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GLOBAL INVARIANTS FOR STRONGLY PSEUDOCONVEX VARIETIES WITH ISOLATED SINGULARITIES: BERGMAN FUNCTIONS

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Abstract. Let M be a strongly pseudoconvex manifold which is a resolution of strongly pseudoconvex variety V with only isolated singularities. We define a Bergman function B_M on M which is a biholomorphic invariant of M. The Bergman function B_M vanishes precisely on the exceptional set of M. Hence B_M can be pushed down and we obtain a Bergman function B_V which is a biholomorphic invariant of V and vanishes precisely on the singularities of V. This Bergman function not only can distinguish analytic structures of isolated singularities, but it can also distinguish the CR structures of the boundaries of V. As an application, we define a continuous numerical invariant on strongly pseudoconvex CR manifolds in $V = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$. We show that our invariant varies continuously in $\mathbb{R}$ when the CR structure of strongly pseudoconvex CR manifold changes in V. Our global numerical invariant is explicitly computable. Moreover we show that the Bergman function allows us to determine the automorphism groups of these CR manifolds.

1. Introduction

The Bergman kernel form is a basic biholomorphic invariant on complex manifolds [Ko]. A lot of work has been done in its explicit computation and asymptotic expansion. However, it seems that there is little attention given to the possible role of the Bergman kernel on analytic spaces, in connection with the study of singularities and CR manifolds. In [L-Y-Y], an initial step in studying the Bergman kernel on a resolution of an isolated 2-dimensional Gorenstein singularity was given. It was shown that the exceptional set of the resolution is exactly the minimal set of the Bergman kernel. Thus the analytic definition of the Bergman kernel contains important topological information on the singularity. However, the Bergman kernel defined in [L-Y-Y] is not a biholomorphically invariant except for the rational double points.

Let M be a complex manifold of dimension n. A real valued C^∞ function ϕ on M is said to be strongly plurisubharmonic if and only if the hermitian form

$$\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}} dz_i \overline{dz_j}$$

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is positive definite with respect to any system of local coordinates $(z_1, \dots, z_n)$. The complex manifold M is said to be strongly pseudoconvex if there is a compact subset $B \subset M$, and a continuous real valued function ϕ on M, which is strongly plurisubharmonic outside B, and such that for each $c \in \mathbb{R}$, the set $B_c = \{x \in M : \phi(x) < c\}$ is relatively compact in M. Note that a strongly pseudoconvex manifold is a modification of a Stein space at a finite many points.

The purpose of this paper is to define a new Bergman function B_M for each strongly pseudoconvex manifold M with dimension $n \geq 2$ which is a resolution of strongly pseudoconvex variety V with only isolated singularities. We show that B_M is a biholomorphically invariant function and B_M vanishes precisely on the exceptional set of M. Hence B_M can be pushed down and we obtain a Bergman function B_V which is a biholomorphic invariant of V and vanishes precisely on the singularities of V.

The invariance properties of a CR manifold X of real dimension $2n-1$ which is a real hypersurface in $\mathbb{C}^n$ with respect to the infinite pseudo-group of biholomorphic transformations were studied extensively by many important mathematicians. The systematic study of such properties for hypersurfaces with nondegenerate Levi form was first made by Catan [Ca] in 1932, and latter by Chern and Moser [Ch-Mo]. A main result of the theory is the existence of a complete system of local differential invariants. On the other hand, by using the Catan method of equivalence, Webster [We] gave a complete characterization when two ellipsoids in $\mathbb{C}^n$ are CR equivalent. In [Fe], Fefferman has shown that a biholomorphic map between two bounded strongly pseudoconvex domains with smooth boundaries extends smoothly to the boundaries. Then Webster's result gives a necessary and sufficient condition for two ellipsoidal domains to be equivalent.

Despite the success of the Chern-Moser theory, the fundamental question of distinguishing two strongly pseudoconvex manifolds remains unsolved. Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$. In 1974, Boutel de Monvel [Bo] (see also Kohn [Koh]) proved that X is CR-embeddable in some $\mathbb{C}^N$ if $\dim X \geq 5$. In this paper, we shall only consider CR embeddable strongly pseudoconvex CR manifolds. Let X be an embeddable compact strongly pseudoconvex CR manifold. In view of a beautiful theorem of Harvey-Lawson [Ha-La], there exists a complex variety V in $\mathbb{C}^N$ such that $\partial V = X$ and V has only normal isolated singularities. Theorem 3.1 below says that we can use the structures of the singularities of V to distinguish the CR structure of X. Thus if two strongly pseudoconvex manifolds bound non-isomorphic singularities, then their CR structures are different. The difficult unsolved CR equivalence problem is: how can one distinguish strongly pseudoconvex CR manifolds X_1 and X_2 when they are lying in the same variety V. If V is $\mathbb{C}^N$, this difficult problem has been considered by leading mathematicians Chern-Moser [Ch-Mo], Fefferman [Fe], Webster [We], etc. Even in this case, it seems that the CR equivalence problem for complete Reinhardt domains (except for the ellipsoidal domains which was solved by Webster) remains open. On the other hand, when V is a singular variety, the CR equivalence problem is basically untouched. One of the purpose of this paper is to offer a novel technique to attack CR equivalence problem. The main observation is that our new Bergman functions put a lot of restriction on biholomorphic maps between strongly pseudoconvex CR manifolds, from which new CR invariants can be constructed and the automorphism groups of the CR manifolds can be determined. We illustrate how our new technique works in a concrete example.

We define a continuous numerical invariant on strongly pseudoconvex CR manifolds in $V = \{(x, y, z) \in \mathbb{C}^2 : xy = z^2\}$. We show that our invariant varies continuously in $\mathbb{R}$ when the CR structure of strongly pseudoconvex CR manifold changes in V. Our global numerical invariant is explicitly computable. Moreover we show that the Bergman function allows us to determine the automorphism groups of these CR manifolds.

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2. Bergman function on strongly pseudoconvex manifold and variety

Let M be a complex n-dimensional manifold. We first recall the definition of the Bergman kernel. Let F be the set of all holomorphic n-forms ϕ on M such that $\left| \int_M \phi \wedge \overline{\phi} \right| < \infty$. ($\phi$ will be called L^2 or square integrable.) F is a separable complex Hilbert space under the inner product $\langle \phi_1, \phi_2 \rangle = (\sqrt{-1})^{n^2} \int_M \phi_1 \wedge \overline{\phi}_2$. The corresponding norm $\langle \phi, \phi \rangle^{\frac{1}{2}}$ will be denoted by $\|\phi\|$. Let $\{\omega_j\}$ be a complete orthonormal basis of F. Then $K(z, \overline{w}) = \Sigma \omega_j(z) \wedge \overline{\omega_j(w)}$ can be shown to converge uniformly on compact subsets to a holomorphic $2n$-form on $M \times \overline{M}$. Here, $\overline{M}$ denotes the conjugate complex manifold obtained by taking the conjugate coordinate charts of M. Further, $K(z, \overline{w})$ is independent of the choice of complete orthonormal basis of F. If each point $z \in M$ is identified with the point $(z, \overline{z}) \in M \times \overline{M}$, then $K(z, \overline{z})$ can be regarded as a $2n$-form on M and is referred to as the Bergman kernel of M. Since the Hilbert space F with its inner product is invariant under biholomorphic maps, so is the Bergman kernel.

Let V be a Stein variety of dimension $n \geq 2$ in $\mathbb{C}^N$ with only irreducible isolated singularities. We assume that ∂V is a strongly pseudoconvex CR manifold. Let $\pi : M \to V$ be a resolution of singularity with E as a exceptional set. We shall define a Bergman function $B_M(z)$ on M which is a biholomorphic invariant of M.

Definition 2.1. *Let F_0 be the set of all L^2 integrable holomorphic n-forms ψ on M vanishing on the exception set E of M. Let $\{\omega_j\}$ be a complete orthonormal basis of F_0. The Bergman kernel vanishing on the exceptional set is defined to be $K_0(z, \overline{z}) = \Sigma \omega_j(z) \wedge \overline{\omega_j(z)}$.*

Lemma 2.2. *F/F_0 is a finite dimensional vector space.*

Proof. Let Ω^n be the sheaf of germs of holomorphic n-forms on M and $\Omega^n(-E)$ be the sheaf of germs of holomorphic n-forms on M vanishing along E. Clearly $F = \Gamma(M, \Omega^n)$ and $F_0 = \Gamma(M, \Omega^n(-E))$. From the short exact sequence

$$0 \longrightarrow \Omega^n(-E) \longrightarrow \Omega^n \longrightarrow \Omega^n\Big|_E \longrightarrow 0,$$

we have $\dim F/F_0 \leq \dim H^0\big(E, \Omega^n\big|_E\big)$. Since $\Omega^n\big|_E$ is a coherent sheaf on a compact analytic set E, $\dim H^0\big(E, \Omega^n\big|_E\big)$ is finite. □

Lemma 2.3. *Bergman kernel vanishing on the exceptional set $K_0(z, \overline{z})$ is independent of the choice of the complete orthonormal basis of F_0 and $K_0(z, \overline{z})$ is invariant under biholomorphic maps.*

Proof. Let $\{\omega_i\}$ and $\{\widetilde{\omega}_i\}$ be two complete orthonormal bases of F_0. In view of Lemma 2.2, there exists $\alpha_1, \dots, \alpha_k$ holomorphic n-forms on M where $k = \dim(F/F_0)$ such that both $\{\alpha_1, \dots, \alpha_k\} \cup \{\omega_i\}$ and $\{\alpha_1, \dots, \alpha_k\} \cup \{\widetilde{\omega}_i\}$ form complete orthonormal basis of F. Since

$$\sum_{i=1}^{k} \alpha_i \wedge \overline{\alpha_i} + \sum \omega_i \wedge \overline{\omega_i} = K(z, \overline{z}) = \sum_{i=1}^{k} \alpha_i \wedge \overline{\alpha_i} + \sum \widetilde{\omega}_i \wedge \overline{\widetilde{\omega}_i},$$

we have $\sum \omega_i \wedge \overline{\omega_i} = \sum \widetilde{\omega}_i \wedge \overline{\widetilde{\omega}_i}$.

Recall that exceptional set E is the maximal compact analytic set in M. Since E is invariant under biholomorphic maps, so is the space F_0 of all L^2-integrable holomorphic n-forms on M vanishing on the exceptional set E. Hence $K_0(z, \overline{z})$ is invariant under biholomorphic maps. □

Definition 2.4. *Let M be a resolution of a strongly pseudoconvex variety V of $\dim n \geq 2$ in $\mathbb{C}^N$ with only irreducible isolated singularity at the origin. The Bergman function B_M on M is defined to be $K_{M,0}/K_M$.*

Theorem 2.5. *B_M is a global function defined on M which is invariant under biholomorphic maps. Moreover, the zero set of B_M is precisely the exceptional set of M.*

Proof. Let $\Phi : M' \longrightarrow M$ be a biholomorphic map. Then

$$\Phi^*(B_M) = \frac{\Phi^* K_{M,0}}{\Phi^* K_M} = \frac{K_{M',0}}{K_{M'}} = B_{M'}.$$

We first observe that there exists a holomorphic n-form which does not vanish on the exceptional set of M. This can be seen as follows. Let $\pi : M \to V$ be the blowing down map. Since V is a Stein space and Ω^n_V, the sheaf of germs of holomorphic n-forms on V, is coherent. There exists a holomorphic n-form ω on V which does not vanish at the singularity set of V. Then $\pi^*\omega$

is a holomorphic n-form which does not vanish on the exceptional set of M. In particular, K_M does not vanish on the exceptional set of M. Clearly $K_{M,0}$ vanishes along exceptional set. To finish the proof, we need to show that $K_{M,0}$ does not vanish outside the exceptional set. For this purpose, given $p \in M - E$, it suffices to produce a holomorphic n-form vanishing along E but not at p. Let $\Omega^n(-E)$ be the sheaf of germs of holomorphic n-forms on M vanishing along E. Since $\Omega^n(-E)$ is coherent and π is proper, $\pi_*\Omega^n(-E)$ is a coherent sheaf on V by Grauert's direct image theorem. As V is a Stein variety, we can find ω in $\Gamma(V, \pi_*\Omega^n(-E))$ which does not vanish at $\pi(p)$. Then $\pi^*\omega$ is a holomorphic n-form vanishing along the exceptional set E but not at p. □

The same argument of the proof of Theorem 1 in [L-Y-Y] will prove the following theorem.

Theorem 2.6. *Let M be a strongly pseudoconvex manifold of dimension $n \geq 2$ with exceptional set E. Let A be compact submanifold containing in E. Let $\pi : M_1 \to M$ be the blow up of M along A. Then we have $K_{M_1}(z, \overline{z}) = \pi^* K_M(z, \overline{z})$ and $K_{M_1,0}(z, \overline{z}) = \pi^* K_{M,0}(z, \overline{z})$. Consequently $B_{M_1}(z) = \pi^* B_M(z)$.*

Let $\pi_i : M_i \longrightarrow V$, $i = 1, 2$, be two resolutions of singularities of V. By Hironaka's theorem [Hi], there exists a resolution $\widetilde{\pi} : \widetilde{M} \longrightarrow V$ of singularities of V such that $\widetilde{M}$ can be obtained from M_i, $i = 1, 2$, by successive blowing up along submanifolds in exceptional set. In view of Theorem 2.5 and Theorem 2.6, the following definition is well defined.

Definition 2.7. *Let V be a strongly pseudoconvex variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Let $\pi : M \to V$ be a resolution of singularities of V. Define the Bergman function B_V on V to be the push forward of the Bergman function B_M by the map π.*

Theorem 2.8. *Let V be a strongly pseudoconvex variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Then the Bergman function B_V on V is invariant under biholomorphic maps and B_V vanishes precisely on the singular set of V.*

Proof. Easy consequence of Theorem 2.5, and Theorem 2.6. □

Theorem 2.9. *Let V be a strongly pseudoconvex variety in $\mathbb{C}^N$ with only isolated normal singularities of dimension $n \geq 2$. Let F_V be the set of all L^2-integrable holomorphic n-forms on $V - S$, where S is the singular part of V. Let $F_{V,0} = \{\omega \in F_V : \omega$ vanishes on $S\}$. Let $K_V(z, \overline{z})$ and $K_{V,0}(z, \overline{z})$ be defined in the usual manner* (cf. Definition 2.1). *Then $B_V = \dfrac{K_V(z, \overline{z})}{K_{V,0}(z, \overline{z})}$ and B_V is a biholomorphical invariant of V.*

Proof. Let $\pi : M \to V$ be a resolution of singularities of V. It is well known (cf. [La], [Ya]) that a holomorphic n-form ω on a deleted neighborhood of the singular set S is L^2-integrable if and only if $\pi^*\omega$ is a holomorphic n-form on a neighborhood of the exceptional set E in M. Thus $\pi^* : F_V \to F_M$ is an isomorphism which sends $F_{V,0}$ onto $F_{M,0}$. The theorem follows easily. □

3. Continuous numerical invariant of strongly pseudoconvex CR manifold

Let X be a strongly pseudoconvex CR manifold of real dimension $2n-1$. It is well known [Bo] that X can be CR embedded into $\mathbb{C}^N$ if $n \geq 3$. For our subsequent discussion, we shall assume that X is of dimension $2n-1$ in $\mathbb{C}^N$. By a theorem of Harvey and Lawson [Ha-La], X is a boundary of a variety V with only isolated normal singularities.

Theorem 3.1. *Let X_1, X_2 be two strongly pseudoconvex CR manifolds of dimension $2n-1$ which bound varieties V_1, V_2 respectively in $\mathbb{C}^N$ with only isolated normal singularities. If $\Phi : X_1 \to X_2$ is a CR-isomorphism, then Φ can be extended to a biholomorphic map from V_1, to V_2.*

Proof. Let $\phi_1, \ldots, \phi_N$ be the component functions of Φ. Then ϕ_i as CR holomorphic function on X can be extended in a one sided neighborhood of X_1 in V_1. By Andreotti and Grauert [An-Gr, Théoréme 15], ϕ_i can be extended holomorphically to $V_1 - S_1$ where S_1 is the singular set of V. Since S_1 consists of only isolated normal singularities, ϕ_i can be extended holomorphically to V_1. Clearly $(\phi_1, \ldots, \phi_N)(V_1)$ is a variety with boundary equal to X_2. By uniqueness of complex Plateau problem, we have $(\phi_1, \ldots, \phi_N)(V_1) = V_2$.

Let $\psi_1, \ldots, \psi_N$ be the component functions of Ψ which is the inverse mapping of Φ. The argument above shows that $\psi_1, \ldots, \psi_N$ can be extended holomorphically to V_2 and $(\psi_1, \ldots, \psi_N)(V_2) = V_1$. Since $(\phi_1, \ldots, \phi_N) \circ (\psi_1, \ldots, \psi_N)$ restrict to X_1 is the identity map, it follows that $(\phi_1, \ldots, \phi_N) \circ (\psi_1, \ldots, \psi_N)$ is the identity map on V_1. □

In view of the above Theorem 3.1, if X_1 and X_2 are two strongly pseudoconvex CR manifolds which bound varieties V_1 and V_2 with non-isomorphic singularities, then X_1 and X_2 are not CR equivalent. Therefore to study the CR equivalence of two strongly pseudoconvex CR manifolds X_1 and X_2, it remains to consider the case when X_1 and X_2 are lying on the same variety V. The purpose of this section is to show that our global invariant Bergman function defined in section 2 can be used to study the CR equivalence problem of strongly pseudoconvex CR manifolds lying on the same variety. As an example, we shall show explicitly that how CR manifolds varies in the variety $\widetilde{V} := \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = xy - z^2 = 0\}$. An explicit resolution $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ can be given in terms of coordinate charts and transition functions as follows:

$$\text{Coordinate charts} \quad : \quad \widetilde{W}_k = \mathbb{C}^2 = \{(u_k, v_k)\},\ k = 0, 1$$

$$\text{Transition functions} \quad : \quad \begin{cases} u_1 = \frac{1}{v_0} \\ v_1 = u_0 v_0^2 \end{cases} \quad \text{or} \quad \begin{cases} u_0 = u_1^2 v_1 \\ v_0 = \frac{1}{u_1} \end{cases}$$

$$\text{Resolution map} \quad : \quad \widetilde{\pi}(u_k, v_k) = (u_k^{k+1} v_k^k, u_k^{1-k} v_k^{2-k}, u_k v_k) \ \text{ or}$$

$$(x, y, z) \quad = \quad (u_0, u_0 v_0^2, u_0 v_0) = (u_1^2 v_1, v_1, u_1 v_1)$$

$$\text{Exceptional set} \quad : \quad E = \pi^{-1}(0) = C_1 = \{u_0 = 0\} \cup \{v_1 = 0\}.$$

We consider $V = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2$ and $\eta(x, y, z) < \epsilon_0,$ where η is a strictly plurisubharmonic function$\}$. Then $M = \widetilde{\pi}^{-1}(V)$ is given by the coordinate charts:

$$W_k = \{(u_k, v_k) : \eta(u_k^{k+1} v_k^k, u_k^{1-k} v_k^{2-k}, u_k v_k) < \epsilon_0\}, \qquad k = 0, 1.$$

Observe that under $\pi : M \to V$, $W_0 \backslash C_1$ is mapped biholomorphically onto $V \backslash y$-axis. In particular $M \backslash W_0$ is of measure zero in the obvious sense. Hence, we may compute integrals on M using the (u_0, v_0) coordinate on the chart W_0 alone.

In what follows, we shall assume that η is a Reinhardt function such that W_0 is a complete Reinhardt domain, i.e. whenever $(u_0, v_0) \in W_0$, then $(\tau_1 u_0, \tau_2, v_0) \in W_0$ for all complex numbers τ_j with $|\tau_j| \leq 1$. The following proposition can be found in Proposition 8 of [L-Y-Y].

Proposition 3.2. *In the above notations, let* $\phi_{\alpha\beta} = u_0^\alpha v_0^\beta du_0 \wedge dv_0,\ \alpha, \beta = 0, 1, 2, \ldots.$ *Assume that* W_0 *is a complete Reinhardt domain. Then*

$$\left\{ \frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} : \alpha \geq \frac{1}{2}\beta \right\}$$

is a complete orthonormal base of F. *In other words, a complete orthonormal base of* F *is of the form:*

$$\left\{ \begin{array}{llll} \frac{1}{\|\phi_{00}\|_M} du_0 \wedge dv_0, & \frac{u_0}{\|\phi_{10}\|_M} du_0 \wedge dv_0, & \frac{u_0 v_0}{\|\phi_{11}\|_M} du_0 \wedge dv_0, & \\ \frac{u_0 v_0^2}{\|\phi_{12}\|_M} du_0 \wedge dv_0, & \frac{u_0^2}{\|\phi_{20}\|_M} du_0 \wedge dv_0, & \frac{u_0^2 v_0}{\|\phi_{21}\|_M} du_0 \wedge dv_0, & \frac{u_0^2 v_0^2}{\|\phi_{22}\|_M} du_0 \wedge dv_0, \\ \frac{u_0^2 v_0^3}{\|\phi_{23}\|_M} du_0 \wedge dv_0, & \frac{u_0^2 v_0^4}{\|\phi_{24}\|_M} du_0 \wedge dv_0, & \frac{u_0^3}{\|\phi_{30}\|_M} du_0 \wedge dv_0, & \frac{u_0^3 v_0}{\|\phi_{31}\|_M} du_0 \wedge dv_0, \\ \frac{u_0^3 v_0^2}{\|\phi_{32}\|_M} du_0 \wedge dv_0, & \frac{u_0^3 v_0^3}{\|\phi_{33}\|_M} du_0 \wedge dv_0, & \frac{u_0^3 v_0^4}{\|\phi_{34}\|_M} du_0 \wedge dv_0, & \frac{u_0^3 v_0^5}{\|\phi_{35}\|_M} du_0 \wedge dv_0, \\ \frac{u_0^3 v_0^6}{\|\phi_{36}\|_M} du_0 \wedge dv_0, \ldots & & & \end{array} \right\}.$$

Observe that except for $\frac{1}{\|\phi_{00}\|} du_0 \wedge dv_0$, all the other holomorphic 2-forms above are vanishing at the exceptional set. Therefore the Bergman kernel vanishing on the exceptional set $K_{M,0}$ and Bergman kernel are given respectively by:

$$\begin{aligned} K_{M,0}((u_0,v_0),(\overline{u}_0,\overline{v}_0)) &= \Theta_M du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0} \\ K_M((u_0,v_0),(\overline{u_0},\overline{v_0})) &= \left(\frac{1}{\|\phi_{00}\|_M^2} + \Theta_M\right) du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0} \end{aligned}$$

where

$$\begin{aligned} \Theta_M = & \frac{|u_0|^2}{\|\phi_{10}\|_M^2} + \frac{|u_0|^2|v_0|^2}{\|\phi_{11}\|_M^2} + \frac{|u_0|^2|v_0|^4}{\|\phi_{12}\|_M^2} + \frac{|u_0|^4}{\|\phi_{20}\|_M^2} + \frac{|u_0|^4|v_0|^2}{\|\phi_{21}\|_M^2} \\ & + \frac{|u_0|^4|v_0|^4}{\|\phi_{22}\|_M^2} + \frac{|u_0|^4|v_0|^6}{\|\phi_{23}\|_M^2} + \frac{|u_0|^4|v_0|^8}{\|\phi_{24}\|_M^2} + \frac{|u_0|^6}{\|\phi_{30}\|_M^2} + \frac{|u_0|^6|v_0|^2}{\|\phi_{31}\|_M^2} + \frac{|u_0|^6|v_0|^4}{\|\phi_{32}\|_M^2} \\ & + \frac{|u_0|^6|v_0|^6}{\|\phi_{33}\|_M^2} + \frac{|u_0|^6|v_0|^8}{\|\phi_{34}\|_M^2} + \frac{|u_0|^6|v_0|^{10}}{\|\phi_{35}\|_M^2} + \frac{|u_0|^6|v_0|^{12}}{\|\phi_{36}\|_M^2} + \cdots \qquad (3.1) \end{aligned}$$

Theorem 3.3. *Assume that W_0 is a complete Reinhardt domain. Then the Bergman function for the strongly pseudoconvex manifold M is given by*

$$\begin{aligned} B_M((u_0,v_0),(\overline{u_0,v_0})) = & \|\phi_{00}\|_M^2\Theta_M\Big[1 - \|\phi_{00}\|_M^2\Theta_M + (\|\phi_{00}\|_M^2\Theta_M)^2 \\ & -(\|\phi_{00}\|_M^2\Theta_M)^3 + (\|\phi_{00}\|_M^2\Theta_M)^4 - \cdots\Big]. \qquad (3.2) \end{aligned}$$

The Bergman function for the strongly pseudoconvex variety V is given by

$$\begin{aligned} B_V((x,y,z),(\overline{x,y,z})) = & \|\phi_{00}\|_M^2\Theta_V\Big[1 - \|\phi_{00}\|_M^2\Theta_V + (\|\phi_{00}\|_M^2\Theta_V)^2 \\ & -(\|\phi_{00}\|_M^2\Theta_V)^3 + (\|\phi_{00}\|_M^2\Theta_V)^4 - \cdots\Big] \qquad (3.3) \end{aligned}$$

where

$$\begin{aligned} \Theta_V = & \frac{|x|^2}{\|\phi_{10}\|_M^2} + \frac{|z|^2}{\|\phi_{11}\|_M^2} + \frac{|y|^2}{\|\phi_{12}\|_M^2} + \frac{|x|^4}{\|\phi_{20}\|_M^2} + \frac{|x|^2|z|^2}{\|\phi_{21}\|_M^2} + \frac{|z|^4}{\|\phi_{22}\|_M^2} \\ & + \frac{|y|^2|z|^2}{\|\phi_{23}\|_M^2} + \frac{|y|^4}{\|\phi_{24}\|_M^2} + \frac{|x|^6}{\|\phi_{30}\|_M^2} + \frac{|x|^5|y|}{\|\phi_{31}\|_M^2} + \frac{|x|^4|y|^2}{\|\phi_{32}\|_M^2} + \frac{|z|^6}{\|\phi_{33}\|_M^2} \\ & + \frac{|x|^2|y|^4}{\|\phi_{34}\|_M^2} + \frac{|x|\,|y|^5}{\|\phi_{35}\|_M^2} + \frac{|y|^6}{\|\phi_{36}\|_M^2} + \cdots \qquad (3.4) \end{aligned}$$

PROOF: $B_M((u_0,v_0),(\overline{u_0,v_0})) = \dfrac{K_{M,0}}{K_M} = \dfrac{\Theta_M}{\frac{1}{\|\phi_{00}\|_M^2} + \Theta_M} = \dfrac{\|\phi_{00}\|_M^2\Theta_M}{1 + \|\phi_{00}\|_M^2\Theta_M}$.

Hence (3.2) follows immediately. Recall that the resolution map is given by $(x, y, z) = (u, u_0 v_0^2, u_0 v_0)$. Then (3.4) and (3.3) follow from (3.1) and (3.2) respectively. *Q.E.D.*

Lemma 3.4. *Any biholomorphism $\Psi = (\psi_1, \psi_2, \psi_3) : V \to V$ has the following representation*

$$\begin{pmatrix} \psi_1(x,y,z) \\ \psi_2(x,y,z) \\ \psi_3(x,y,z) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \text{higher order terms in } x, y \text{ and } z$$

where the constants a_{ij} satisfy the following equations

$$a_{11}a_{21} - a_{31}^2 = 0 \tag{3.5}$$
$$a_{12}a_{22} - a_{32}^2 = 0 \tag{3.6}$$
$$a_{13}a_{23} - a_{33}^2 + a_{11}a_{22} + a_{12}a_{21} - 2a_{31}a_{32} = 0 \tag{3.7}$$
$$a_{11}a_{23} + a_{13}a_{21} - 2a_{31}a_{33} = 0 \tag{3.8}$$
$$a_{12}a_{23} + a_{13}a_{22} - 2a_{32}a_{33} = 0 \tag{3.9}$$
$$\det(a_{ij}) \neq 0. \tag{3.10}$$

Proof. Since $\Psi : V \to V$, we have $\psi_1(x,y,z)\psi_2(x,y,z) - \psi_3^2(x,y,z) = 0$. By looking at the quadratic part of this equation, we obtain

$$(a_{11}\,x + a_{12}\,y + a_{13}\,z)(a_{21}\,x + a_{22}\,y + a_{23}\,z) - (a_{31}\,x + a_{32}\,y + a_{33}\,z)^2 = 0$$

which implies

$$\begin{aligned} &(a_{11}a_{21} - a_{31}^2)x^2 + (a_{11}a_{22} - a_{32}^2)y^2 + (a_{13}a_{23} - a_{33}^2)z^2 \\ &+(a_{11}a_{22} + a_{12}a_{21} - 2a_{31}a_{32})xy + (a_{11}a_{23} + a_{13}a_{21} - 2a_{31}a_{33})xz \\ &+(a_{12}a_{23} + a_{13}a_{22} - 2a_{32}a_{33})yz = 0. \end{aligned} \tag{3.11}$$

Since $z^2 = xy$, (3.5)-(3.9) follows from (3.11). (3.10) is a consequence of the fact that Ψ is a biholomorphism. □

Proposition 3.5. *Let $V_i = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ and $\eta_i(x,y,z) < \epsilon_0$, where η_i is a strictly plurisubharmonic Reinhardt function$\}$ for $i = 1, 2$. Let $M_i = \widetilde{\pi}^{-1}(V_i)$, $i = 1, 2$. Suppose that $\Psi : V_1 \to V_2$ is a biholomorphic map given by $\Psi(x,y,z) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z) +$*

higher order terms. Then

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}|a_{11}|^2+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}|a_{31}|^2+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}|a_{21}|^2=\frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{10}\|^2_{M_1}} \tag{3.12}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}|a_{12}|^2+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}|a_{32}|^2+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}|a_{22}|^2=\frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{12}\|^2_{M_1}} \tag{3.13}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}|a_{13}|^2+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}|a_{33}|^2+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}|a_{23}|^2=\frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{11}\|^2_{M_1}} \tag{3.14}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}a_{11}\overline{a_{12}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}a_{31}\overline{a_{32}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}a_{21}\overline{a_{22}}=0 \tag{3.15}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}a_{11}\overline{a_{13}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}a_{31}\overline{a_{33}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}a_{21}\overline{a_{23}}=0 \tag{3.16}$$

$$\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{10}\|^2_{M_2}}a_{12}\overline{a_{13}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}a_{32}\overline{a_{33}}+\frac{\|\phi_{00}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}a_{22}\overline{a_{23}}=0 \tag{3.17}$$

Proof. $B_{V_1}((x,y,z),(\overline{x,y,z}\,)) = \|\phi_{00}\|^2_{M_1}\Theta_{V_1} - \|\phi_{00}\|^4_{M_1}\Theta^2_{V_1} + \|\phi_{00}\|^6_{M_1}\Theta^3_{V_1}\dots$

$$= \frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{10}\|^2_{M_1}}|x|^2 + \frac{\|\phi_{00}\|^2_{M_1}|z|^2}{\|\phi_{11}\|^3_{M_1}} + \frac{\|\phi_{00}\|^2_{M_1}}{\|\Phi_{12}\|^2_{M_1}}|y|^2 + \text{ higher order term.}$$

In view of Theorem 2.8, we have

$$B_{V_1}((x,y,z)),(\overline{x,y,z}\,)) = B_{V_2}(\Psi(x,y,z),\overline{\Psi(x,y,z)}\,)$$

which implies

$$\begin{aligned}
&\frac{\|\phi_{00}\|_{M_1}^2|x|^2}{\|\phi_{10}\|_{M_1}^2}+\frac{\|\phi_{00}\|_{M_1}^2|z|^2}{\|\phi_{11}\|_{M_1}^2}+\frac{\|\phi_{00}\|_{M_1}^2|y|^2}{\|\phi_{12}\|_{M_1}^2}\\
&=\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}|a_{11}x+a_{12}y+a_{13}z|^2+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}|a_{31}x+a_{32}y+a_{33}z|^2\\
&\ +\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}|a_{21}x+a_{22}y+a_{23}z|^2\\
&=\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}|a_{11}|^2+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}|a_{31}|^2+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}|a_{21}|^2\right)|x|^2\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}|a_{12}|^2+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}|a_{32}|^2+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}|a_{22}|^2\right)|y|^2\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}|a_{13}|^2+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}|a_{33}|^2+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}|a_{23}|^2\right)|z|^2\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}a_{11}\overline{a_{12}}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}a_{31}\overline{a_{32}}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}a_{21}\overline{a_{22}}\right)x\overline{y}\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}\overline{a_{11}}a_{12}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}\overline{a_{31}}a_{33}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\overline{a_{21}}a_{22}\right)\overline{x}y\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}a_{11}\overline{a_{13}}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}a_{31}\overline{a_{33}}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}a_{21}\overline{a_{23}}\right)x\overline{z}\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}a_{11}\overline{a_{13}}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}\overline{a_{31}}a_{33}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\overline{a_{21}}a_{23}\right)\overline{x}z\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}a_{12}\overline{a_{13}}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}a_{32}\overline{a_{33}}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}a_{22}\overline{a_{23}}\right)y\overline{z}\\
&\ +\left(\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}\overline{a_{12}}a_{13}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}\overline{a_{32}}a_{33}+\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\overline{a_{22}}a_{23}\right)y\overline{z}
\end{aligned}$$

(3.12)-(3.17) follows immediately. □

The following theorem gives a continuous numerical invariant for strongly pseudoconvex CR manifolds lying in $\widetilde{V}=\{(x,y,z)\in\mathbb{C}^3 : xy=z^2\}$.

Theorem 3.6. *Let $V=\{(x,y,z)\in\mathbb{C}^3 : xy=z^2$ and $\eta(x,y,z)<\epsilon_0$ where η is a strictly plurisubharmonic Reinhardt function$\}$ such that $X=\partial V$ is a smooth CR manifold. Let $M=\widetilde{\pi}^{-1}(V)$. With the notation in Proposition* 3.2*, $\nu_X:=\dfrac{\|\phi_{11}\|_M^2}{\|\phi_{10}\|_M\|\phi_{12}\|_M}$ is a CR invariant of X in V, i.e. if X_1 and X_2 are two such strongly pseudoconvex CR manifolds in V which are CR equivalent, then*

$\frac{\|\Phi_{11}\|^2_{M_1}}{\|\phi_{10}\|_{M_1}\|\phi_{12}\|_{M_1}} = \frac{\|\phi_{11}\|^2_{M_2}}{\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}$, *where* M_1 *and* M_2 *are strongly pseudo-convex CR manifolds which have* X_1 *and* X_2 *as boundaries respectively.*

Proof. Let $V_i = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ and $\eta_i(x,y,z) < \epsilon_0$, where η_i is a strictly plurisubharmonic Reinhardt function$\}$ and $\partial V_i = X_i$, $i = 1,2$. If X_i is CR equivalent to X_2, then V_1 is biholomorphic equivalent to V_2 by Theorem 3.1. Theorem 3.6 follows from the following Theorem 3.7. □

Theorem 3.7. *Let* $V_i = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ *and* $\eta_i(x,y,z) < \epsilon_0$, *where* η_i *is a strictly plurisubharmonic Reinhardt function*$\}$ *and* $M_i = \widetilde{\pi}^{-1}(V_i)$ *is the resolution of singularity of* V_i, $i = 1,2$. *If there exists a biholomorphic map* Ψ *from* V_1 *to* V_2 *and* $\frac{\|\phi_{11}\|^2_{M_2}}{\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}} \neq \frac{1}{2}$, *then* $\frac{\|\phi_{11}\|^2_{M_1}}{\|\phi_{10}\|_{M_1}\|\phi_{12}\|_{M_1}} = \frac{\|\phi_{11}\|^2_{M_2}}{\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}$.

Proof. The same argument as in Lemma 3.4 will show that Ψ can be written as $(\psi_1,\psi_2,\psi_3) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z)+$ higher order terms such that (3.5)-(3.10) hold. By Proposition 3.5, we know that (3.12)-(3.17) also hold. We have three cases to consider: Case 1, $a_{31} \neq 0$ and $a_{32} \neq 0$; Case 2, $a_{31} = 0$; Case 3, $a_{32} = 0$.

Case 1: $a_{31} \neq 0$ and $a_{32} \neq 0$. In view of (3.5) and (3.6), we have $a_{11} \neq 0, a_{21} \neq 0, a_{12} \neq 0$ and $a_{22} \neq 0$ in this case.

$$(3.5) \implies \frac{a_{11}}{a_{31}} = \frac{a_{31}}{a_{21}} := r_1 \neq 0$$

$$\implies a_{11} = r_1 a_{31}, \quad a_{21} = \frac{1}{r_1} a_{31} \tag{3.18}$$

$$(3.6) \implies \frac{a_{22}}{a_{32}} = \frac{a_{32}}{a_{12}} := r_2 \neq 0$$

$$\implies a_{22} = r_2 a_{32}, \quad a_{12} = \frac{1}{r_2} a_{32} \tag{3.19}$$

$$(3.8) \text{ and } (3.18) \implies r_1 a_{23} + \frac{1}{r_1} a_{13} - 2a_{33} = 0 \tag{3.20}$$

$$(3.9) \text{ and } (3.19) \implies \frac{1}{r_2} a_{23} + r_2 a_{13} - 2a_{33} = 0 \tag{3.21}$$

$$(3.20) \text{ and } (3.21) \implies (r_1 - \frac{1}{r_2}) a_{23} + (\frac{1}{r_1} - r_2) a_{13} = 0. \tag{3.22}$$

There are two cases to be considered.

Case 1 (a): $r_1 - \dfrac{1}{r_2} = 0$, i.e. $r_2 = \dfrac{1}{r_1}$

$$(3.23) \qquad (3.19) \implies a_{22} = \frac{1}{r_1}a_{32}, \quad a_{12} = r_1 a_{32}$$

$$(3.24) \qquad (3.20) \implies a_{33} = \frac{1}{2}r_1 a_{23} + \frac{1}{2r_1}a_{13}$$

$$(3.25) \qquad (3.7), (3.18) \text{ and } (3.23) \implies a_{13}a_{23} - a_{33}^2 = 0$$

$$(3.26) \qquad (3.25) \text{ and } (3.24) \implies a_{13} = r_1^2 a_{23}$$

$$(3.27) \qquad (3.24) \text{ and } (3.26) \implies a_{33} = r_1 a_{23} = \frac{1}{r_1}a_{13}$$

(3.15), (3.18) and (3.23) imply

$$(3.28) \qquad \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}|r_1|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\frac{1}{|r_1|^2} = 0,$$

which is a contradiction because the left hand side of (3.28) is positive. Hence Case 1 (a) cannot happen.

Case 1 (b): $r_1 - \dfrac{1}{r_2} \neq 0$

$$(3.29) \qquad (3.22) \implies a_{23} = \frac{r_2}{r_1}a_{13}$$

$$(3.30) \qquad (3.20) \text{ and } (3.29) \implies a_{33} = \left(\frac{r_2}{2} + \frac{1}{2r_1}\right)a_{13}.$$

In view of (3.29) and (3.30), we have $a_{13} \neq 0$ because $\det(a_{ij}) \neq 0$. (3.15), (3.18) and (3.19) imply

$$(3.31) \qquad \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}\frac{r_1}{\overline{r_2}} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\frac{\overline{r_2}}{r_1} = 0$$

(3.16) and (3.18), (3.29) and (3.30) imply

$$(3.32) \qquad \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}r_1 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}\left(\frac{\overline{r_2}}{2} + \frac{1}{2\overline{r_1}}\right) + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\frac{\overline{r_2}}{|r_1|^2} = 0$$

(3.17), (3.19), (3.29) and (3.30) imply

$$(3.33) \qquad \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2}\frac{1}{r_2} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2}\left(\frac{\overline{r_2}}{2} + \frac{1}{2\overline{r_1}}\right) + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2}\frac{|r_2|^2}{\overline{r_1}} = 0$$

Clearly (3.32), (3.33) and (3.34) imply

$$\det\begin{pmatrix} \frac{r_1}{\overline{r_2}} & 1 & \frac{\overline{r_2}}{r_1} \\ r_1 & \frac{\overline{r_2}}{2}+\frac{1}{2\overline{r_1}} & \frac{\overline{r_2}}{|r_1|^2} \\ \frac{1}{r_2} & \frac{\overline{r_2}}{2}+\frac{1}{2\overline{r_1}} & \frac{|r_2|^2}{r_1} \end{pmatrix} = 0$$

$$\Longrightarrow (\overline{r_1 r_2}+1)(r_1 r_2 + \overline{r_2 r_2})(r_1 r_2 - 1) + 2\overline{r_1 r_2}(1 - r_1 r_2)(1 + r_1 r_2) = 0.$$

Since $r_1 - \dfrac{1}{r_2} \neq 0$, i.e. $r_1 r_2 - 1 \neq 0$, we have

$$\begin{aligned} & (\overline{r_1 r_2}+1)(r_1 r_2 + \overline{r_1 r_2}) - 2\overline{r_1 r_2}(1 + r_1 r_2) = 0 \\ \Longrightarrow \quad & (r_1 r_2 - \overline{r_1 r_2})(1 - \overline{r_1 r_2}) = 0. \end{aligned}$$

Since $\overline{r_1 r_2} - 1 \neq 0$, we have

$$r_1 r_2 = \overline{r_1 r_2} \tag{3.34}$$

Let $\alpha = \frac{r_1}{\overline{r_2}}$. Then $\alpha = \overline{\alpha}$, $r_1 = \alpha\overline{r_2}$, $\overline{r_1} = \alpha r_2$. (3.31), (3.32) and (3.33) can be rewritten as

$$\alpha^2 + \frac{\|\phi_{10}\|^2_{M_2}}{\|\phi_{11}\|^2_{M_2}}\alpha + \frac{\|\phi_{10}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}} = 0 \tag{3.35}$$

$$\frac{\alpha^3|r_2|^2}{\|\phi_{10}\|^2_{M_2}} + \frac{1}{\|\phi_{11}\|^2_{M_2}}\left(\frac{\alpha^2|r_2|^2}{2} + \frac{\alpha}{2}\right) + \frac{1}{\|\phi_{12}\|^2_{M_2}} = 0 \tag{3.36}$$

$$\frac{\alpha}{\|\phi_{10}\|^2_{M_2}} + \frac{1}{\|\phi_{11}\|^2_{M_2}}\left(\frac{\alpha|r_2|^2}{2} + \frac{1}{2}\right) + \frac{|r_2|^2}{\|\phi_{12}\|^2_{M_2}} = 0 \tag{3.37}$$

(3.36) $-\ \alpha$(3.37) $\Rightarrow$

$$\frac{\alpha^2(\alpha|r_2|^2 - 1)}{\|\phi_{10}\|^2_{M_2}} + \frac{1 - \alpha|r_2|^2}{\|\phi_{12}\|^2_{M_2}} = 0 \tag{3.38}$$

$$\text{i.e. } 1 - \alpha|r_2|^2 = 0 \qquad \text{or} \qquad \alpha^2 = \frac{\|\phi_{10}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}.$$

If $\alpha^2 = \dfrac{\|\phi_{10}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}$, then (3.35) implies $\alpha = \dfrac{-2\|\phi_{11}\|^2_{M_2}}{\|\phi_{12}\|^2_{M_2}}$. It follows easily that $\dfrac{\|\phi_{11}\|^2_{M_2}}{\|\phi_{12}\|_{M_2}\|\phi_{10}\|_{M_2}} = \dfrac{1}{2}$, a contradiction to our hypothesis. Hence we conclude that

$$\alpha|r_2|^2 = 1. \tag{3.39}$$

Putting (3.39) in (3.37), we get

$$\frac{1}{|r_2|^2\|\phi_{10}\|^2_{M_2}} + \frac{1}{\|\phi_{11}\|^2_{M_2}} + \frac{|r_2|^2}{\|\phi_{12}\|^2_{M_2}} = 0$$

which is absurd since the left hand side is positive. Thus Case 1 (b) cannot occur also.

Case 2: $a_{31} = 0$. By (3.5), we have either $a_{11} = 0$ or $a_{21} = 0$.

Case 2 (a): $a_{31} = 0$ and $a_{11} = 0$. Since $\det(a_{ij}) \neq 0$, we have $a_{21} \neq 0$.

$$(3.40) \qquad (3.15) \implies a_{21}\overline{a_{22}} = 0 \Longrightarrow a_{22} = 0$$
$$(3.41) \qquad (3.8) \implies a_{13}a_{21} = 0 \Longrightarrow a_{13} = 0$$
$$(3.42) \qquad (3.6) \text{ and } (3.40) \implies a_{32} = 0.$$

Since $\det(a_{ij}) \neq 0$ and $a_{11} = 0 = a_{13}$, we have $a_{12} \neq 0$.

$$(3.43) \qquad (3.9), (3.40) \text{ and } (3.42) \implies a_{12}a_{23} = 0 \Longrightarrow a_{23} = 0$$
$$(3.44) \qquad (3.7) \text{ and } (3.43) \implies -a_{33}^2 + a_{12}a_{21} = 0$$
$$(3.45) \qquad (3.12) \implies |a_{21}|^2 = \frac{\|\phi_{12}\|_{M_2}^2}{\|\phi_{00}\|_{M_2}^2} \cdot \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{10}\|_{M_1}^2}$$
$$(3.46) \qquad (3.13) \implies |a_{12}|^2 = \frac{\|\phi_{10}\|_{M_2}^2}{\|\phi_{00}\|_{M_2}^2} \cdot \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{12}\|_{M_1}^2}$$
$$(3.47) \qquad (3.14) \implies |a_{33}|^2 = \frac{\|\phi_{11}\|_{M_2}^2}{\|\phi_{00}\|_{M_2}^2} \cdot \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{11}\|_{M_1}^2}$$

(3.44), (3.45), (3.46) and (3.47) imply

$$(3.48) \qquad \frac{\|\phi_{11}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}\|\phi_{10}\|_{M_2}} = \frac{\|\phi_{11}\|_{M_1}^2}{\|\phi_{10}\|_{M_1}\|\phi_{12}\|_{M_1}}.$$

Case 2 (b): $a_{31} = 0$ and $a_{21} = 0$. Since $\det(a_{ij}) \neq 0$, we have $a_{11} \neq 0$.

$$(3.49) \qquad (3.15) \implies a_{11}\overline{a_{12}} = 0 \Longrightarrow a_{12} = 0$$
$$(3.50) \qquad (3.8) \implies a_{11}a_{23} = 0 \Longrightarrow a_{23} = 0$$
$$(3.51) \qquad (3.6) \text{ and } (3.48) \implies a_{32} = 0.$$

Since $a_{32} = 0 = a_{12}$ and $\det(a_{ij}) \neq 0$, we have $a_{22} \neq 0$.

$$(3.52)\qquad (3.9), (3.51) \text{ and } (3.49) \implies a_{13}a_{22} = 0 \implies a_{13} = 0$$

$$(3.53)\qquad (3.7), (3.52) \text{ and } (3.49) \implies -a_{33}^2 + a_{11}a_{22} = 0$$

$$(3.54)\qquad (3.12) \implies |a_{11}|^2 = \frac{\|\phi_{10}\|^2_{M_2}}{\|\phi_{00}\|^2_{M_2}} \cdot \frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{10}\|^2_{M_1}}$$

$$(3.55)\qquad (3.13) \implies |a_{22}|^2 = \frac{\|\phi_{12}\|^2_{M_2}}{\|\phi_{00}\|^2_{M_2}} \cdot \frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{12}\|^2_{M_1}}$$

$$(3.56)\qquad (3.14) \implies |a_{33}|^2 = \frac{\|\phi_{11}\|^2_{M_2}}{\|\phi_{00}\|^2_{M_2}} \cdot \frac{\|\phi_{00}\|^2_{M_1}}{\|\phi_{11}\|^2_{M_1}}$$

(3.53), (3.54), (3.55) and (3.56) imply (3.48).

Case 3: $a_{32} = 0$. By (3.6), we have either $a_{12} = 0$ or $a_{22} = 0$.

Case 3 (a): $a_{32} = 0$ and $a_{12} = 0$. By the same argument as above, we can show that all a_{ij} are zero except a_{11}, a_{22} and a_{33}. Moreover a_{11}, a_{22}, a_{33} satisfy (3.53), (3.54), (3.55) and (3.56) so that (3.48) holds.

Case 3 (b): $a_{32} = 0$ and $a_{22} = 0$. By the same argument as above, we can show that all a_{ij} are zero except a_{12}, a_{21} and a_{33}. Moreover, a_{12}, a_{21}, a_{33} satisfy (3.44), (3.45), (3.46) and (3.47) so that (3.48) holds. □

Corollary 3.8. *Let $V_i = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2$ and $\eta_i(x, y, z) < \epsilon_0$, where η_i is a strictly plurisubharmonic Reinhardt function$\}$. If the CR invariant ν_{X_2} in Theorem 3.6 is not equal to 0.5, then the biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3) : V_1 \to V_2$ must be one of the following forms:*

(1) $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z) +$ *higher order terms and* $a_{33}^2 = a_{11}a_{22}$.

(2) $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z) +$ *higher order terms and* $a_{33}^2 = a_{12}a_{21}$.

PROOF: It is clear from the proof of Theorem 3.7.

4. Explicit computation of new CR invariant

Let a be positive real number. We shall follow the notations in our previous section. Let $V_a = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2,\ a|x|^2 + |y|^2 + |z|^2 < \epsilon_0\}$. Recall that $(x, y, z) = (u_0, u_0v_0^2, u_0v_0)$. Then M_a be the resolution of V_a with coordinate chart $W_0 = \{(u_0, v_0) : a|u_0|^2 + |u_0|^2|v_0|^4 + |u_0|^2|v_0|^2 < \epsilon_0\}$. Next write $u_0 = re^{i\theta}$

and $v_0 = \rho e^{i\phi}$. Then

$$\begin{aligned}\|\phi_{\alpha\beta}\|^2_{M_a} &= \int_{M_a} \phi_{\alpha\beta} \wedge \overline{\phi_{\alpha\beta}} = \int_{W_0} |u_0^\alpha|^2 |v_0^\beta|^2 du_0 \wedge dv_0 \wedge d\,\overline{u_0} \wedge d\,\overline{v_0} \\ &= 2\pi \int_0^{2\pi} \iint_D r^{2\alpha+1} \rho^{2\beta+1} dr d\rho\, d\theta\end{aligned}$$

where $D = \{(r,\rho) : r \geq 0,\ \rho \geq 0,\ ar^2 + r^2\rho^4 + r^2\rho^2 < \epsilon_0\}$. In particular

$$\begin{aligned}\|\phi_{1\beta}\|^2_{M_a} &= 2\pi \int_0^{2\pi} \int_0^\infty \int_0^{\frac{\sqrt{\epsilon_0}}{\sqrt{a+\rho^2+\rho^4}}} r^3 \rho^{2\beta+1} dr d\rho d\theta \\ &= 2\pi \int_0^{2\pi} \int_0^\infty \frac{\epsilon_0^2 \rho^{2\beta+1}}{(a+\rho^2+\rho^4)^2} d\rho d\theta.\end{aligned}$$

Therefore the new CR invariant for the CR manifold $X_a := \partial V_a$ is

$$\begin{aligned}\nu_a &:= \frac{\|\phi_{11}\|^2_{M_a}}{\|\phi_{10}\|_{M_a} \|\phi_{12}\|_{M_a}} \\ &= \frac{\int_0^{2\pi}\int_0^\infty \frac{\rho^3}{(a+\rho^2+\rho^4)^2} d\rho d\theta}{\left(\int_0^{2\pi}\int_0^\infty \frac{\rho}{(a+\rho^2+\rho^4)^2} d\rho d\theta\right)^{\frac{1}{2}} \left(\int_0^{2\pi}\int_0^\infty \frac{\rho^5}{(a+\rho^2+\rho^4)^2} d\rho d\theta\right)^{\frac{1}{2}}}.\end{aligned}$$

Case 1: $a = \dfrac{1}{4}$

$$\begin{aligned}I_1 &= \int_0^{2\pi}\int_0^\infty \frac{\rho}{(\rho^4+\rho^2+\frac{1}{4})^2} d\rho d\theta = \pi \int_0^\infty \frac{1}{(x+\frac{1}{2})^4} dx = \frac{8}{3}\pi \\ I_2 &= \int_0^{2\pi}\int_0^\infty \frac{\rho^3}{(\rho^4+\rho^2+\frac{1}{4})^2} d\rho d\theta = \pi \int_0^\infty \frac{x}{(x+\frac{1}{2})^4} dx = \frac{2}{3}\pi \\ I_3 &= \int_0^{2\pi}\int_0^\infty \frac{\rho^5}{(\rho^4+\rho^2+\frac{1}{4})^2} d\rho d\theta = \pi \int_0^\infty \frac{x^2}{(x+\frac{1}{2})^4} dx = \frac{2}{3}\pi \\ \nu_{\frac{1}{4}} &= \frac{I_2}{\sqrt{I_1 I_3}} = \frac{\frac{2}{3}\pi}{\sqrt{\frac{8}{3}\pi \cdot \frac{2}{3}\pi}} = \frac{1}{2}.\end{aligned}$$

Case 2: $a > \dfrac{1}{4}$

$$
\begin{aligned}
I_1 &= \int_0^{2\pi}\int_0^{\infty} \frac{\rho}{(\rho^4+\rho^2+a)^2} d\rho d\theta = \pi \int_0^{\infty} \frac{dx}{(x^2+x+a)^2} \\
&= \pi\left[\frac{2x+1}{(4a-1)(x^2+x+a)} + \frac{4}{(4a-1)^{\frac{3}{2}}} \arctan\left(\frac{x+1}{\sqrt{4a-1}}\right)\right]\Bigg|_0^{\infty} \\
&= \pi\left[\frac{\pi}{(4a-1)^{\frac{3}{2}}} - \frac{1}{a(4a-1)} - \frac{4}{(4a-1)^{\frac{3}{2}}} \arctan\left(\frac{1}{\sqrt{4a-1}}\right)\right] \\
I_2 &= \int_0^{2\pi}\int_0^{\infty} \frac{\rho^3}{(\rho^4+\rho^2+a)^2} d\rho d\theta = \pi \int_0^{\infty} \frac{x dx}{(x^2+x+a)^2} \\
&= \pi\left[-\frac{2a+x}{(4a-1)(x^2+x+a)} - \frac{2}{(4a-1)^{\frac{3}{2}}} \arctan\left(\frac{2x+1}{\sqrt{4a-1}}\right)\right]\Bigg|_0^{\infty} \\
&= \pi\left[\frac{2}{4a-1} - \frac{\pi}{(4a-1)^{\frac{3}{2}}} + \frac{2}{(4a-1)^{\frac{3}{2}}} \arctan\left(\frac{1}{\sqrt{4a-1}}\right)\right] \\
I_3 &= \int_0^{2\pi}\int_0^{\infty} \frac{\rho^5}{(\rho^4+\rho^2+a)^2} d\rho d\theta = \pi \int_0^{\infty} \frac{x^2 dx}{(x^2+x+a)^2} \\
&= \pi\left[\frac{(1-2a)x+a}{(4a-1)(x^2+x+a)} + \frac{4a}{(4a-1)^{\frac{3}{2}}} \arctan\left(\frac{2x+1}{\sqrt{4a-1}}\right)\right]\Bigg|_0^{\infty} \\
&= \pi\left[\frac{2a\pi}{(4a-1)^{\frac{3}{2}}} - \frac{1}{4a-1} - \frac{4a}{(4a-1)^{\frac{3}{2}}} \arctan\left(\frac{1}{\sqrt{4a-1}}\right)\right].
\end{aligned}
$$

If $a = \dfrac{1}{2}$, then $I_1 = \pi(\pi-2)$, $I_2 = \pi\left(2 - \dfrac{\pi}{2}\right)$ and $I_3 = \pi\left(\dfrac{\pi}{2} - 1\right)$. We have $\nu_{\frac{1}{2}} = \dfrac{2-\frac{\pi}{2}}{\sqrt{(\pi-2)\left(\frac{\pi}{2}-1\right)}} = \dfrac{\sqrt{2}(4-\pi)}{2(\pi-2)} \simeq 0.5317007373 > 0.5.$

Similarly if $a = 1 > \dfrac{1}{4}$, then $I_1 = \frac{\pi}{6}\left(\dfrac{4\pi}{3\sqrt{3}} - 1\right)$, $I_2 = \frac{\pi}{3}\left(1 - \dfrac{\pi}{3\sqrt{3}}\right)$, $I_3 = I_1$ and $\nu_1 \simeq 0.55753024284785 > 0.5$.

Case 3: $a < \frac{1}{4}$

$$\begin{aligned}
I_1 &= \int_0^{2\pi}\int_0^{\infty} \frac{\rho}{(\rho^4+\rho^2+a)} d\rho d\theta = \pi \int_0^{\infty} \frac{dx}{(x^2+x+a)^2} \\
&= \pi\left[\frac{1}{a(1-4a)} + \frac{2}{(1-4a)^{\frac{3}{2}}} \ln \frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right] \\
I_2 &= \int_0^{2\pi}\int_0^{\infty} \frac{\rho^3}{(\rho^4+\rho^2+a)^2} d\rho d\theta = \pi \int_0^{\infty} \frac{x}{(x^2+x+a)^2} dx \\
&= \pi\left[-\frac{2}{1-4a} - \frac{1}{(1-4a)^{\frac{3}{2}}} \ln \frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right] \\
I_3 &= \int_0^{2\pi}\int_0^{\infty} \frac{\rho^5}{(\rho^4+\rho^2+a)^2} d\rho d\theta = \pi \int_0^{\infty} \frac{x^2}{(x^2+x+a)^2} dx \\
&= \frac{\pi}{2}\left[\frac{1}{1-4a} + \frac{2a}{(1-4a)^{\frac{3}{2}}} \ln \frac{1-\sqrt{1-4a}}{1+\sqrt{1-4a}}\right].
\end{aligned}$$

If $a = \frac{3}{16} < \frac{1}{4}$, then $I_1 = \pi\big(\frac{64}{3} - 16\ln 3\big)$, $I_2 = \pi(8\ln 3 - 8)$ and $I_3 = \pi(4 - 3\ln 3)$. Hence

$$\begin{aligned}
\nu_{\frac{3}{16}} &= \frac{8\ln 3 - 8}{\sqrt{(\frac{64}{3} - 16\ln 3)(4 - 3\ln 3)}} = \frac{2\sqrt{3}(\ln 3 - 1)}{4 - 3\ln 3} \\
&\simeq 0.4851191037356 < 0.5.
\end{aligned}$$

Similarly one has the following results:

$$\begin{aligned}
a = \frac{2}{9} &< \frac{1}{4}, & \nu_{\frac{2}{9}} &\simeq 0.4940697511735 \\
a = \frac{15}{64} &< \frac{1}{4}, & \nu_{\frac{15}{64}} &\simeq 0.49674784168264 \\
a = \frac{6}{25} &< \frac{1}{4}, & \nu_{\frac{6}{25}} &\simeq 0.49794879674687.
\end{aligned}$$

5. Automorphism group of compact CR manifold

In this section, we shall show that our Bergman function can be used to determine the automorphism group of a compact CR manifold.

Theorem 5.1. *Let $X = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2,\ \eta(x,y.z) < \epsilon_0\}$ where η is a strictly plurisubharmonic Reinhardt function and ν_X be the CR invariant defined in the previous section, where $X = \partial V$. Then the automorphism group of X for $\nu_X \neq \frac{1}{2}$ consists of biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3)$ of the following forms:*

(1) $(\psi_1, \psi_2, \psi_3) = (e^{i\theta_1}x, e^{i\theta_2}y, e^{i\frac{\theta_1+\theta_2}{2}}z)$
(2) $(\psi_1, \psi_2, \psi_3) = (e^{i\theta_1}y, e^{i\theta_2}x, e^{i\frac{\theta_1+\theta_2}{2}}z)$.

Proof. In view of Corollary 3.8, we know that $\Psi = (\psi_1, \psi_2, \psi_3)$ must be one of the following forms:

(1) $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z)+$ higher order terms and $a_{33}^2 = a_{11}a_{22}$
(2) $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)+$ higher order terms and $a_{33}^2 = a_{12}a_{21}$.

Recall that the Bergman function $B_V = \|\phi_{00}\|_M^2 \Theta_V - \|\phi_{00}\|_M^4 \Theta_V^2 + \cdots$. where M is the minimal resolution of V as described in section 3, and Θ_V is given by (3.4). In view of Theorem 2.8, we have

$$B_V((x,y,z), \overline{(x,y,z)}) = B_V(\Psi(x,y,z), \overline{\Psi(x,y,z)}). \tag{5.1}$$

Putting (1) and (2) in (5.1) and comparing the 3rd order terms in (5.1), we see easily the 2nd order terms of (ψ_1, ψ_2, ψ_3) are zero. Repeating this argument, we see that (ψ_1, ψ_2, ψ_3) has only linear terms. Using (5.1) again, we obtain $|a_{11}| = |a_{22}| = |a_{12}| = |a_{21}| = 1$. □

In Corollary 3.8, we need to assume that our invariant ν_X is not $\frac{1}{2}$. If we consider only automorphism instead of biholomorphism, we can deal with the case $\nu_X = \frac{1}{2}$.

Theorem 5.2. *Let $V = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ and $\eta(x,y,z) < \epsilon_0$, where η is a strictly plurisubharmonic Reinhardt function$\}$. Suppose that $\nu_X = \frac{1}{2}$ where ν_X is the CR invariant of $X = \partial V$ defined in Theorem* 3.6*. Then the automorphism $\psi = (\psi_1, \psi_2, \psi_3) : X \to X$ must be one of the following forms:*

(1) $(\psi_1, \psi_2, \psi_3) = (e^{i\theta_1}x, e^{i\theta_2}y, e^{i\frac{\theta_1+\theta_2}{2}}z)$
(2) $(\psi_1, \psi_2, \psi_3) = (e^{i\theta_1}y, e^{i\theta_2}x, e^{i\frac{\theta_1+\theta_2}{2}}z)$
(3) $$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \alpha r e^{-i\theta} a_{31} & \frac{1}{r} e^{-i\theta} a_{32} & a_{13} \\ \frac{1}{\alpha r} e^{i\theta} a_{31} & r e^{i\theta} a_{32} & \frac{1}{\alpha} e^{2i\theta} a_{13} \\ a_{31} & a_{32} & \frac{\alpha r^2+1}{2\alpha r} e^{i\theta} a_{13} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} +$$
higher order terms
where $\alpha = \dfrac{-2\|\phi_{11}\|_M^2}{\|\phi_{12}\|_M^2}$

$$\begin{aligned} a_{31} &= \frac{r}{|\alpha| r^2 + 1} e^{i\theta_{31}} \\ a_{32} &= \frac{r\|\phi_{10}\|_M}{\|\phi_{12}\|_M + r^2\|\phi_{10}\|_M} e^{i\theta_{32}} \\ a_{13} &= \frac{2r\|\phi_{10}\|_M}{r^2\|\phi_{10}\|_M + \|\phi_{12}\|_M} e^{i(\frac{\pi}{2} + \frac{\theta_{31}}{2} + \frac{\theta_{32}}{2} - \theta)}. \end{aligned}$$

Proof. The proof is the same as the proof of Theorem 3.7 except in case 1(b) where we did use the assumption $\nu_{X_2} \neq \frac{1}{2}$. Therefore to prove Theorem 5.2, it is sufficient to prove that statement (3) above occurs in case 1(b) of the proof of Theorem 3.7 under the assumption

$$\nu_X = \frac{\|\phi_{11}\|_M^2}{\|\phi_{10}\|_M \|\phi_{12}\|_M} = \frac{1}{2}. \tag{5.2}$$

Recall that in case 1(b) of the proof of Theorem 3.7, we have

$$a_{11} = r_1 a_{31}, \qquad a_{21} = \tfrac{1}{r_1} a_{31} \tag{5.3}$$

$$a_{22} = r_2 a_{32}, \qquad a_{12} = \tfrac{1}{r_2} a_{32} \tag{5.4}$$

$$a_{23} = \frac{r_2}{r_1} a_{13}, \quad a_{33} = \left(\tfrac{r_2}{2} + \tfrac{1}{2r_1}\right) a_{13} \tag{5.5}$$

$$\alpha = \frac{r_1}{\bar{r}_2} = \frac{-2\|\phi_{11}\|_M^2}{\|\phi_{12}\|_M^2} = \frac{-\|\phi_{10}\|_M}{\|\phi_{12}\|_M}. \tag{5.6}$$

Let

$$r_2 = re^{i\theta}. \tag{5.7}$$

Then

$$r_1 = \alpha r e^{-i\theta} = \frac{-\|\phi_{10}\|_M}{\|\phi_{12}\|_M} r e^{i\theta} \tag{5.8}$$

and

$$|r_1| = \frac{\|\phi_{10}\|_M}{\|\phi_{12}\|_M} r. \tag{5.9}$$

Putting (3.29), (3.30), (3.18) and (3.19) into (3.7), we get

$$-a_{13}^2 \left(\frac{r_2}{2} - \frac{1}{2r_1}\right)^2 + a_{31}a_{32}\left(r_1 r_2 + \frac{1}{r_1 r_2} - 2\right) = 0. \tag{5.10}$$

Putting (3.18) into (3.12), we get

$$|a_{31}|^2 \|\phi_{00}\|_M^2 \left[\frac{|r_1|^2}{\|\phi_{10}\|_M^2} + \frac{1}{\|\phi_{11}\|_M^2} + \frac{1}{|r_1|^2 \|\phi_{12}\|_M^2}\right] = \frac{\|\phi_{00}\|_M^2}{\|\phi_{10}\|_M^2} \tag{5.11}$$

(5.2) and (5.11) imply

$$|a_{31}| = \frac{|r_1|\, \|\phi_{12}\|_M}{|r_1|^2 \|\phi_{12}\|_M + \|\phi_{10}\|_M} = \frac{r\|\phi_{10}\|_M}{|\alpha| r^2 \|\phi_{10}\|_M + \|\phi_{00}\|_M} = \frac{r}{|\alpha| r^2 + 1}. \tag{5.12}$$

Putting (3.19) into (3.13), we get

$$|a_{32}| = \frac{|r_2|\, \|\phi_{10}\|_M}{\|\phi_{12}\|_M + |r_2|^2 \|\phi_{10}\|_M} = \frac{r\|\phi_{10}\|_M}{\|\phi_{12}\|_M + |r^2| \|\phi_{10}\|_M}. \tag{5.13}$$

Putting (3.29) and (3.30) into (3.14), we get

$$\begin{aligned} |a_{13}|^2 &= \frac{2|r_1|^2 \|\phi_{10}\|_M \|\phi_{12}\|_M}{|r_1|^2 \|\phi_{12}\|_M^2 + \frac{1}{2}(r_1 r_2 + 1)^2 \|\phi_{12}\|_M \|\phi_{10}\|_M + |r_2|^2 \|\phi_{10}\|_M^2} \\ &= \frac{4r^2 \|\phi_{10}\|_M^2}{(r^2 \|\phi_{10}\|_M + \|\phi_{12}\|_M)^2}. \end{aligned} \tag{5.14}$$

This implies

$$|a_{13}| = \frac{2r^2\|\phi_{10}\|_M^2}{r^2\|\phi_{10}\|_M + \|\phi_{12}\|_M} \tag{5.15}$$

(5.9) implies

$$a_{13}^2 = a_{31}a_{32}\frac{4r_1}{r_2}. \tag{5.16}$$

Putting (5.12), (5.13) and (5.14) into (5.16), we get

$$\frac{4r^2\|\phi_{10}\|_M^2}{(r^2\|\phi_{10}\|_M + \|\phi_{12}\|_M)^2}e^{2i\theta_{13}} = \frac{r}{|\alpha|r^2+1}\frac{r\|\phi_{10}\|_M}{\|\phi_{12}\|_M + r^2\|\phi_{10}\|_M}e^{i(\theta_{31}+\theta_{32})}4\frac{\alpha r e^{-i\theta}}{re^{i\theta}}.$$

It follows that

$$e^{2i\theta_{13}} = e^{-2i\theta+i\pi+i(\theta_{31}+\theta_{32})}.$$

Hence

$$\theta_{13} = \frac{\pi}{2} + \frac{\theta_{31}}{2} + \frac{\theta_{32}}{2} - \theta \tag{5.17}$$

□

Corollary 5.3. *Let $V = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2$ and $\eta(x,y,z) < \epsilon_0$, where η is a strictly plurisubharmonic Reinhardt function$\}$. Suppose that $\nu_X = \frac{1}{2}$ where ν_X is the CR invariant of $X = \partial V$ defined in Theorem* 3.6*. If case* (3) *of Theorem* 5.2 *occurs, then the CR automorphism group of X contains a* 4*-dimensional linear subgroup of the following form:*

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}\alpha r e^{-i\theta}a_{31} & \frac{1}{r}e^{-i\theta}a_{32} & a_{13}\\ \frac{1}{\alpha r}e^{i\theta}a_{31} & re^{i\theta}a_{32} & \frac{1}{\alpha}e^{2i\theta}a_{13}\\ a_{31} & a_{32} & \frac{\alpha r^2+1}{2\alpha r}e^{i\theta}a_{13}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

where

$$\begin{aligned}\alpha &= \frac{-2\|\phi_{11}\|_M^2}{\|\phi_{12}\|_M^2}\\ a_{31} &= \frac{r}{|\alpha|r^2+1}e^{i\theta_{31}}\\ a_{32} &= \frac{r\|\phi_{10}\|_M}{\|\phi_{12}\|_M + r^2\|\phi_{10}\|_M}e^{i\theta_{32}}\\ a_{13} &= \frac{2r\|\phi_{10}\|_M}{r^2\|\phi_{10}\|_M + \|\phi_{12}\|_M}e^{i(\frac{\pi}{2}+\frac{\theta_{31}}{2}+\frac{\theta_{32}}{2}-\theta)}.\end{aligned}$$

Now we are ready to compute the automorphism group of X even if $\nu_X = \frac{1}{2}$.

Theorem 5.4. *Let $X_a = \{(x,y,z) \in \mathbb{C}^3 : a|x|^2 + |y|^2 + |z|^2 = \epsilon_0\}$ and ν_a be the CR invariant defined in the previous section. Then $\nu_{1/4} = \nu(X_{1/4}) = \frac{1}{2}$ and the automorphism group of $X_{1/4}$ consists of biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3)$ of the following forms:*

(1) $(\psi_1, \psi_2, \psi_3) = (e^{i\theta_1}x, e^{i\theta_2}y, e^{i\frac{\theta_1+\theta_2}{2}}z)$
(2) $(\psi_1, \psi_2, \psi_3) = (e^{i\theta_1}y, e^{i\theta_2}x, e^{i\frac{\theta_1+\theta_2}{2}}z)$.

Proof. We only need to prove that the automorphism group of $X_{1/4}$ does not have element of the form in Corollary 5.3. Suppose on the contrary that automorphism of the form in Corollary 5.3 does exist. Since $(0, \sqrt{\epsilon_0}, 0)$ is in $X_{1/4}$, we have $(\frac{\sqrt{\epsilon_0}}{r}e^{-i\theta}a_{32}, \sqrt{\epsilon_0}re^{i\theta}a_{32}, \sqrt{\epsilon_0}a_{32})$ in $X_{1/4}$. Hence

$$\frac{1}{4}\frac{\epsilon_0}{r^2}|a_{32}|^2 + \epsilon_0 r^2|a_{32}|^2 + \epsilon_0|a_{32}|^2 = \epsilon_0.$$

This implies

$$\left(\frac{1}{4r^2} + r^2 + 1\right)\frac{r^2\|\phi_{10}\|_M^2}{(\|\phi_{12}\|_M + r^2\|\phi_{10}\|_M^2} = 1.$$

It follows that

$$r^2 = \frac{\|\phi_{10}\|_M + 2\|\phi_{12}\|_M}{4\|\phi_{10}\|_M}.$$

This is absurd. □

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