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EXISTENCE OF RELATIVE PERIODIC ORBITS NEAR RELATIVE EQUILIBRIA

Viktor Ginzburg and Eugene Lerman

Abstract. We show existence of relative periodic orbits (a.k.a. relative nonlinear normal modes) near relative equilibria of a symmetric Hamiltonian system under an appropriate assumption on the Hessian of the Hamiltonian. This gives a relative version of the Moser-Weinstein theorem.

1. Introduction

In this paper we discuss a generalization of the Weinstein-Moser theorem [Mo, W1, W2] on the existence of nonlinear normal modes (i.e., periodic orbits) near an equilibrium in a Hamiltonian system to a theorem on the existence of relative periodic orbits (r.p.o.'s) near a relative equilibrium of a symmetric Hamiltonian system.

More specifically let (M, ω_M) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding equivariant moment map $\Phi : M \to \mathfrak{g}^*$. Let $h \in C^\infty(M)^G$ be a G-invariant Hamiltonian. We will refer to the quadruple $(M, \omega_M, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)$ as a **symmetric Hamiltonian system**. The main result of the paper is the following theorem (the terms used in the statement are explained below):

Theorem 1. *Let $(M, \omega_M, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)$ be a symmetric Hamiltonian system. Suppose $x \in M$ is a positive definite relative equilibrium of the system and let $\mu = \Phi(x)$. Then for every sufficiently small $E > 0$ the set $\{h = E + h(x)\} \cap \Phi^{-1}(\mu)$ (if nonempty) contains a relative periodic orbit of h.*

This theorem strengthens or complements numerous other results about the existence of relative periodic orbits; see, e.g., [LT, O1, O2].

We now define the relevant terms. Recall that for every invariant function $h \in C^\infty(M)^G$ the restriction $h|_{\Phi^{-1}(\mu)}$ descends to a continuous function h_μ on the symplectic quotient

$$M//_\mu G := \Phi^{-1}(\mu)/G_\mu,$$

where G_μ denotes the stabilizer of $\mu \in \mathfrak{g}^*$ under the coadjoint action. If G_μ acts freely on $\Phi^{-1}(\mu)$ then the symplectic quotient $M//_\mu G$ is a symplectic manifold

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and the flow (of the Hamiltonian vector field) of h on $\Phi^{-1}(\mu)$ descends to the flow of h_μ on the quotient. More generally the quotient $M//_\mu G$ is a symplectic stratified space: the quotient is a union of strata, which are symplectic manifolds and which fit together in a locally simple manner. In this case the flow of h descends to a flow on $M//_\mu G$, which is, on each stratum, the flow of the restriction of h_μ to the stratum.[1]

A point $x \in \Phi^{-1}(\mu)$ is a **relative equilibrium** of h (strictly speaking of the symmetric Hamiltonian system $(M, \omega, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)$, if its image $[x] \in M//_\mu G$ is stationary under the flow of h_μ. If x is a relative equilibrium then there is a vector $\xi \in \mathfrak{g}$ such that $dh(x) = d(\langle \Phi, \xi\rangle)(x)$. It is not hard to see that ξ is in the Lie algebra $\mathfrak{g}_\mu$ of the stabilizer of μ and that ξ is unique modulo $\mathfrak{g}_x$, the Lie algebra of the stabilizer of x. If we set

$$h^\xi := h - \langle \Phi, \xi\rangle,$$

then $d(h^\xi)(x) = 0$. Hence the Hessian $d^2(h^\xi)(x)$ is well-defined. The *symplectic slice* at $x \in M$ is, by definition, the vector space

$$V := T_x(G\cdot x)^\omega/(T_x(G\cdot x) \cap T_x(G\cdot x)^\omega),$$

where $\cdot^\omega$ denotes the symplectic perpendicular. Note that V is naturally a symplectic representation of G_x, the stabilizer of x. Note also that $T_x(G\cdot x)^\omega = \ker d\Phi_x$, so V is isomorphic to a maximal symplectic subspace of $\ker d\Phi_x$. In particular if G_μ acts freely at x then V models the symplectic quotient $M//_\mu G$ near $[x]$. A computation shows that $T_x(G\cdot x)\cap T_x(G\cdot x)^\omega = T_x(G_\mu\cdot x)$ and that $h^\xi(g\cdot x) = h^\xi(x)$ for all $g \in G_\mu$. Hence the subspace $T_x(G_\mu \cdot x)$ lies in the kernel of the quadratic from $d^2(h^\xi)(x)$. It follows that the Hessian $d^2(h^\xi)(x)$ descends to a well-defined quadratic form q on the symplectic slice V (which depends on ξ). We say that the relative equilibrium x is **positive definite** if q is positive definite for some choice of ξ. When the action is free, q can be thought of as the restriction of $d^2\left(h\mid_{\Phi^{-1}(\mu)}\right)$ to the normal V in $\Phi^{-1}(\mu)$ to the orbit $G\cdot x$. An integral curve $\gamma(t) \subset \Phi^{-1}(\mu)$ of h is a **relative periodic orbit** if its projection $[\gamma(t)] \subset M//_\mu G$ is periodic.

The motivation for Theorem 1 comes from a result of Weinstein [W1] generalizing a classical theorem of Liapunov which asserts that if x is an equilibrium of a Hamiltonian h on a symplectic manifold (M, ω) and if the Hessian $d^2h(x)$ of h at x is positive definite, then for every $E > 0$ sufficiently small the energy surface

$$\{h = h(x) + E\}$$

carries at least $\frac{1}{2}\dim M$ periodic orbits. Now suppose $(M, \omega, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)$ is a symmetric Hamiltonian system, $\mu \in \mathfrak{g}^*$ is a point and the action

[1]The facts referred to above were first proved in [SL] in the special case of G being compact and $\mu = 0$. The case where G acts properly and the coadjoint orbit through μ is closed was dealt with in [BL]. The assumption on the coadjoint orbit was subsequently removed in [LW] following a suggestion in [O1].

of G_μ on $\Phi^{-1}(\mu)$ is free, so that the symplectic quotient $M//_\mu G$ is smooth. If a relative equilibrium $x \in \Phi^{-1}(\mu)$ is positive definite then the Hessian of h_μ at $[x]$ is positive definite. Hence by Weinstein's theorem applied to h_μ at $[x]$, for every $E > 0$ sufficiently small the energy surface

$$\{h_\mu = h_\mu([x]) + E\}$$

carries at least $\frac{1}{2} \dim M//_\mu G$ periodic orbits. In other words, under these natural assumptions the manifolds

$$\{h = h(x) + E\} \cap \Phi^{-1}(\mu)$$

carry relative periodic orbits of h. It is natural to ask what happens if G_μ does not act freely at or near the relative equilibrium x of h. To address this issue let us recall where the strata of the symplectic quotients come from. For a subgroup H of G the set $M_{(H)}$ of points of orbit type (H) is defined by

$$M_{(H)} := \{m \in M \mid \text{ the stabilizer } G_m \text{ is conjugate to } H \text{ in } G\}.$$

Since the action of G is proper, $M_{(H)}$ is a manifold. Moreover the set

$$(M//_\mu G)_{(H)} := \left(M_{(H)} \cap \Phi^{-1}(\mu)\right)/G_\mu$$

is naturally a symplectic manifold [BL, SL]. Now if $x \in \Phi^{-1}(\mu)$ is a relative equilibrium and G_x is the stabilizer of x, then $(M//_\mu G)_{(G_x)}$ is the stratum of the quotient $M//_\mu G$ containing $[x]$. Hence by Weinstein's theorem if $h_\mu|_{(M//_\mu G)_{(G_x)}}$ has a positive definite Hessian at $[x]$ then the energy surface

$$\{h_\mu = h_\mu([x]) + E\}$$

contains at least $\frac{1}{2}\dim(M//_\mu G)_{(G_x)}$ periodic orbits of $h_\mu|_{(M//_\mu G)_{(G_x)}}$ for all $E > 0$ sufficiently small.

The trivial observation above raises a natural question. Suppose the stratum containing the point $[x]$ is *not* open in the quotient $M//_\mu G$. Under suitable assumptions on the second partials of h at a relative equilibrium x, must the set

$$\{h_\mu = h_\mu([x]) + E\} \subset M//_\mu G$$

contain more periodic orbits of the flow of h_μ than $\frac{1}{2}(\dim M//_\mu G)_{(H)}$? For example, are there periodic orbits of h_μ in nearby strata? Theorem 1 in effect affirmatively answer the question in a special case: the stratum of $M//_\mu G$ passing through $[x]$ is a single point $\{[x]\}$.

Keeping in mind that $V = T_xM$, when x is a fixed point of the action, note that the following is a special case of Theorem 1 above:

Theorem 2. *Let $K \to Sp(V,\omega)$ denote a symplectic representation of a compact Lie group K on a symplectic vector space (V,ω), let $\Phi : V \to \mathfrak{k}^*$ denote the associated homogeneous moment map. Let $h \in C^\infty(V)^K$ be an invariant function with $dh(0) = 0$ and the quadratic form $q := d^2h(0)$ positive definite. Then for every $E > 0$ sufficiently small there is a relatively periodic orbit of h on the set*

$$\{h = h(0) + E\} \cap \{\Phi = 0\},$$

provided the set in question is non-empty.

We will show that in fact Theorem 2 implies Theorem 1.

Theorem 3. *Let x be a relative equilibrium of a symmetric Hamiltonian system $(M, \omega, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)$. Denote the stabilizer of x by G_x, the stabilizer of $\mu = \Phi(x)$ by G_μ, the symplectic slice at x by V and the moment map associated to the symplectic representation of G_x on V by Φ_V.*

Then there exists a Hamiltonian $h_V \in C^\infty(V)^{G_x}$ with $dh_V(0) = 0$ so that for any $E \in \mathbb{R}$ sufficiently small and for any G_x-relatively periodic orbit of h_V in

$$\{h_V = E\} \cap \Phi_V^{-1}(0)$$

sufficiently close to 0 there is a G-relatively periodic orbit of h in

$$\{h = h(x) + E\} \cap \Phi^{-1}(\mu).$$

Moreover, if x is a positive definite relative equilibrium of h, then h_V can be chosen so that the the Hessian $d^2h_V(0)$ is positive definite.

Clearly Theorem 2 and Theorem 3 together imply Theorem 1. We will then reduce the proof of Theorem 2 to

Theorem 4. *Let Q be a compact manifold with a contact form α whose Reeb flow generates a torus action. Then for any contact form β C^2-close to α, the Reeb flow of β has at least one periodic orbit.*

A note on notation. Throughout the paper the Lie algebra of a Lie group denoted by a capital Roman letter will be denoted by the same small letter in the fraktur font: thus $\mathfrak{g}$ denotes the Lie algebra of a Lie group G etc. The identity element of a Lie group is denoted by 1. The natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ will be denoted by $\langle \cdot, \cdot \rangle$.

When a Lie group G acts on a manifold M we denote the action by an element $g \in G$ on a point $x \in G$ by $g \cdot x$; $G \cdot x$ denotes the G-orbit of x and so on. The vector field induced on M by an element X of the Lie algebra $\mathfrak{g}$ of G is denoted by X_M. The isotropy group of a point $x \in M$ is denoted by G_x; the Lie algebra of G_x is denoted by $\mathfrak{g}_x$ and is referred to as the isotropy Lie algebra of x. We recall that $\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\}$. The image of a point $x \in M$ in M/G under the orbit map is denoted by $[x]$.

If P is a principal G-bundle then $[p, m]$ denotes the point in the associated bundle $P \times_G M = (P \times M)/G$ which is the orbit of $(p, m) \in P \times M$.

If ω is a differential form on a manifold M and Y is a vector field on M, the contraction of ω by Y is denoted by $\iota(Y)\omega$.

2. Reducing non-linear to linear: proof of Theorem 3

2.1. Facts about symplectic quotients. In this subsection we gather a few facts [SL, BL, LW] about symplectic quotients that we will need in the proof of Theorem 3. As we mentioned in the introduction, the symplectic quotient at $\mu \in \mathfrak{g}^*$ for a proper Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) is the topological space

$$M//_\mu G := \Phi^{-1}(\mu)/G_\mu$$

where as before $\Phi : M \to \mathfrak{g}^*$ denotes the associated equivariant moment map. We define the set of **smooth functions** $C^\infty(M//_\mu G)$ by

$$C^\infty(M//_\mu G) = \{f \in C^0(M//_\mu G) \mid \pi_\mu^* f \in C^\infty(M)^G|_{\Phi^{-1}(\mu)}\},$$

where $\pi_\mu : \Phi^{-1}(\mu) \to M//_\mu G$ denotes the orbit map. Note that since π_μ is surjective, given $h \in C^\infty(M)^G$ there is a *unique* $h_\mu \in C^\infty(M//_\mu G)$ with

$$h|_{\Phi^{-1}(\mu)} = \pi_\mu^* h_\mu.$$

One often refers to h_μ as the **reduction** of the Hamiltonian h at μ.

Theorem 2.1 (Arms-Cushman-Gotay [ACG])**.** *The Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ induces a Poisson bracket $\{\cdot,\cdot\}_\mu$ on $C^\infty(M//_\mu G)$ so that*

$$\pi_\mu^* : C^\infty(M//_\mu G) \to C^\infty(M)^G|_{\Phi^{-1}(\mu)}$$

is a Poisson map.

Definition 2.2. We say that a curve $\gamma : I \to M//_\mu G$ is **smooth** (C^∞) if $f \circ \gamma : I \to \mathbb{R}$ is smooth for any $f \in C^\infty(M//_\mu G)$ (I, of course, is an interval). A curve $\gamma : I \to M//_\mu G$ is an **integral curve** of a function $f \in C^\infty(M//_\mu G)$ if for any $k \in C^\infty(M//_\mu G)$ we have

$$\frac{d}{dt}(k \circ \gamma)(t) = \{f, k\}_\mu(\gamma(t)).$$

The following fact is an easy consequence of the well-known result that for proper group actions (the only kind of actions we consider) smooth invariant functions separate orbits:

Proposition 2.3. *Integral curves of functions in $C^\infty(M//_\mu G)$ are unique.*

It is also not hard to see that Theorem 2.1 implies that if γ is an integral curve of an invariant Hamiltonian $h \in C^\infty(M)^G$ lying in $\Phi^{-1}(\mu)$ then $\pi_\mu \circ \gamma$ is an integral curve of the corresponding reduced Hamiltonian $h_\mu \in C^\infty(M//_\mu G)$ (as above $\pi_\mu^* h_\mu = h|_{\Phi^{-1}(\mu)}$). Combining this with Proposition 2.3 we get

Lemma 2.4. *Let $h \in C^\infty(M)^G$ be an invariant Hamiltonian and $h_\mu \in C^\infty(M//_\mu G)$ the corresponding reduced Hamiltonian at μ. If $\gamma : I \to M//_\mu G$ is an integral curve of h_μ there exists an integral curve $\tilde{\gamma} : I \to \Phi^{-1}(\mu)$ of h so that*

$$\pi_\mu \circ \tilde{\gamma} = \gamma.$$

Note that $\tilde{\gamma}$ is not unique: for any $a \in G_\mu$ the curve $a \cdot \tilde{\gamma}$ is also an integral curve of γ projecting down to γ. We will need one more fact about integral curves of functions in symplectic quotients, which is an easy consequence of Proposition 2.3.

Lemma 2.5. *Suppose $\tau : M//_\mu G \to M'//_{\mu'} G'$ is a continuous map between two symplectic quotients such that the pull-back τ^* maps $C^\infty(M'//_{\mu'}G')$ to $C^\infty(M//_\mu G)$ preserving the Poisson brackets (i.e, τ is a morphism of symplectic quotients). Then for any $h' \in C^\infty(M'//_{\mu'}G')$ if γ is an integral curve of $\tau^* h'$ then $\tau \circ \gamma$ is an integral curve h'.*

Proof. Since γ is an integral curve of $\tau^* h'$

$$\frac{d}{dt}((\tau^* f') \circ \gamma)(t) = \{\tau^* h', \tau^* f'\}_\mu(\gamma(t)).$$

for any $f' \in C^\infty(M'//_{\mu'}G')$. Hence

$$\frac{d}{dt} f'(\tau \circ \gamma) = \{\tau^* h', \tau^* f'\}_\mu(\gamma) = \tau^*(\{h', f'\}_{\mu'})(\gamma) = \{h', f'\}_{\mu'}(\tau \circ \gamma).$$

□

This ends our digression on the subject of symplectic quotients.

In proving Theorem 3 we will argue that there is a G_x-equivariant symplectic embedding $\sigma : \mathcal{V} \to \mathcal{U}$ of a G_x-invariant neighborhood $\mathcal{V}$ of 0 in V into a G-invariant neighborhood $\mathcal{U}$ of x in M which induces a morphism

$$\bar\sigma : \mathcal{V}//G_x \to \mathcal{U}//_\mu G$$

of symplectic quotients so that $\bar\sigma$ embeds $\mathcal{V}//G_x$ as a connected component of $\mathcal{U}//_\mu G$. Note that for σ to induce $\bar\sigma$ we would want σ to map $\Phi_V^{-1}(0) \cap \mathcal{V}$ into $\Phi^{-1}(\mu) \cap \mathcal{U}$ in such a way that the diagram

$$\begin{array}{ccc} \Phi_V^{-1}(0) \cap \mathcal{V} & \xrightarrow{\sigma} & \Phi^{-1}(\mu) \cap \mathcal{U} \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_\mu} \\ \mathcal{V}//_0 G_x & \xrightarrow{\bar\sigma} & \mathcal{U}//_\mu G \end{array} \tag{2.1}$$

commutes, where π_0, π_μ are the respective orbit maps. As before given $f \in C^\infty(\mathcal{V})^{G_x}$ we denote by f_0 the unique function in $C^\infty(\mathcal{V}//_0 G_x)$ with $f|_{\Phi_V^{-1}(0)\cap\mathcal{V}} = \pi_0^* f_0$ and similarly $h_\mu \in C^\infty(\mathcal{U}//_\mu G)$ is determined by $\pi_\mu^* h_\mu = h|_{\Phi^{-1}(\mu)}$. Then the commutativity of (2.1) implies:

$$(\sigma^* h)_0 = \bar\sigma^* h_\mu$$

for any $h \in C^\infty(\mathcal{U})^G$.

Suppose next that we have constructed $\sigma : \mathcal{V} \to \mathcal{U}$ with the desired properties. Given $h \in C^\infty(M)^G$ and any $\xi \in \mathfrak{g}_\mu$ let

$$h_V = \sigma^*(h - \langle \Phi, \xi \rangle).$$

Then $h_V \in C^\infty(\mathcal{V})^{G_x}$. Moreover, since (2.1) commutes,

$$(h_V)_0 = (\sigma^*(h - \langle \Phi, \xi \rangle))_0 = \bar\sigma^*(h - \langle \Phi, \xi \rangle)_\mu = \bar\sigma^* h_\mu - \langle \mu, \xi \rangle.$$

In other words,

$$(h_V)_0 = \bar\sigma^* h_\mu + \text{ constant.} \tag{2.2}$$

Lemma 2.5 and equation (2.2) imply that if γ_V is an integral curve of h_V in $\mathcal{V} \cap \Phi_V^{-1}(0) \cap \{h_V = h_V(0) + E\}$ then $\pi_0 \circ \gamma_V$ is an integral curve of $(h_V)_0$ in $\{(h_V)_0 = (h_V)(\pi_0(0)) + E\}$. Hence $\bar{\sigma} \circ \pi_0 \circ \gamma_V$ is an integral curve of h_μ in $\{h_\mu = h_\mu(\pi_\mu(x)) + E\}$. It follows that there is an integral curve γ of h in $\{h = h(x) + E\} \cap \mathcal{U} \cap \Phi^{-1}(\mu)$ with

$$\pi_\mu \circ \gamma = \bar{\sigma} \circ \pi_0 \circ \gamma_V.$$

If γ_V is a G_x-relative periodic orbit of h_V then $\pi_0 \circ \gamma_V$ is a periodic orbit of $(h_V)_0$. Consequently γ is a G-relative periodic orbit of h.

Finally note that if additionally we can arrange for

$$d\sigma_0(T_0\mathcal{V}) \subset T_x(G \cdot x)^\omega,$$

then since σ is symplectic

$$d\sigma_0(T_0\mathcal{V}) \cap T_x(G_\mu \cdot x) = \{0\}.$$

Consequently,
if $\xi \in \mathfrak{g}_\mu$ is such that $d(h - \langle \Phi, \xi \rangle)(x) = 0$ and $d^2(h - \langle \Phi, \xi \rangle)(x)|_{T_x(G\cdot x)^\omega}$ is positive semi-definite of maximal rank, then

$$d^2(h - \langle \Phi, \xi \rangle)(x)|_{d\sigma_0(T_0\mathcal{V})}$$

is positive definite. Therefore the Hessian

$$d^2(\sigma^*(h - \langle \Phi, \xi \rangle))(0)$$

is positive definite as well. We conclude that in order to prove Theorem 3 it is enough to construct the embedding σ with the desired properties. In other words it is enough to prove:

Proposition 2.6. *Let $(M, \omega, \Phi : M \to \mathfrak{g}^*)$ be a symplectic manifold with a proper Hamiltonian action of a Lie group G. Fix a point x in M. Let $\mu = \Phi(x)$, let $\Phi_V : V \to \mathfrak{g}_x^*$ be the homogeneous moment map associated with the symplectic slice representation $G_x \to \mathrm{Sp}(V, \omega_V)$. There exists a G_x-invariant neighborhood $\mathcal{V}$ of 0 in V, a G-invariant neighborhood $\mathcal{U}$ of x in M and a G_x-equivariant embedding $\sigma : \mathcal{V} \to \mathcal{U}$ with*

$$\sigma(\Phi_V^{-1}(0) \cap \mathcal{V}) \subset \Phi^{-1}(\mu) \cap \mathcal{U}$$

such that the composition

$$\Phi_V^{-1}(0) \cap \mathcal{V} \xrightarrow{\sigma} \Phi^{-1}(\mu) \cap \mathcal{U} \xrightarrow{\pi_\mu} (\Phi^{-1}(\mu) \cap \mathcal{U})/G_\mu = \mathcal{U}/\!/_\mu G$$

drops down to

$$\bar{\sigma} : \mathcal{V}/\!/_0 G_x = (\Phi_V^{-1}(0) \cap \mathcal{V})/G_x \to \mathcal{U}/\!/_\mu G$$

making (2.1) commute. Moreover,

1. *$\bar{\sigma}(\mathcal{V}/\!/_0 G_x)$ is a connected component of $\mathcal{U}/\!/_\mu G$ and $\bar{\sigma}$ is a homeomorphism onto its image;*
2. *the pull-back $\bar{\sigma}^*$ sends $C^\infty(\mathcal{U}/\!/_\mu G)$ isomorphically to $C^\infty(\mathcal{V}/\!/_0 G_x)$ (as Poisson algebras).*

Additionally

$$d\sigma_0(T_0\mathcal{V}) \subset T_x(G \cdot x)^\omega.$$

Proposition 2.6 will follow from

Proposition 2.7. *As above let $(M, \omega, \Phi : M \to \mathfrak{g}^*)$ be a symplectic manifold with a proper Hamiltonian action of a Lie group G. Fix a point x in M. Let $\mu = \Phi(x)$, let $\Phi_V : V \to \mathfrak{g}_x^*$ be the homogeneous moment map associated with the symplectic slice representation $G_x \to \mathrm{Sp}(V, \omega_V)$. There exists a slice Σ at x for the action of G on M, a G_x-invariant neighborhood $\mathcal{V}$ of 0 in V and a G_x-equivariant embedding $\sigma : \mathcal{V} \to \Sigma$ so that*

1. *$\sigma(\mathcal{V})$ is closed in Σ;*
2. *$\sigma(\Phi_V^{-1}(0) \cap \mathcal{V}) = \Sigma \cap \Phi^{-1}(\mu)$;*
3. *$\sigma^*\omega = \omega_V$ and $\Phi \circ \sigma = i \circ \Phi_V + \mu$ where $i : \mathfrak{g}_x^* \to \mathfrak{g}^*$ is a G_x-equivariant injection;*
4. *$G_\mu \cdot (\Sigma \cap \Phi^{-1}(\mu))$ is a connected component of $\Phi^{-1}(\mu) \cap \mathcal{U}$, where $\mathcal{U} = G \cdot \Sigma$.*

Proof of Proposition 2.7. Our proof of Proposition 2.7 uses the Bates-Lerman version [BL][pp. 212–215] of the local normal form theorem for moment maps of Marle, Guillemin and Sternberg:

Theorem 2.8 ([BL]). *Let (M, ω) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding equivariant moment map $\Phi : M \to \mathfrak{g}^*$. Fix $x \in M$, let $\mu = \Phi(x)$, (V, ω_V) the symplectic slice at x, $\Phi_V : V \to \mathfrak{g}_x^*$ the associated homogeneous moment map. Choose a G_x-equivariant splitting*

$$\mathfrak{g}^* = \mathfrak{g}_x^* \oplus (\mathfrak{g}_\mu/\mathfrak{g}_x)^* \oplus \mathfrak{g}_\mu^\circ$$

($\mathfrak{g}_\mu^\circ$ denotes the annihilator of $\mathfrak{g}_\mu$ in $\mathfrak{g}^*$) and thereby G_x-equivariant injections

$$i : \mathfrak{g}_x^* \to \mathfrak{g}^*, \quad j : (\mathfrak{g}_\mu/\mathfrak{g}_x)^* \to \mathfrak{g}^*.$$

Let

$$Y = G \times_{G_x} ((\mathfrak{g}_\mu/\mathfrak{g}_x)^* \times V);$$

it is a homogeneous vector bundle over G/G_x. There exists a closed 2-form ω_Y on Y which is non-degenerate in a neighborhood of the zero section $G \times_{G_x} (\{(0,0)|\}) = G \cdot [1, 0, 0]$ (1 denotes the identity in G) such that

1. *a G-invariant neighborhood of $G \cdot x$ in (M, ω) is G-equivariantly symplectomorphic to a neighborhood of $G \cdot [1, 0, 0]$ in (Y, ω_Y);*
2. *the moment map Φ_Y for the action of G on (Y, ω_Y) is given by*

   $$\Phi_Y([g, \eta, v]) = g \cdot (\mu + j(\eta) + i(\Phi_V(x)))$$

   for all $(g, \eta, v) \in G \times (\mathfrak{g}_\mu/\mathfrak{g}_x)^* \times V$;
3. *the embedding $\iota : V \to Y$, $\iota(v) = [1, 0, v]$ is symplectic: $\iota^*\omega_Y = \omega_V$.*

Hence, we can assume without loss of generality that $(M, \omega, \Phi) = (Y, \omega_Y, \Phi_Y)$ and $x = [1, 0, 0]$. Note that the embedding $\kappa : (\mathfrak{g}_\mu/\mathfrak{g}_x)^* \times V \to Y$, $\kappa(\eta, v) = [1, \eta, v]$ is a slice at x for the action of G on Y.

We now argue that for a small enough G_x-invariant neighborhood $\mathcal{V}$ of 0 in V and a small enough G_x-invariant neighborhood $\mathcal{W}$ of 0 in $(\mathfrak{g}_\mu/\mathfrak{g}_x)^*$

$$\Sigma := \kappa(\mathcal{W} \times \mathcal{V}) \subset Y$$

is the desired slice and

$$\sigma = \kappa|_{\{0\}\times\mathcal{V}} : \mathcal{V} \to \Sigma, \quad \sigma(v) = [1, 0, v]$$

is the desired embedding.

Note that no matter how $\mathcal{V}$ and $\mathcal{W}$ are chosen we automatically have that $\sigma(\mathcal{V})$ is closed in Σ and $\sigma^*\omega_Y = \omega_V$. Hence $\sigma(\mathcal{V})$ is closed in $\mathcal{U} := G \cdot \Sigma$, which is a G-invariant neighborhood of x. Note also that

$$\Phi_Y \circ \sigma(v) = \Phi_Y([1, 0, v]) = \mu + i(\Phi_V(v)).$$

Next we make our choice of $\mathcal{V}$ and $\mathcal{W}$ and prove that the resulting embedding σ has all the desired properties. For this purpose, factor $\Phi_Y : Y \to \mathfrak{g}^*$ as a sequence of maps (we identify $\mathfrak{g}_\mu^*$ with $j((\mathfrak{g}_\mu/\mathfrak{g}_x)^*) \oplus i(\mathfrak{g}_x^*) \subset \mathfrak{g}^*$):

$$G \times_{G_x} ((\mathfrak{g}_\mu/\mathfrak{g}_x)^* \times V) \xrightarrow{F_1} G \times_{G_x} ((\mathfrak{g}_\mu/\mathfrak{g}_x)^* \times \mathfrak{g}_x^*) \xrightarrow{F_2} G \times_{G_\mu} \mathfrak{g}_\mu^* \xrightarrow{\mathcal{E}} \mathfrak{g}^*,$$

where

$$\begin{aligned} F_1([g, \eta, v]) &= [g, \eta, \Phi_V(v)], \\ F_2([g, \eta, \nu]) &= [g, j(\eta) + i(\nu)] \text{ and} \\ \mathcal{E}([g, \vartheta]) &= g \cdot (\mu + \vartheta). \end{aligned}$$

Since the tangent space $T_\mu(G \cdot \mu)$ is canonically isomorphic to the annihilator $\mathfrak{g}_\mu^\circ$ and since $\mathfrak{g}^* = \mathfrak{g}_\mu^* \oplus \mathfrak{g}_\mu$, the vector bundle $G \times_{G_\mu} \mathfrak{g}_\mu^*$ is the normal bundle for the embedding $G \cdot \mu \hookrightarrow \mathfrak{g}^*$ and $\mathcal{E} : G \times_{G_\mu} \mathfrak{g}_\mu^* \to \mathfrak{g}^*$ is the exponential map for a flat G_x-invariant metric on $\mathfrak{g}_x^*$. Therefore $\mathcal{E}$ is a local diffeomorphism near the zero section. In particular there is a G_x-invariant neighborhood $\mathcal{O}$ of $[1, 0] \in G \times_{G_\mu} \mathfrak{g}_\mu^*$ so that $\mathcal{E}|_{\mathcal{O}}$ is a diffeomorphism onto its image. Let $\mathcal{O}' = (F_2 \circ F_1)^{-1}(\mathcal{O})$. Then

$$\begin{aligned} \mathcal{O}' \cap \Phi_Y^{-1}(\mu) &= \mathcal{O}' \cap (F_2 \circ F_1)^{-1}((\mathcal{E}|_{\mathcal{O}})^{-1}(\mu)) \\ &= \mathcal{O}' \cap F_1^{-1}(F_2^{-1}([1, 0])) \\ &= \mathcal{O}' \cap F_1^{-1}(G_\mu \times_{G_x} \{(0, 0)\}) \\ &= \mathcal{O}' \cap G_\mu \times_{G_x} (\{0\} \times \Phi_V^{-1}(0)). \end{aligned}$$

We may take $\mathcal{O}$ to be of the form $\mathcal{A} \times_{G_\mu} (\mathcal{W} \times \mathcal{V}')$ where $\mathcal{A} \subset G$ is a $G_x \times G_\mu$-invariant neighborhood of 1, $\mathcal{W} \subset (\mathfrak{g}_\mu/\mathfrak{g}_x)^*$ is a G_x-invariant neighborhood of 0 and $\mathcal{V}' \subset \mathfrak{g}_x^*$ is a convex G_x-invariant neighborhood of 0. We take $\mathcal{V} = \Phi_V^{-1}(\mathcal{V}')$. Then $\mathcal{V} \cap \Phi_V^{-1}(0)$ is connected. With the choices above, $\mathcal{O}' = \mathcal{A} \times_{G_x} (\mathcal{W} \times \mathcal{V})$ and

$$\mathcal{O}' \cap \Phi_Y^{-1}(\mu) = (\mathcal{A} \times_{G_x} (\mathcal{W} \times \mathcal{V})) \cap \left(G_\mu \times_{G_x} (\{0\} \times (\Phi_V^{-1}(0) \cap \mathcal{V}))\right).$$

Since $G_\mu \times_{G_x} (\{0\} \times (\Phi_V^{-1}(0) \cap \mathcal{V}))$ is closed in $G \times_{G_x} (\mathcal{W} \times \mathcal{V}) = G \cdot \Sigma = \mathcal{U}$ and since $\Phi_V^{-1}(0) \cap \mathcal{V}$ is connected, $G_\mu \cdot \sigma(\Phi_V^{-1}(0) \cap \mathcal{V}) = G_\mu \times_{G_x} (\{0\} \times (\Phi_V^{-1}(0) \cap \mathcal{V}))$ is a connected component of $\Phi_Y^{-1}(\mu) \cap \mathcal{U}$. This proves property (4).

Note that $\Sigma = \{[1, \eta, v] \mid \eta \in \mathcal{W}, v \in \mathcal{V}\} \subset \mathcal{A} \times_{G_x} (\mathcal{W} \times \mathcal{V})$. Hence $\Phi_Y^{-1}(\mu) \cap \Sigma = (\Phi_Y^{-1}(\mu) \cap \mathcal{O}') \cap \Sigma = \{[1, 0, v] \mid v \in \Phi_V^{-1}(0) \cap \mathcal{V}\}$, i.e.,

$$\sigma(\Phi_V^{-1}(0) \cap \mathcal{V}) = \Phi_Y^{-1}(\mu) \cap \Sigma,$$

which proves property (2) and thereby finishes the proof of Proposition 2.7. □

Proof of Proposition 2.6. We continue to use the notation above. Since $\sigma(\Phi_V^{-1}(0) \cap \mathcal{V}) = \Phi^{-1}(\mu) \cap \Sigma$ and since $\sigma : \mathcal{V} \to \Sigma$ is a closed embedding, the restriction $\sigma|_{\Phi_V^{-1}(0) \cap \mathcal{V}} : \Phi_V^{-1}(0) \cap \mathcal{V} \to \Phi^{-1}(\mu) \cap \Sigma$ is a G_x-equivariant homeomorphism. Hence

$$\bar{\sigma} : (\Phi_V^{-1}(0) \cap \mathcal{V})/G_x \to (\Phi^{-1}(\mu) \cap \Sigma)/G_x$$

is a homeomorphism as well. Since Σ is a slice at x for the action of G on M, it is also a slice for the action of G_μ. Consequently

$$(G_\mu \cdot (\Phi^{-1}(\mu) \cap \Sigma))/G_\mu \cong (\Phi^{-1}(\mu) \cap \Sigma)/G_x.$$

Since $G_\mu \cdot (\Phi^{-1}(\mu) \cap \Sigma)$ is a component of $\Phi^{-1}(\mu) \cap \mathcal{U}$, $\bar{\sigma} : \mathcal{V}/\!/_0 G_x \to \mathcal{U}/\!/_\mu G$ is a homeomorphism onto its image. Moreover, the diagram (2.1) commutes.

We now argue that $\bar{\sigma}$ pulls back the smooth functions in $C^\infty(\mathcal{V}/\!/G_x)$ to smooth functions in $C^\infty(\mathcal{U}/\!/_\mu G)$ and that the pull-back is an isomorphism of Poisson algebras. Since Σ is a slice and $\mathcal{U} = G \cdot \Sigma$, the restriction

$$C^\infty(\mathcal{U})^G \to C^\infty(\Sigma)^{G_x}, \quad f \mapsto f|_\Sigma$$

is a bijection. Since $\sigma(\Phi_V^{-1}(0) \cap \mathcal{V}) \subset \Phi^{-1}(\mu) \cap \mathcal{U}$ and since

$$\begin{array}{ccc} C^\infty(\mathcal{U})^G & \xrightarrow{\sigma^*} & C^\infty(\mathcal{V})^{G_x} \\ \big\downarrow & & \big\downarrow \\ C^\infty(\mathcal{U})^G|_{\Phi^{-1}(\mu)\cap\mathcal{U}} & \xrightarrow{\bar{\sigma}^*} & C^\infty(\mathcal{V})^{G_x}|_{\Phi_V^{-1}(0)\cap\mathcal{V}} \end{array}$$

commutes, where the vertical arrows are restrictions, and since the top arrow is surjective, the bottom arrow is surjective as well. Since Σ is a slice and $\mathcal{U} = G \cdot \Sigma$, any function $f \in C^\infty(\mathcal{U})^G|_{\Phi^{-1}(\mu)\cap\mathcal{U}}$ is uniquely defined by its values on $\Phi^{-1}(\mu) \cap \Sigma = \sigma(\Phi_V^{-1}(0) \cap \mathcal{V})$. Hence the bottom arrow $\bar{\sigma}^*$ is also

injective. Since $\sigma : \mathcal{V} \to \mathcal{U}$ is symplectic, $\sigma^* : C^\infty(\mathcal{U}) \to C^\infty(\mathcal{V})$ is Poisson. Hence $\sigma^* : C^\infty(\mathcal{U})^G \to C^\infty(\mathcal{V})^{G_x}$ is also Poisson. Consequently $\sigma^* : C^\infty(\mathcal{U})^G|_{\Phi^{-1}(\mu)\cap\mathcal{U}} \to C^\infty(\mathcal{V})^{G_x}|_{\Phi_V^{-1}(0)\cap\mathcal{V}}$ is Poisson as well. We conclude that

$$\bar{\sigma}^* : C^\infty(\mathcal{U}/\!/_\mu G) \to C^\infty(\mathcal{V}/\!/_0 G_x)$$

is an isomorphism of Poisson algebras.

Finally since $\Phi \circ \sigma = i \circ \Phi_V + \mu$,

$$d\Phi_x \circ d\sigma_0 = i \circ d(\Phi_V)_0.$$

Since Φ_V is quadratic homogeneous, $d(\Phi_V)_0 = 0$. Hence

$$d\sigma_0(T_0\mathcal{V}) \subset \ker d\Phi_x = T_x(G\cdot x)^\omega.$$

□

This concludes our proof of Theorem 3 as well.

3. From invariant Hamiltonians on vector spaces to Reeb flows: Theorem 4 implies Theorem 2

Remark 3.1. Theorem 2 is easily seen to be true in a special case: the set V^K of K-fixed vectors is a subspace of V of positive dimension. Indeed, since h is K-invariant its Hamiltonian flow preserves the symplectic subspace V^K, which is contained in the zero level set $\Phi^{-1}(0)$ of the moment map. Moreover the flow of h in V^K is the Hamiltonian flow of the restriction $h|_{V^K}$. Hence Weinstein's theorem applied to the Hamiltonian system $(V^K, \omega|_{V^K}, h|_{V^K})$ guarantees that for any $E > 0$ sufficiently small there are at least $\frac{1}{2}\dim V^K$ periodic orbits of $h|_{V^K}$ in the surface

$$\{h|_{V^K} = E\} = \{h = E\} \cap V^K \subset \{h = E\} \cap \{\Phi = 0\}.$$

To show that Theorem 4 implies Theorem 2 we first need to digress on the subject of contact quotients.

3.1. Facts about contact quotients. Suppose that a Lie group G acts properly on a manifold Σ, preserving a contact form β. The associated moment map $\Psi : \Sigma \to \mathfrak{g}^*$ is defined by

$$\langle \Psi(x), \xi\rangle = \beta_x(\xi_M(x))$$

for all $x \in \Sigma$, all $\xi \in \mathfrak{g}^*$. The map Ψ is G-equivariant. The contact quotient at zero $\Sigma//G$ is, by definition, the set

$$\Sigma//G := \Psi^{-1}(0)/G.$$

Just as in the case of symplectic quotients the contact quotients are stratified spaces [LW] with the stratification induced by the orbit type decomposition:

$$\Sigma//G := \coprod_{H<G} (\Psi^{-1}(0) \cap \Sigma_{(H)})/G,$$

where the disjoint union is taken over conjugacy classes of subgroups of G. Additionally each stratum

$$(\Sigma//G)_{(H)} := (\Psi^{-1}(0) \cap \Sigma_{(H)})/G$$

is a contact manifold and the contact form $\beta_{(H)}$ on each stratum is induced by the contact form β on Σ [Wi, Theorem 3, p. 4256]. More precisely for each subgroup H of G the set $\Psi^{-1}(0)\cap\Sigma_{(H)}$ is a manifold and

$$\pi_{(H)}^*\beta_{(H)} = \beta|_{\Psi^{-1}(0)\cap\Sigma_{(H)}},$$

where $\pi_{(H)} : \Psi^{-1}(0) \cap \Sigma_{(H)} \to (\Psi^{-1}(0) \cap \Sigma_{(H)})/G = (\Sigma//G)_{(H)}$ is the orbit map. It is not hard to see that the flow of the Reeb vector field X of β preserves the moment map and the orbit type decomposition, hence descends to a strata-preserving flow on the quotient $\Sigma//G$. Also, on each stratum the induced flow is the Reeb flow of the induced contact form $\beta_{(H)}$.

We're now ready to prove that Theorem 4 implies Theorem 2. It is no loss of generality to assume that $h(0) = 0$. Since the quadratic form $q = d^2h(0)$ is positive definite, the energy surface

$$\{q = E\},$$

$E > 0$, is a K-invariant hypersurface star-shaped about 0. Hence

$$\alpha_E := \iota(R)\omega|_{\{q=E\}}$$

is a K-invariant contact form, where $R(v) = v$ denotes the radial vector field on V. For $E > 0$ sufficiently small, the K-invariant set

$$\{h = E\}$$

is a hypersurface which is C^2-close to $\{q = E\}$. Hence

$$\beta_E := \iota(R)\omega|_{\{h=E\}}$$

is also a K-invariant contact form. By the implicit function theorem, for $E > 0$ sufficiently small, there is a function $f : \{q = E\} \to (0, \infty)$, which is C^2 close to 1, so that

$$\phi : \{q = E\} \to \{h = E\} \quad \phi(x) = f(x)x$$

is a K-equivariant diffeomorphism. Since $\phi^*\beta_E = f^2\alpha_E$, the manifolds $\{q = E\}$ and $\{h = E\}$ are K-equivariantly contactomorphic. Moreover, under the identification ϕ the two contact forms α_E and β_E are C^2-close (again, provided E is small). Note that the two associated contact moment maps are $\Phi|_{\{q=E\}}$ and $\Phi|_{\{h=E\}}$ respectively.

Up to re-parameterization the integral curves of the Hamiltonian vector field of h in $\{h = E\}$ are the integral curves of the Reeb vector field X_E of β_E. Similarly the integral curves of the Hamiltonian vector field of q on $\{q = E\}$ are the integral curves of the Reeb vector field Y_E of α_E. In particular the relatively periodic orbits of h on $\{h = E\}$ are relatively periodic orbits of X_E. Since the hypersurface $\{h = E\}$ is compact and since the orbit type decomposition of the contact quotient $\{h = E\}//G$ is a stratification, the minimal strata of the quotient are compact. Let $Q = (\{h = E\}//G)_{(L)}$ be one such stratum. Then the relatively periodic orbits of h in $\{h = E\} \cap \Phi^{-1}(0) \cap V_{(L)}$ descend to periodic orbits of the Reeb vector field X of the contact form $\beta_{(L)}$ on $Q = (\{h = E\} \cap \Phi^{-1}(0) \cap V_{(L)})/G$. Therefore to prove Theorem 2 it is enough to establish the existence of periodic orbits of X. For this, according to Theorem 4, it suffices to establish the existence of a contact form α on Q whose Reeb vector field Y generates a torus action and such that $\beta_{(L)}$ is C^2 close to α when $E > 0$ is small enough.

The form α, of course, is the form induced by α_E. Let us prove that it does have the desired properties. Since $\phi : \{q = E\} \to \{h = E\}$ is an equivariant contactomorphism it induces an identification of the contact manifold Q with $(\{q = E\} \cap \Phi^{-1}(0) \cap V_{(L)})/G$. Moreover, since α_E and β_E are C^2-close, the induced forms $\beta_{(L)}$ and $\alpha = \alpha_{(L)}$ are C^2-close as well. Since q is definite, its Hamiltonian flow generates a linear symplectic action of a torus $\mathbb{T}$ on V. The restriction of this action to $\{q = E\}$ is also generated by the Reeb vector field Y_E of α_E. Since q is K-invariant, the action of $\mathbb{T}$ commutes with the action of K and preserves the moment map Φ. Hence it descends to an action of $\mathbb{T}$ on Q. Moreover, since the Reeb vector field of α_E descends to the Reeb vector field of $\alpha_{(L)}$ on Q, the induced action of $\mathbb{T}$ on Q is generated by the Reeb vector field of $\alpha_{(L)}$. We conclude that Theorem 4 implies Theorem 2.

4. Perturbations of Reeb flows: proof of Theorem 4

In the proof of Theorem 4 we will need the following elementary result.

Lemma 4.1. *Let ϕ_t be a dense one-parameter subgroup in a torus $\mathbb{T}$ and let H be a subgroup of $\mathbb{T}$ topologically generated by an element ϕ^τ, $\tau > 0$. Then either H has codimension one in $\mathbb{T}$ or $H = \mathbb{T}$.*

Proof of Lemma 4.1. It suffices to show that the map

$$[0, \tau] \times H \to \mathbb{T}, \quad F(t, h) = \phi^t \cdot h$$

is onto $\mathbb{T}$. Pick $g \in \mathbb{T}$. Assume first that g is in the one-parameter subgroup, i.e., $g = \phi^t$ for some t. Then we have $t = k\tau + t'$ with $0 \le t' < \tau$ and, clearly,

$$g = \phi^{t'} \cdot [(\phi^\tau)^k] = F\left(t', (\phi^\tau)^k\right).$$

Hence g is in the image of F.

Let now g be in $\mathbb{T}$, but not in the one-parameter subgroup ϕ^t. Then there exists a sequence $t_r \to \pm\infty$ such that $\phi^{t_r} \to g$. (This sequence must go to positive or negative infinity, for otherwise g would be in the one-parameter subgroup.) Assume that $t_r \to \infty$; the case of negative infinity can be dealt with in a similar fashion. As above, we write

$$t_r = k_r\tau + t'_r,$$

where $k_r \to \infty$ as $r \to \infty$ and $0 < t'_r < \tau$.

The elements $(\phi^\tau)^{k_r}$ are in H and, since H is compact, we may assume that $(\phi^\tau)^{k_r} \to h \in H$ by passing if necessary to a subsequence. Furthermore, by passing if necessary to a subsequence again, we may assume that $t'_r \to t' \in [0, \tau]$.

We claim now that $g = F(t', h)$. To see this note that as above

$$\phi^{t_r} = \phi^{k_r\tau + t'_r} = \phi^{t'_r} \cdot [(\phi^\tau)^{k_r}].$$

As r goes to infinity, $\phi^{t'_r} \to \phi^{t'}$ and the second term goes to h. Hence,

$$g = \phi^{t'} \cdot h = F(t', h).$$

This completes the proof of the lemma. □

Proof of Theorem 4. First, let us set notation. We denote by X the Reeb vector field of α and by ϕ^t its Reeb flow. By the hypotheses of the theorem, the flow ϕ^t generates an action of a torus $\mathbb{T}$ on Q. We will view ϕ^t as a dense one-parameter subgroup of $\mathbb{T}$. The points on periodic orbits of X will be referred to as periodic points. We break up the proof of the theorem into four steps. Steps 1–3 concern exclusively properties of the Reeb flow of α. The perturbed form β enters the proof only at the last step.

1. We claim that the periodic points of X are exactly the points $x \in Q$ whose stabilizers $\mathbb{T}_x$ have codimension one in $\mathbb{T}$.

Indeed, let $x \in Q$ be a periodic point, i.e., $\phi^T(x) = x$ for some $T > 0$. Since ϕ^t is dense in $\mathbb{T}$, the Reeb orbit through x is dense in the $\mathbb{T}$-orbit through x. Since x is a periodic point, the Reeb orbit is closed and thus equal to the $\mathbb{T}$-orbit. Hence, $\mathbb{T}/\mathbb{T}_x$ is a circle and thus $\mathbb{T}_x$ has codimension one. Conversely, if $\mathbb{T}_x$ has codimension one, the Reeb orbit through x must be dense in the $\mathbb{T}$-orbit and hence equal to the $\mathbb{T}$-orbit because the latter is a circle.

2. Let now N be a minimal stratum of the $\mathbb{T}$-action, which is comprised entirely of periodic points. We claim that such a stratum exists, is a smooth submanifold, and all points of N have the same period, i.e., the $\mathbb{T}$-action on N factors through a free circle action.

Since the $\mathbb{T}$-action has no fixed points, periodic points lie in minimal strata of the action. Furthermore, the Reeb flow of α has at least one periodic orbit (in fact, at least two unless Q is a circle); this follows, for example, from a theorem of Banyaga and Rukimbira, [BR]. Now it suffices to take as N a minimal stratum containing a periodic point. The fact that N is smooth is a general result about compact group actions. Finally, all points in N have the same stabilizer $\mathbb{T}_x$ and the action of the circle $\mathbb{T}/\mathbb{T}_x$ on N is free because N is minimal. The period T of $x \in N$ is the first $T > 0$ such that $\phi^T \in \mathbb{T}_x$.

3. We claim that N is a non-degenerate invariant submanifold for the Reeb flow of α.

Let $x \in N$. We need to show that the linearization $d\phi^T$ on the normal space ν_x to N at x does not have unit as an eigenvalue. By definition, this linearization is just the linearized action of $\phi^T \in \mathbb{T}_x$ on ν_x. As is well known, the isotropy representation of $\mathbb{T}_x$ on ν_x contains no trivial representations in its decomposition into the sum of irreducible representations. Hence, it suffices to show that the subgroup generated by ϕ^T is dense in $\mathbb{T}_x$.

Let m be the first positive integer such that $\left(\phi^T\right)^m$ is in $\mathbb{T}_x^0$, the connected component of identity in $\mathbb{T}_x$. (Such an integer m exists because $\mathbb{T}_x/\mathbb{T}_x^0$ is a finite subgroup of the circle $\mathbb{T}/\mathbb{T}_x^0$.) Since $\mathbb{T}_x/\mathbb{T}_x^0$ is finite cyclic, it suffices to show that the subgroup H topologically generated by ϕ^{mT} is equal to $\mathbb{T}_x^0$. This follows immediately from Lemma 4.1. Indeed, by the lemma, the group H is either equal to $\mathbb{T}$ or has codimension one in $\mathbb{T}$. Since $H \subset \mathbb{T}_x^0$ and $\mathbb{T}_x^0$ has codimension one, the group H must have codimension one. Thus H is a closed subgroup of $\mathbb{T}_x^0$ of the same dimension as $\mathbb{T}_x^0$ and hence $H = \mathbb{T}_x^0$.

4. Now we invoke the following theorem due to Kerman [K] (p. 967). Let $Crit(P)$ be the minimal possible number of critical points of a smooth function on a compact manifold P.

Theorem 4.2 (Kerman, [K])**.** *Let Q be a compact odd-dimensional manifold, X a non-vanishing vector field on Q, and N a non-degenerate periodic submanifold of X. Let Ω be a closed maximally non-degenerate two-form on Q whose kernel is C^2-close to X and such that the class $[\Omega|_N]$ is in the image of the pull-back from $H^2(N/S^1)$ to $H^2(N)$. Then Ω has at least $Crit(N/S^1)$ closed characteristics near N.*

Applying this theorem to Q, N and X as above, and $\Omega = d\beta$ we obtain the required result. □

Remark 4.3. In fact, our proof of Theorem 4 establishes the existence of two distinct periodic orbits when Q is not a circle. As a consequence, in the setting of Theorems 1 and 2 there exist at least two distinct relative periodic orbits unless Q is a circle.

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Department of Mathematics, University of California at Santa Cruz, Santa Cruz, CA 95064

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

E-mail address: ginzburg@math.ucsc.edu

E-mail address: lerman@math.uiuc.edu