uses the Lie-Weinstein property. The most famous of these are probably the Symplectic Leaves Correspondence Theorem [14, 1] and Weinstein's theorem on transverse Poisson structures [14, Theorem 8.1]. Here we give an example showing that these theorems fail if one does not assume a local (Lie-Weinstein) hypothesis. Of course, by the theorem above the hypothesis that P_1 and P_2 are of complementary dimension is also sufficient.

Example 9. Let

$$M = \mathbb{T}^2 \times \mathbb{T}^3 \times \mathbb{R} \tag{7}$$

with coordinates (θ, ϕ, x), where $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$, $\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{T}^3$, and $\theta_j, \phi_j \in \mathbb{R}/2\pi\mathbb{Z}$. Let $\pi_1 : M \to \mathbb{T}^3 \times \mathbb{R}$ and $\pi_2 : M \to \mathbb{R}$ be given by

$$\pi_1(\theta, \phi, x) = (\phi, x), \quad \text{and} \quad \pi_2(\theta, \phi, x) = x.$$

We endow M with a Poisson structure whose Poisson tensor can be written in block form as

$$B^\sharp = \begin{bmatrix} 0 & A & B \\ -A^T & C & 0 \\ -B^T & 0 & 0 \end{bmatrix}.$$

We assume that the entries of A, B, C are independent over $\mathbb{Q}$, excluding those forced to vanish. An argument along similar lines to that for Example 6 shows that $P_1 \overset{\pi_1}{\leftarrow} M \overset{\pi_2}{\rightarrow} P_2$ forms a regular Howe dual pair. However, since the dimensions of $P_1 = \mathbb{T}^3 \times \mathbb{R}$ and $P_2 = \mathbb{R}$ differ by an odd number, it is clear that they cannot have the property of having anti-isomorphic transverse Poisson structures, up to a product with a symplectic factor. Indeed, the Poisson structure on P_2 is of course trivial, while that on P_1 is of rank 2.

This example also shows that in the absence of the local Lie-Weinstein hypothesis, the correspondence between the symplectic leaves of two Poisson manifolds in duality may fail. In order to see this take $x \in \mathbb{R}$. This is a symplectic leaf in $\mathbb{R}$, and if the Symplectic Leaves Correspondence theorem were valid then $\pi_1(\pi_2^{-1}(x)) \simeq \mathbb{T}^3$ would be a symplectic leaf in $\mathbb{T}^2 \times \mathbb{T}^3 \times \mathbb{R}$. This is obviously impossible as $\mathbb{T}^3$ is of odd dimension.

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