

AN EXAMPLE OF FOURIER–MUKAI PARTNERS OF MINIMAL ELLIPTIC SURFACES

HOKUTO UEHARA

ABSTRACT. Let X and Y be smooth projective varieties over \mathbb{C} . We say that X and Y are *D-equivalent* (or, X is a *Fourier–Mukai partner* of Y) if their derived categories of bounded complexes of coherent sheaves are equivalent as triangulated categories. The aim of this short note is to find an example of mutually D-equivalent but not isomorphic relatively minimal elliptic surfaces.

1. Introduction

Let X be a smooth projective variety over \mathbb{C} . The derived category $D(X)$ of X is a triangulated category whose objects are bounded complexes of coherent sheaves on X . A *Fourier–Mukai (FM) transform* relating smooth projective varieties X and Y is an equivalence of triangulated categories $\Phi : D(X) \rightarrow D(Y)$. If there exists an FM transform relating X and Y , we call X an *FM partner* of Y . We also say that X and Y are *D-equivalent*. Moreover we say that X and Y are *K-equivalent* if there exist a smooth projective variety Z and birational morphisms $f : Z \rightarrow X$, $g : Z \rightarrow Y$ such that $f^*K_X \sim g^*K_Y$. It is conjectured by Kawamata (Conjecture 1.2 in [7]) that given birationally equivalent smooth projective varieties X and Y , they are D-equivalent if and only if they are K-equivalent. In this note, we construct a counterexample to his conjecture. More precisely, we have:

Main Theorem . (i) *Let p be a positive integer. Then there is a rational elliptic surface $S(p)$ such that $S(p)$ has a singular fiber of type ${}_pI_0$ and at least three non-multiple singular fibers of different Kodaira’s types.*
(ii) *Let N be a positive integer and p a prime number such that $p > 6(N-1)+1$. Then there are rational elliptic surfaces T_i , ($1 \leq i \leq N$) such that $T_i \not\cong T_j$ for $i \neq j$ and every T_i is an FM partner of $S(p)$. As a special case, $S = S(11)$ has an FM partner T such that $T \not\cong S$. These S and T are birational, D-equivalent but not K-equivalent.*

Note that if X and Y are K-equivalent, they are isomorphic in codimension 1 (Lemma 4.2, [7]). In particular, if surfaces S and T are not isomorphic, they are not K-equivalent. Hence, in (ii), the statement for $p = 11$ follows from the one for arbitrary p .

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Before ending Introduction, we give a few remarks to Main Theorem. For a smooth projective variety X , it is an interesting problem to find the set of isomorphic classes of FM partners of X . In connection with this problem, we have the following.

Theorem 1.1 (Theorem 1.1, [2] and Theorem 1.6, [7]). *Assume that X and Y are D -equivalent smooth projective surfaces but not isomorphic to each other. Then we know that one of the following holds.*

- (i) X and Y are $K3$ surfaces.
- (ii) X and Y are abelian surfaces.
- (iii) X and Y are elliptic surfaces with the non-zero Kodaira dimension $\kappa(X) = \kappa(Y)$.

Using Theorem 1.1, we obtain the complete answer to the problem mentioned above in dimension 2 ([2], see also [7]). It is well-known that the cases (i) and (ii) in Theorem 1.1 really occur. More strongly, we have:

Theorem 1.2 ([8] and [6]). *Let N be a positive integer. Then there are $K3$ (respectively, abelian) surfaces T_i , ($1 \leq i \leq N$) such that $T_i \not\cong T_j$ for $i \neq j$ and all T_i 's are D -equivalent each other.*

Our Main Theorem means that the case (iii) in Theorem 1.1 really occurs, and a similar result to Theorem 1.2 is true for elliptic surfaces.

In contrast to Main Theorem and Theorem 1.2, it is predicted that given a smooth projective variety X , the set of isomorphic classes of FM partners of X is finite. Actually this is known for the 2-dimensional case ([2] and [7]).

Notation and conventions. All varieties are defined over \mathbb{C} and “elliptic surface” always means “relatively minimal elliptic surface” in this note. For a set I , we denote by $|I|$ the cardinality of I .

2. The proof of Main Theorem

We need some standard notation and results before giving the proof. Let $\pi : S \rightarrow C$ be an elliptic surface. For an object E of $D(S)$, we define the fiber degree of E

$$d(E) = c_1(E) \cdot f,$$

where f is a general fiber of π . Let us denote by $\lambda_{S/C}$ the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, $\lambda_{S/C}$ is the smallest number d such that there is a holomorphic d -section of π . For integers $a > 0$ and i with i coprime to $a\lambda_{S/C}$, by [1] there exists a smooth, 2-dimensional component $J_S(a, i)$ of the moduli space of pure dimension one stable sheaves on S , the general point of which represents a rank a , degree i stable vector bundle supported on a smooth fiber of π . There is a natural morphism $J_S(a, i) \rightarrow C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of S to the point x . This morphism is a minimal elliptic fibration ([1]). Put $J^i(S) := J_S(1, i)$. Obviously, $J^0(S) \cong J(S)$, the Jacobian surface associated to S , and $J^1(S) \cong S$.

Fix an elliptic surface with a section $\pi : B \rightarrow C$. Let $\eta = \text{Spec } k$ be the generic point of C , where $k = k(C)$ is the function field of C , and let \bar{k} be the algebraic closure of k . Put $\bar{\eta} = \text{Spec } \bar{k}$. We define the *Weil–Chatelet group* $WC(B)$ by the Galois cohomology $H^1(G, B_\eta(\bar{k}))$. Here $G = \text{Gal}(\bar{k}/k)$ and $B_\eta(\bar{k})$ is the group of points of the elliptic curve B_η defined over \bar{k} . Suppose that we are given a pair (S, φ) , where S is an elliptic surface $S \rightarrow C$ and φ is an isomorphism $J(S) \rightarrow B$ over C , fixing their 0-sections. Then we have a morphism

$$B_\eta \times S_\eta \rightarrow J(S)_\eta \times S_\eta \rightarrow S_\eta.$$

Here the first morphism is induced by $\varphi^{-1} \times id_S$ and the second is given by translation. We obtain a principal homogeneous space S_η of B_η . Since this correspondence is invertible and the group $H^1(G, B_\eta(\bar{k}))$ classifies isomorphic classes of principal homogeneous spaces of B_η , we know that $WC(B)$ consists of all isomorphic classes of pairs (S, φ) . Here two pairs (S, φ) and (S', φ') are *isomorphic* if there is an isomorphism $\alpha : S \rightarrow S'$ over C , such that $\varphi' \circ \alpha_* = \varphi$, where $\alpha_* : J(S) \rightarrow J(S')$ is the isomorphism induced by α (fixing 0-sections).

$$\begin{array}{ccc} J(S) & \xrightarrow{\alpha_*} & J(S') \\ \varphi \downarrow & & \downarrow \varphi' \\ B & \xlongequal{\quad} & B \end{array}$$

There is a short exact sequence (page 185, [4] or page 38, [3])

$$0 \rightarrow \text{Sh}(B) \rightarrow WC(B) \rightarrow \bigoplus_{t \in C} H_1(B_t, \mathbb{Q}/\mathbb{Z}) \rightarrow 0,$$

if B is not the product $C \times E$, where E is an elliptic curve. The group $\text{Sh}(B)$ is called *Tate–Shafarevich group* and it is the subgroup of $WC(B)$ which consists of all isomorphic classes of pairs (S, φ) such that S does not have multiple fibers. For a rational surface B , it is known that $\text{Sh}(B)$ is trivial (Example 1.5.12, [5]).

Now we are in position to prove Main Theorem.

Proof. (i) By the Persson’s list [9], there is a rational elliptic surface $B \rightarrow C$ having a section and three singular fibers of type III^* , I_2 , I_1 (there are many other choices for B). Fix a point $t_0 \in C$ such that B_{t_0} is smooth. Take an element $\xi = (\xi_t)$ of $WC(B) \cong \bigoplus_{t \in C} H_1(B_t, \mathbb{Q}/\mathbb{Z})$ such that ξ_{t_0} is of order p and $\xi_t = 0$ for other t . Then the surface $\pi : S(p) \rightarrow C$ corresponding to ξ is an elliptic surface with desired singular fibers. We can check that $S(p)$ is rational, for instance, by Proposition 1.3.23, [5].

(ii) Put $S = S(p)$. Because every (-1) -curve on S is a p -section of π , we know that $\lambda_{S/C} = p$. For $i \in \mathbb{Z}$, there is an isomorphism $\varphi_i : J(J^i(S)) \rightarrow B$ such that $(J^i(S), \varphi_i)$ corresponds to $i\xi \in WC(B)$ ([3], page 38). By Theorem 2.2, each $J^i(S)$ is mutually D-equivalent for $1 \leq i < p$. We can also conclude that $J^i(S)$ is rational, since $\kappa(J^i(S)) = -\infty$ and the Euler numbers $e(J^i(S))$

and $e(S)$ coincide by Proposition 2.3, [2] (we can check the rationality also by using Proposition 1.3.23, [5]). Put

$$I = \{1, \dots, p-1\}, \quad I(a) = \{i \in I \mid J^i(S) \cong J^a(S)\}$$

for $a \in I$. Then there are $i_1, \dots, i_M \in I$ such that $I = \coprod_{k=1}^M I(i_k)$ (disjoint union).

Claim 2.1. *For all $a \in I$, $|I(a)| \leq 6$.*

If Claim 2.1 is true, we have $6M \geq |I| = p-1$. By the assumption $p > 6(N-1)+1$, we have $M \geq N$, which completes the proof of Main Theorem.

Let us start the proof of Claim 2.1.

Step 1. For each $i \in I(a)$, we fix an isomorphism $\alpha_i : J^a(S) \rightarrow J^i(S)$. Because the rational surface $J^a(S)$ has a unique elliptic fibration, there exists $\delta \in \text{Aut } C$ such that the following diagram is commutative.

$$\begin{array}{ccc} J^a(S) & \xrightarrow{\alpha_i} & J^i(S) \\ \downarrow & & \downarrow \\ C & \xrightarrow{\delta} & C \end{array}$$

This makes the following diagram commutative.

$$\begin{array}{ccc} J(J^a(S)) & \xrightarrow{\varphi_a^{-1} \circ \varphi_i \circ \alpha_{i*}} & J(J^a(S)) \\ \downarrow & & \downarrow \\ C & \xrightarrow{\delta} & C \end{array}$$

By our assumption, $J(J^a(S))$ has at least three singular fibers of different Kodaira's types. Hence δ must be the identity on $C \cong \mathbb{P}^1$ and then we can say that every α_i is an isomorphism over C .

Step 2. By Step 1, we know that $\varphi_i \circ \alpha_{i*} \circ \varphi_a^{-1}$ is an automorphism of B over C , fixing the 0-section. Put $\gamma_i = \varphi_i \circ \alpha_{i*} \circ \varphi_a^{-1}$.

$$\begin{array}{ccc} J(J^a(S)) & \xrightarrow{\alpha_{i*}} & J(J^i(S)) \\ \varphi_a \downarrow & & \downarrow \varphi_i \\ B & \xrightarrow{\gamma_i} & B \end{array}$$

Suppose $\gamma_i = \gamma_j$ for $i, j \in I(a)$, then by the isomorphism $\alpha_j \circ \alpha_i^{-1}$, we see that $(J^i(S), \varphi_i)$ is isomorphic to $(J^j(S), \varphi_j)$ and hence $i\xi = j\xi$ in $WC(B)$. Because the order of ξ is p , we obtain $i = j$. Since the order of the group of automorphism of B over C fixing the 0-section is at most 6, we get $|I(a)| \leq 6$. This finishes the proof. \square

Theorem 2.2 (Proposition 4.4, [2]). *Let $\pi : S \rightarrow C$ be an elliptic surface and T a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.*

- (i) T is an FM partner of S .
- (ii) T is isomorphic to $J^b(S)$ for some integer b with $(b, \lambda_{S/C}) = 1$.

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DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN
E-mail address: hokuto@kurims.kyoto-u.ac.jp