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RESONANCES AND SCATTERING POLES ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. On an asymptotically hyperbolic manifold (X, g), we show that the poles (called resonances) of the meromorphic extension of the resolvent $(\Delta_g - \lambda(n-\lambda))^{-1}$ coincide, with multiplicities, with the poles (called scattering poles) of the renormalized scattering operator, except for the points of $\frac{n}{2} - \mathbb{N}$. At each $\lambda_k := \frac{n}{2} - k$ with $k \in \mathbb{N}$, the resonance multiplicity $m(\lambda_k)$ and the scattering pole multiplicity $\nu(\lambda_k)$ do not always coincide: $\nu(\lambda_k) - m(\lambda_k)$ is the dimension of the kernel of a differential operator on the boundary $\partial\bar{X}$ introduced by Graham and Zworski; in the asymptotically Einstein case, this operator is the k-th conformal Laplacian.

1. Introduction

The purpose of this work is to give a 'more direct' proof of the result of Borthwick and Perry [1] about the equivalence between resolvent resonances and scattering poles, notably in order to analyze the special points $(\frac{n-k}{2})_{k\in\mathbb{N}}$ that they did not deal with. This problem is especially interesting on convex co-compact hyperbolic quotients since these are the scattering poles (not the resonances) which appear in the divisor of Selberg's zeta function associated to the group (cf. Patterson-Perry [14]).

Let $\bar{X} = X \cup \partial\bar{X}$ a $n+1$-dimensional smooth compact manifold with boundary and x a defining function for the boundary, that is a smooth function x on $\bar{X}$ such that

$$x \geq 0, \quad \partial\bar{X} = \{m \in \bar{X}, x(m) = 0\}, \quad dx|_{\partial\bar{X}} \neq 0$$

We say that a smooth metric g on the interior X of $\bar{X}$ is *conformally compact* if $x^2 g$ extends smoothly as a metric to $\bar{X}$. An *asymptotically hyperbolic manifold* is a conformally compact manifold such that for all $y \in \partial\bar{X}$, all sectional curvatures at $m \in X$ converge to -1 as $m \to y$. Notice that convex co-compact hyperbolic quotients are included in this class of manifolds. An asymptotically hyperbolic manifold is necessarily complete and the spectrum of its Laplacian Δ_g acting on functions consists of absolutely continuous spectrum $[\frac{n^2}{4}, \infty)$ and a

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finite set of eigenvalues $\sigma_{pp}(\Delta_g) \subset (0, \frac{n^2}{4})$. The resolvent $(\Delta_g - z)^{-1}$ is a meromorphic family on $\mathbb{C} \setminus [\frac{n^2}{4}, \infty)$ of bounded operators and the new parameter $z = \lambda(n-\lambda)$ with $\Re(\lambda) > \frac{n}{2}$ induces a modified resolvent

$$R(\lambda) := (\Delta_g - \lambda(n-\lambda))^{-1}$$

which is meromorphic on $\{\Re(\lambda) > \frac{n}{2}\}$, its poles being the points λ_e such that $\lambda_e(n-\lambda_e) \in \sigma_{pp}(\Delta_g)$. Mazzeo and Melrose [12] have constructed the finite-meromorphic extension (i.e. with poles whose residue is a finite rank operator) of $R(\lambda)$ on $\mathbb{C} \setminus \frac{1}{2}(n - \mathbb{N})$. We proved in a previous work [6] that this extension is finite-meromorphic on $\mathbb{C}$ if and only if the metric is even in the sense that there exists a boundary defining function x such that the metric can be expressed by

$$g = \frac{dx^2 + h(x^2, y, dy)}{x^2} \tag{1.1}$$

in the collar $[0, \epsilon) \times \partial\bar{X}$ induced by x, with $h(z, y, dy)$ smooth up to $\{z = 0\}$. We will only consider these cases of even metrics to simplify the statements, but our result works as long as the studied singularity is a pole of finite multiplicity for the resolvent.

The poles of the extension $R(\lambda)$ are called *resonances* and the multiplicity of a resonance λ_0 is defined by

$$m(\lambda_0) := \operatorname{rank} \int_{C(\lambda_0,\epsilon)} (n-2\lambda)R(\lambda)d\lambda = \operatorname{rank}\big(\operatorname{Res}_{\lambda_0}((n-2\lambda)R(\lambda))\big)$$

where $C(\lambda_0, \epsilon)$ is a circle around λ_0 with radius $\epsilon > 0$ chosen sufficiently small to avoid other resonances in $D(\lambda_0, \epsilon)$ and Res means the residue. In other words, this is the rank of the residue at $z_0 = \lambda_0(n-\lambda_0)$ of the resolvent as a function of $z = \lambda(n-\lambda)$.

The scattering operator $S(\lambda)$ is the operator on $\partial\bar{X}$ defined as follows: let $\lambda \in \{\Re(\lambda) = \frac{n}{2}\}$ and $\lambda \neq \frac{n}{2}$, for all $f_0 \in C^\infty(\partial\bar{X})$ there exists a unique solution $F(\lambda)$ of the problem

$$(\Delta_g - \lambda(n-\lambda))F(\lambda) = 0, \quad F(\lambda) = x^\lambda f_- + x^{n-\lambda} f_+$$

$$f_-, f_+ \in C^\infty(\bar{X}), \quad f_+|_{\partial\bar{X}} = f_0$$

we then set $S(\lambda)$ the operator $S(\lambda) : f_0 \to f_-|_{\partial\bar{X}}$. In fact we should use half-densities and define $S(\lambda)$ on conormal bundles on $\partial\bar{X}$ to get invariance with respect to x, but this is dropped here. Joshi and Sá Barreto showed [10] that this family of operators extends meromorphically in $\mathbb{C} \setminus \frac{1}{2}(n - \mathbb{N})$ in the sense of pseudo-differential operators on $\partial\bar{X}$ and that $S(\lambda)$ has the principal symbol

$$\sigma_0(S(\lambda)) = c(\lambda)\sigma_0\left(\Lambda^{2\lambda-n}\right), \text{ with } \Lambda := (1+\Delta_{h_0})^{\frac{1}{2}}, \quad c(\lambda) := 2^{n-2\lambda}\frac{\Gamma(\frac{n}{2}-\lambda)}{\Gamma(\lambda-\frac{n}{2})} \tag{1.2}$$

and $h_0 := x^2 g|_{T\partial\bar{X}}$, which leads to the factorization (see [16, 9, 14, 1, 6] for a similar approach)

$$\widetilde{S}(\lambda) := c(n-\lambda)\Lambda^{-\lambda+\frac{n}{2}} S(\lambda)\Lambda^{-\lambda+\frac{n}{2}} = 1 + K(\lambda) \tag{1.3}$$

with $K(\lambda)$ compact finite-meromorphic. It is clear that the poles of $S(\lambda)$ and $\widetilde{S}(\lambda)$ coincide except for the points of $\frac{n}{2} + \mathbb{Z}$. A pole λ_0 of $\widetilde{S}(\lambda)$ is called a *scattering pole* and we define its multiplicity by

$$\nu(\lambda_0) := -\mathrm{Tr}\left(\frac{1}{2\pi i}\int_{C(\lambda_0,\epsilon)} \widetilde{S}'(\lambda)\widetilde{S}^{-1}(\lambda)d\lambda\right) = -\mathrm{Tr}\big(\mathrm{Res}_{\lambda_0}(\widetilde{S}'(\lambda)\widetilde{S}^{-1}(\lambda))\big).$$

Using a method close to that of Guillopé-Zworski [9] and Gohberg-Sigal theory [4], we then obtain the

Theorem 1.1. *Let (X, g) be an asymptotically hyperbolic manifold with g even in the sense of (1.1) and let $\lambda_0 \in \{\Re(\lambda) < \frac{n}{2}\}$ such that*

$$\lambda_0 \notin \{\lambda \in \mathbb{C}; \lambda(n-\lambda) \in \sigma_{pp}(\Delta_g)\} \cap \frac{1}{2}(n - \mathbb{N}).$$

Then λ_0 is a pole of $R(\lambda)$ if and only if it is a pole of $S(\lambda)$ and we have

$$m(\lambda_0) = m(n-\lambda_0) + \nu(\lambda_0) - 1\!\!1_{\frac{n}{2}-\mathbb{N}}(\lambda_0) \dim\ker Res_{n-\lambda_0} S(\lambda) \tag{1.4}$$

where $1\!\!1_{\frac{n}{2}-\mathbb{N}}$ is the characteristic function of $\frac{n}{2} - \mathbb{N}$ and Res means the residue.

Remark 1: the term $m(n-\lambda_0)$ vanishes when $\lambda_0(n-\lambda_0) \notin \sigma_{pp}(\Delta_g)$ and (1.4) can be extended to the line $\{\Re(\lambda) = \frac{n}{2}\}$ by using that $R(\lambda)$ and $\widetilde{S}(\lambda)$ are continuous on this line except possibly at $\frac{n}{2}$, where only $R(\lambda)$ can have a pole; in this case $\nu(\lambda_0) = 0$ and (1.4) is satisfied.

Remark 2: the additional term introduced at $\lambda_0 = \frac{n}{2} - k$ is exactly the dimension of the kernel of the operator p_{2k} defined by Graham-Zworski in [5, Prop. 3.5]. Therefore it only depends on the $2k$ first derivatives of the metric at the boundary. When the manifold is asymptotically Einstein, this is

$$\dim\ker \mathrm{Res}_{\frac{n}{2}+k} S(\lambda) = \dim\ker P_k$$

P_k being the k-th conformally invariant power of the Laplacian on $(\partial\bar{X}, h_0 = x^2 g|_{T\partial\bar{X}})$ (cf. [5] for a definition), which only depends on the conformal class of h_0. If n is even, it is worth noting that $\dim\ker p_n \geq 1$ since p_n always annihilates constants. Moreover, if $(\partial\bar{X}, h_0)$ is conformally flat with (X, g) asymptotically Einstein, the additional term is $\dim\ker P_k = H_0(\partial\bar{X})$, the number of connected components of the boundary.

The recent formula obtained by Patterson-Perry [14] and Bunke-Olbrich [2] for the divisor at $\lambda_0 \in \mathbb{C}$ of Selberg's zeta function on a convex co-compact hyperbolic quotient always makes the 'spectral term' $\nu(\lambda_0)$ appear and an additional 'topological term' (an integer multiple of the Euler characteristic) comes when $\lambda_0 \in -\mathbb{N}_0$. As a matter of fact, the 'spectral term' at $\lambda_0 = \frac{n}{2} - k$ (with $k \in \mathbb{N}$) could be splitted in a 'resonance term' $m(\lambda_0)$ and a 'conformal term' $\dim\ker p_{2k}$

with p_{2k} the residue of $S(\lambda)$ at $\frac{n}{2}+k$. Notice also that for $\lambda_0 \in \frac{n}{2}-\mathbb{N}$, $m(\lambda_0)$ can be 0 though $\nu(\lambda_0)$ is not (this is the case of $\mathbb{H}^{n+1}$ when $n+1$ is odd).

Moreover the Poisson formula obtained by Perry [17] for convex co-compact quotients is used to derive a lower bound of poles of $\widetilde{S}(\lambda)$ (with multiplicity $\nu(\lambda_0)$) in a disc $D(\frac{n}{2}, R) \subset \mathbb{C}$ with radius R. It is clear that the number of these poles is bigger than the number of resonances, in view of Theorem 1.1. In the trivial case of $\mathbb{H}^{n+1}$ with $n+1$ odd, we notably have no resonance though the number of poles of $\widetilde{S}(\lambda)$ in $D(\frac{n}{2}, R)$ is CR^{n+1}. However, in dimension $n+1=2$, the explicit formula of the scattering matrix for a hyperbolic funnel by Guillopé-Zworski [8] or the work of Bunke-Olbrich [3, Prop. 4.3] show that the conformal term cancels, so $\nu(\lambda_0) = m(\lambda_0)$ (modulo the discrete spectrum).

To conclude it would be interesting to understand more deeply the role played by the dimension of the kernels of the conformal Laplacians on such quotients, in particular they also appear in the expression of the zero-trace of the wave operator in this case (see [7]).

2. Background on multiplicities

Let $\mathcal{H}_1$, $\mathcal{H}_2$ some Hilbert spaces. If $M(\lambda)$ is meromorphic on an open set $U \subset \mathbb{C}$ with values in the space $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ of bounded linear operators and if λ_0 is a pole of $M(\lambda)$, there exists a neighborhood V_{λ_0} of λ_0, an integer $p > 0$ and some $(M_i)_{i=1,\dots,p}$ in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that for $\lambda \in V_{\lambda_0} \setminus \{\lambda_0\}$

$$M(\lambda) = \Xi_{\lambda_0}(M(\lambda)) + H(\lambda), \tag{2.1}$$

$$\Xi_{\lambda_0}(M(\lambda)) = \sum_{i=1}^{p} M_i(\lambda-\lambda_0)^{-i}, \quad H(\lambda) \in \mathcal{H}ol(V_{\lambda_0}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)).$$

We will call $\Xi_{\lambda_0}(M(\lambda))$ the polar part of $M(\lambda)$ at λ_0, p the order of the pole λ_0, $M_1 = \mathrm{Res}_{\lambda_0} M(\lambda)$ the residue of $M(\lambda)$ at λ_0, $m_{\lambda_0}(M(\lambda)) := \mathrm{rank} M_1$ the multiplicity of λ_0 and

$$\mathrm{Rank}_{\lambda_0} M(\lambda) := \dim \sum_{i=1}^{p} \mathrm{Im}(M_i)$$

the total polar rank of $M(\lambda)$ at λ_0. Finally, a meromorphic family of operators in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ whose poles have finite total polar rank will be called finite-meromorphic.

Assume now that $\mathcal{H}_1 = \mathcal{H}_2$; taking essentially Gohberg-Sigal notations [4], a root function of $M(\lambda)$ at λ_0 is a function $\varphi(\lambda) \in \mathcal{H}ol(V_{\lambda_0}, \mathcal{H}_1)$ such that $\lim_{\lambda\to\lambda_0} M(\lambda)\varphi(\lambda) = 0$ and $\varphi(\lambda_0) \neq 0$, the vanishing order of $M(\lambda)\varphi(\lambda)$ being called the multiplicity of $\varphi(\lambda)$. The vector $\varphi_0 := \varphi(\lambda_0)$ is called an eigenvector of $M(\lambda)$ at λ_0 and the set of eigenvectors of $M(\lambda)$ at λ_0 form a vector subspace of $\mathcal{H}_1$ denoted $\ker_{\lambda_0} M(\lambda)$. The rank of an eigenvector φ_0 is defined as being the supremum of the multiplicities of the root functions $\varphi(\lambda)$ of $M(\lambda)$ at λ_0 such that $\varphi(\lambda_0) = \varphi_0$. If $\dim \ker_{\lambda_0} M(\lambda) = \alpha < \infty$ and the ranks of all eigenvectors are finite, a canonical system of eigenvectors is a basis $(\varphi_0^{(i)})_{i=1,\dots,\alpha}$ of $\ker_{\lambda_0} M(\lambda)$ such that the ranks of $\varphi_0^{(i)}$ have the following property: the rank of $\varphi_0^{(1)}$ is the maximum of the ranks of all eigenvectors of $M(\lambda)$ at λ_0 and the rank of $\varphi_0^{(i)}$ is the maximum of the ranks of all eigenvectors in a direct complement of $\mathrm{Vect}(\varphi_0^{(1)}, \dots, \varphi_0^{(i-1)})$ in $\ker_{\lambda_0} M(\lambda)$. A canonical system of eigenvectors is not unique but the family of ranks of its eigenvectors does not depend on the choice of the canonical system. We then define the partial null multiplicities $r_i := \mathrm{rank}(\varphi_0^{(i)})$ of $M(\lambda)$ at λ_0 and the null multiplicity

$$N_{\lambda_0}(M(\lambda)) = \sum_{i=1}^{\alpha} r_i$$

of $M(\lambda)$ at λ_0.

Assume that $M(\lambda)$ is a meromorphic family of Fredholm operators in $\mathcal{L}(\mathcal{H}_1)$ and λ_0 a pole of finite total polar rank. If the index of $(M(\lambda) - \Xi_{\lambda_0}(M(\lambda))|_{\lambda=\lambda_0}$ is 0, Gohberg and Sigal [4] show that there exist holomorphically invertible operators $U_1(\lambda)$ and $U_2(\lambda)$ near λ_0, orthogonal projections $(P_l)_{l=0,\dots,m}$ and non-zero integers $(k_l)_{l=1,\dots,m}$ such that

$$M(\lambda) = U_1(\lambda)\left(P_0 + \sum_{l=1}^{m} (\lambda - \lambda_0)^{k_l} P_l\right) U_2(\lambda), \tag{2.2}$$

$$P_i P_j = \delta_{ij} P_j, \quad \mathrm{rank}(P_l) = 1 \text{ for } l = 1, \dots, m, \quad \dim(1 - P_0) < \infty.$$

If moreover $M(\lambda)$ has a meromorphic inverse $M^{-1}(\lambda)$ (i.e. when $P_0 + \sum_{l=1}^{m} P_l = 1$) then λ_0 is at most a pole of finite total polar rank of $M^{-1}(\lambda)$ and

$$M^{-1}(\lambda) = U_2^{-1}(\lambda)\left(P_0 + \sum_{l=1}^{m} (\lambda - \lambda_0)^{-k_l} P_l\right) U_1^{-1}(\lambda). \tag{2.3}$$

It is important to notice that the set of partial null multiplicities remains invariant under multiplication by a holomophically invertible family of operators (cf. [4]). In view of (2.2) and (2.3), it is then easy to see that

$$\dim \ker_{\lambda_0} M(\lambda) = \sharp\{l; k_l > 0\}, \quad \dim \ker_{\lambda_0} M^{-1}(\lambda) = \sharp\{l; k_l < 0\}$$

and that the set of partial null multiplicities of $M(\lambda)$ (resp. $M^{-1}(\lambda)$) at λ_0 is $\{k_l; k_l > 0\}$ (resp. $\{k_l; k_l < 0\}$). We deduce

$$N_{\lambda_0}(M(\lambda)) = \sum_{k_l > 0} k_l, \quad N_{\lambda_0}(M^{-1}(\lambda)) = \sum_{k_l < 0} k_l$$

and from the factorization (2.2) Gohberg-Sigal [4] obtain the generalized logarithmic residue theorem

$$\mathrm{Tr}\left(\mathrm{Res}_{\lambda_0}(M'(\lambda) M^{-1}(\lambda))\right) = N_{\lambda_0}(M(\lambda)) - N_{\lambda_0}(M^{-1}(\lambda)). \tag{2.4}$$

This integer is essentially the order of the zero or the pole of $\det(M(\lambda))$ at λ_0 (when $\det(M(\lambda))$ exists).

To conclude, let $M(\lambda)$ be a meromorphic family of Fredholm operators with index 0 in $\mathcal{L}(\mathcal{H}_1)$ and λ_0 a pole of finite total polar rank. We write $M(\lambda)$ as in (2.2) and if $L(\lambda) := (\lambda-\lambda_0)^{-1}M(\lambda)$, we deduce that $\dim\ker_{\lambda_0} L(\lambda) = \sharp\{l; k_l > 1\}$, the set of partial null multiplicities of $L(\lambda)$ at λ_0 is $\{k_l - 1; k_l > 1\}$ and

(2.5)
$$N_{\lambda_0}(L(\lambda)) = \sum_{k_l>1}(k_l-1) = \sum_{k_l>0}(k_l-1) = N_{\lambda_0}(M(\lambda)) - \dim\ker_{\lambda_0} M(\lambda).$$

This formula will be essential for what follows since the scattering operator $S(\lambda)$ is not finite-meromorphic near $\frac{n}{2}+k$ (with $k\in\mathbb{N}$) whereas $(\lambda-\frac{n}{2}-k)S(\lambda)$ is.

3. Resonances and scattering poles

3.1. Stretched products, half-densities. To begin, let us introduce a few notations and recall some basic facts about stretched products and singular half-densities (the reader can refer to Mazzeo-Melrose [12], Melrose [13] for details). Let $\bar{X}$ a smooth compact manifold with boundary and x a boundary defining function. The manifold $\bar{X}\times\bar{X}$ is a smooth manifold with corners, whose boundary hypersurfaces are diffeomorphic to $\partial\bar{X}\times\bar{X}$ and $\bar{X}\times\partial\bar{X}$, and defined by the functions π_L^*x, π_R^*x (π_L and π_R being the left and right projections from $\bar{X}\times\bar{X}$ onto $\bar{X}$). For notational simplicity, we now write x,x' instead of π_L^*x, π_R^*x and let

$$\delta_{\partial\bar{X}} := \{(m,m)\in\partial\bar{X}\times\partial\bar{X}; m\in\partial\bar{X}\}.$$

The blow-up of $\bar{X}\times\bar{X}$ along the diagonal $\delta_{\partial\bar{X}}$ of $\partial\bar{X}\times\partial\bar{X}$ will be noted $\bar{X}\times_0\bar{X}$ and the blow-down map

$$\beta : \bar{X}\times_0\bar{X} \to \bar{X}\times\bar{X}.$$

This manifold with corners has three boundary hypersurfaces $\mathcal{T},\mathcal{B},\mathcal{F}$ defined by some functions ρ,ρ',R such that $\beta^*(x) = R\rho$, $\beta^*(x') = R\rho'$. Globally, $\delta_{\partial\bar{X}}$ is replaced by a larger manifold, namely by its doubly inward-pointing spherical normal bundle of $\delta_{\partial\bar{X}}$, whose each fiber is a quarter of sphere. From local coordinates (x,y,x',y') on $\bar{X}\times\bar{X}$, this amounts to introducing polar coordinates (R,ρ,ρ',ω,y) around $\delta_{\partial\bar{X}}$:

$$R := (x^2+x'^2+|y-y'|^2)^{\frac{1}{2}}, \quad (\rho,\rho',\omega) := \left(\frac{x}{R},\frac{x'}{R},\frac{y-y'}{R}\right)$$

with $R,\rho,\rho'\in[0,\infty)$. In these polar coordinates the Schwartz kernel of $R(\lambda)$ has a better description.

Using evident identifications induced by the inclusions

$$\delta_{\partial\bar{X}} \subset \partial\bar{X}\times\partial\bar{X} \subset \partial\bar{X}\times\bar{X} \subset \bar{X}\times\bar{X},$$

we denote by $\partial\bar{X} \times_0 \bar{X}$ the blow-up of $\partial\bar{X} \times \bar{X}$ along $\delta_{\partial\bar{X}}$ and $\partial\bar{X} \times_0 \partial\bar{X}$ the blow-up of $\partial\bar{X} \times \partial\bar{X}$ along $\delta_{\partial\bar{X}}$. $\widetilde{\beta}$ and β_∂ are the associated blow-down map

$$\widetilde{\beta} : \partial\bar{X} \times_0 \bar{X} \to \partial\bar{X} \times \bar{X}, \quad \beta_\partial : \partial\bar{X} \times_0 \partial\bar{X} \to \partial\bar{X} \times \partial\bar{X}$$

with $\widetilde{\beta} = \beta|_{\mathcal{T}}$ and $\beta_\partial = \beta|_{\mathcal{B}\cap\mathcal{T}}$. Note that $r := R|_{\mathcal{B}\cap\mathcal{T}}$ is a defining function of the boundary of $\partial\bar{X} \times_0 \partial\bar{X}$ (which is the lift of $\delta_{\partial\bar{X}}$ under β_∂).

Let $\Gamma_0^{\frac{1}{2}}(\bar{X})$ the line bundle of singular half-densities on $\bar{X}$, trivialized by $\nu := |dvol_g|^{\frac{1}{2}}$, and $\Gamma^{\frac{1}{2}}(\partial\bar{X})$ the bundle of half densities on $\partial\bar{X}$, trivialized by $\nu_0 := |dvol_{h_0}|^{\frac{1}{2}}$ (where $h_0 = x^2 g|_{T\partial\bar{X}}$). From these bundles, one can construct the bundles $\Gamma_0^{\frac{1}{2}}(\bar{X} \times \bar{X})$, $\Gamma_0^{\frac{1}{2}}(\partial\bar{X} \times \bar{X})$ and $\Gamma^{\frac{1}{2}}(\partial\bar{X} \times \partial\bar{X})$ by tensor products and the bundles $\Gamma_0^{\frac{1}{2}}(\bar{X} \times_0 \bar{X})$, $\Gamma_0^{\frac{1}{2}}(\partial\bar{X} \times_0 \bar{X})$ and $\Gamma^{\frac{1}{2}}(\partial\bar{X} \times_0 \partial\bar{X})$ by lifting under β, $\widetilde{\beta}$ and β_∂ the three previous bundles. If M denotes $\bar{X}$, $\bar{X} \times \bar{X}$ or $\partial\bar{X} \times \bar{X}$, we write $\dot{C}^\infty(M, \Gamma_0^{\frac{1}{2}})$ the space of smooths sections of $\Gamma_0^{\frac{1}{2}}(M)$ that vanish to all order at all the boundary hypersurfaces of M, and $C^{-\infty}(M, \Gamma_0^{\frac{1}{2}})$ is its topological dual. The Hilbert space $L^2(\bar{X}, \Gamma_0^{\frac{1}{2}})$ and $L^2(\partial\bar{X}, \Gamma^{\frac{1}{2}})$ are isomorphic to $L^2(X, dvol_g)$ and $L^2(\partial\bar{X}, dvol_{h_0})$, they will be denoted $L^2(X)$, $L^2(\partial\bar{X})$.

For $\alpha \in \mathbb{R}$, let $x^\alpha L^2(X) := \{f \in C^{-\infty}(\bar{X}, \Gamma_0^{\frac{1}{2}}); x^{-\alpha} f \in L^2(X)\}$ and we let $\langle .,. \rangle$ be the symmetric non-degenerate products

$$\langle u, v \rangle := \int_X uv \text{ on } L^2(X), \quad \langle u, v \rangle := \int_{\partial\bar{X}} uv \text{ on } L^2(\partial\bar{X}).$$

We can check by using the first pairing that the dual space of $x^\alpha L^2(X)$ is isomorphic to $x^{-\alpha} L^2(X)$. We shall also use the following tensorial notation for $E = x^\alpha L^2(X)$ (resp. $E = L^2(\partial\bar{X})$), $\psi, \phi \in E'$

$$\phi \otimes \psi : \left\{ \begin{array}{ccc} E & \to & E' \\ f & \to & \phi\langle \psi, f \rangle \end{array} \right. .$$

3.2. Resolvent. From [12, 6], we know that on an asymptotically hyperbolic manifold (X, g) with g even, the modified resolvent

$$R(\lambda) := (\Delta_g - \lambda(n - \lambda))^{-1}$$

extends for all $N > 0$ to a finite-meromorphic family of operators in $\{\Re(\lambda) > \frac{n}{2} - N\}$ with values in $\mathcal{L}(x^N L^2(X), x^{-N} L^2(X))$, whose poles, the resonances, form a discrete set $\mathcal{R}$ in $\mathbb{C}$. Moreover $R(\lambda)$ is a continuous operator from $\dot{C}^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}})$ to $C^{-\infty}(\bar{X}, \Gamma_0^{\frac{1}{2}})$, its associated Schwartz kernel being

$$r(\lambda) = r_0(\lambda) + r_1(\lambda) + r_2(\lambda) \in C^{-\infty}(\bar{X} \times \bar{X}, \Gamma_0^{\frac{1}{2}})$$

with (see [12] or [1, Th. 2.1]):

$$\beta^*(r_0(\lambda)) \in I^{-2}(\bar{X} \times_0 \bar{X}, \Gamma_0^{\frac{1}{2}}),$$

$$\beta^*(r_1(\lambda)) \in \rho^\lambda {\rho'}^\lambda C^\infty(\bar{X} \times_0 \bar{X}, \Gamma_0^{\frac{1}{2}}), \quad r_2(\lambda) \in x^\lambda {x'}^\lambda C^\infty(\bar{X} \times \bar{X}, \Gamma_0^{\frac{1}{2}}), \tag{3.1}$$

where $I^{-2}(\bar{X} \times_0 \bar{X}, \Gamma_0^{\frac{1}{2}})$ denotes the set of conormal distributions of order -2 on $\bar{X} \times_0 \bar{X}$ associated to the closure of the lifted interior diagonal

$$\overline{\beta^{-1}(\{(m,m) \in \bar{X} \times \bar{X}; m \in X\})}$$

and vanishing to infinite order at $\mathcal{B} \cup \mathcal{T}$ (note that the lifted interior diagonal only intersects the topological boundary of $\bar{X} \times_0 \bar{X}$ at $\mathcal{F}$, and it does transversally). Moreover, $(\rho\rho')^{-\lambda}\beta^*(r_1(\lambda))$ and $(xx')^{-\lambda} r_2(\lambda)$ are meromorphic in $\lambda \in \mathbb{C}$ and $r_0(\lambda)$ is the kernel of a holomorphic family of operators

$$R_0(\lambda) \in \mathcal{H}ol(\mathbb{C}, \mathcal{L}(x^\alpha L^2(X), x^{-\alpha} L^2(X))), \quad \forall \alpha \geq 0.$$

Note also that Patterson-Perry arguments [14, Lem.4.9] prove that $R(\lambda)$ does not have poles on the line $\{\Re(\lambda) = \frac{n}{2}\}$, except maybe $\lambda = \frac{n}{2}$. The set of poles of $R(\lambda)$ in the half plane $\{\Re(\lambda) > \frac{n}{2}\}$ is $\{\lambda_e; \Re(\lambda_e) > \frac{n}{2}, \lambda_e(n-\lambda_e) \in \sigma_{pp}(\Delta_g)\}$, they are first-order poles and their residue is

$$\mathrm{Res}_{\lambda_e} R(\lambda) = (2\lambda_e - n)^{-1} \sum_{k=1}^{p} \phi_k \otimes \phi_k, \quad \phi_k \in x^{\lambda_e} C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}), \tag{3.2}$$

where $(\phi_k)_{k=1,\dots,p}$ are the normalized eigenfunctions of Δ_g for the eigenvalue $\lambda_e(n-\lambda_e)$. One can see by a Taylor expansion at $x = 0$ of the eigenvector equation that if $x^{-\lambda_e + \frac{n}{2}} \phi_k|_{\partial\bar{X}} = 0$ then $\phi_k \in \dot{C}^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}})$, which is excluded according to Mazzeo's results [11].

To simplify the notations, we shall set $z(\lambda) := \lambda(n-\lambda)$ the holomorphically invertible function from $\Re(\lambda) < \frac{n}{2}$ to $\mathbb{C} \setminus [\frac{n^2}{4}, \infty)$.

For the poles of $R(\lambda)$ in $\{\Re(\lambda) < \frac{n}{2}\}$, we use Lemma 2.4 and 2.11 of [9] to show the

Lemma 3.1. *Let $\lambda_0 \in \mathcal{R}$ and N such that $\frac{n}{2} > \Re(\lambda_0) > \frac{n}{2} - N$, then in a neighbourhood V_{λ_0} of λ_0 we have the decomposition*

$$R(\lambda) = {}^t\Phi F_1(\lambda) \left(\sum_{j=1}^{m} (z(\lambda) - z(\lambda_0))^{k_j} P_j \right) F_2(\lambda) \Phi + H(\lambda), \tag{3.3}$$

with $m \in \mathbb{N}$, $k_1, \dots k_m \in -\mathbb{N}$,

$$H(\lambda) \in \mathcal{H}ol(V_{\lambda_0}, \mathcal{L}(x^N L^2(X), x^{-N} L^2(X))), \quad F_i(\lambda) \in \mathcal{H}ol(V_{\lambda_0}, \mathcal{L}(\mathbb{C}^q)),$$

where $q = -\sum_{j=1}^m k_j = m_{\lambda_0}(z'(\lambda) R(\lambda))$ is the multiplicity of the resonance λ_0, $(P_j)_{j=1,\dots,m}$ are some orthogonal projections on $\mathbb{C}^q$ such that $P_i P_j = \delta_{ij} P_j$ and rank$(P_j) = 1$, Φ is defined by

$$\Phi : \begin{cases} x^N L^2(X) & \to & \mathbb{C}^q \\ f & \to & (\langle \psi_l, f \rangle)_{l=1,\dots,q} \end{cases},$$

$(\psi_l)_{l=1,\dots,q}$ *being a basis of* $Im(A)$ *with* $A := Res_{\lambda_0}(z'(\lambda)R(\lambda))$. *Moreover we have*

$$Im(A) \subset \sum_{j=0}^{p-1} x^{\lambda_0}\log^j(x)C^\infty(\bar{X},\Gamma_0^{\frac{1}{2}}) \tag{3.4}$$

with p *the order of the pole* λ_0 *of* $R(\lambda)$.

Proof. it suffices to use Lemmas 2.4 and 2.11 of [9] but we factorize the resolvent and not the scattering operator. The arguments used in these lemmas are essentially that the polar part of $R(\lambda)$ be expressed by

$$\Xi_{\lambda_0}(R(\lambda)) = \Xi_{\lambda_0}\left(\sum_{i=1}^{p}\frac{(\Delta_g - z(\lambda_0))^{i-1}A}{(z(\lambda)-z(\lambda_0))^i}\right)$$

and the factorization into its Jordan form of the nilpotent matrix of $\Delta_g - z(\lambda_0)$ acting on $\mathrm{Im}(A)$. Observe that the elliptic regularity implies that the elements of $\mathrm{Im}(A)$ are smooth in X.

To study the structure of the Schwartz kernel a_j of A_j, we first consider the following operator

$$\widetilde{R}(\lambda) := x^{-\lambda+\frac{n}{2}}R(\lambda)x^{-\lambda+\frac{n}{2}} \tag{3.5}$$

in a disc $D(\lambda_0,\epsilon)$ around λ_0 with radius ϵ. If ϵ is taken sufficiently small, $\widetilde{R}(\lambda)$ is meromorphic in this disc with values in $\mathcal{L}(x^{2\epsilon}L^2(X), x^{-2\epsilon}L^2(X))$, λ_0 is the only pole and its order is p. The Schwartz kernel $(xx')^{-\lambda+\frac{n}{2}}r(\lambda)$ of $\widetilde{R}(\lambda)$ is meromorphic and its polar part at λ_0 is the same as the one of $(xx')^{-\lambda+\frac{n}{2}}(r_1(\lambda)+r_2(\lambda))$ since $r_0(\lambda)$ is holomorphic in $\mathbb{C}$. We then can easily check [6, Prop. 3.3] that we have in V_{λ_0}

$$\Xi_{\lambda_0}(\widetilde{R}(\lambda)) = \sum_{j=-p}^{-1} B_j(\lambda-\lambda_0)^j \tag{3.6}$$

where $B_j \in \mathcal{L}(x^{2\epsilon}L^2(X), x^{-2\epsilon}L^2(X))$ has a Schwartz kernel of the form

(3.7)

$$b_j(x,y,x',y') = \sum_{i=1}^{r_j}\psi_{ji}(x,y)\varphi_{ji}(x',y')\left|\frac{dxdydx'dy'}{x^{n+1}x'^{n+1}}\right|^{\frac{1}{2}}, \quad \psi_{ij},\varphi_{ij} \in x^{\frac{n}{2}}C^\infty(\bar{X}).$$

Observe now that $x^{\lambda-\frac{n}{2}}$ has the following Taylor expansion at λ_0

$$x^{\lambda-\frac{n}{2}} = x^{\lambda_0-\frac{n}{2}}\sum_{j=0}^{p-1}\log^j(x)\frac{(\lambda-\lambda_0)^j}{j!} + O((\lambda-\lambda_0)^p)$$

in the sense of operators of $\mathcal{L}(x^N L^2(X), x^{2\epsilon}L^2(X))$ and $\mathcal{L}(x^{-2\epsilon}L^2(X), x^{-N}L^2(X))$. We deduce that $z'(\lambda)R(\lambda)$ has a residue A satisfying

$$\mathrm{Im}(A) \subset \sum_{j=0}^{p-1} x^{\lambda_0}\log^j(x)C^\infty(\bar{X},\Gamma_0^{\frac{1}{2}})$$

and we are done. □

3.3. Scattering matrix. Joshi and Sá Barreto [10] have shown that the scattering matrix $S(\lambda)$ (defined in the introduction) has the following Schwartz kernel

$$s(\lambda) := (2\lambda - n)\,(\beta_\partial)_*\left(\beta^*\left(x^{-\lambda+\frac{n}{2}}x'^{-\lambda+\frac{n}{2}}r(\lambda)\right)|_{\mathcal{T}\cap\mathcal{B}}\right) \tag{3.8}$$

Following (3.1) and (3.8) we have in $\mathbb{C}\setminus(\mathcal{R}\cup(\frac{n}{2}+\mathbb{N}))$

$$s(\lambda) = (\beta_\partial)_*\left(r^{-2\lambda}k_1(\lambda)\right) + k_2(\lambda), \tag{3.9}$$

$$k_1(\lambda) \in C^\infty(\partial\bar{X}\times_0\partial\bar{X},\Gamma^{\frac{1}{2}}),\quad k_2(\lambda)\in C^\infty(\partial\bar{X}\times\partial\bar{X},\Gamma^{\frac{1}{2}})$$

where $k_1(\lambda)$ and $k_2(\lambda)$ are meromorphic in $\lambda\in\mathbb{C}$. Outside its poles, $s(\lambda)$ is a conormal distribution of order -2λ associated to $\delta_{\partial\bar{X}}$ and $S(\lambda)$ is a pseudo-differential operator of order $2\lambda-n$ on $\partial\bar{X}$. In the sense of Shubin [18, Def. 11.2], $S(\lambda)$ is a holomorphic family in $\left\{\Re(\lambda)<\frac{n}{2}\right\}\setminus\mathcal{R}$ of zeroth order pseudo-differential operators. We then deduce that $S(\lambda)$ is holomorphic in the same open set, with values in $\mathcal{L}(L^2(\partial\bar{X}))$. Recall the functional equation satisfied by $S(\lambda)$ (cf. [5])

$$S(\lambda)^{-1} = S(n-\lambda) = S(\lambda)^*,\quad \Re(\lambda)=\frac{n}{2},\quad \lambda\neq\frac{n}{2} \tag{3.10}$$

which also proves that $S(\lambda)$ is regular on the line $\{\Re(\lambda)=\frac{n}{2}\}$. Furthermore, (3.10) holds also for $\widetilde{S}(\lambda)$ and by analytic extension we have on $\mathbb{C}\setminus\mathcal{R}$

$$\widetilde{S}^{-1}(\lambda) = \widetilde{S}(n-\lambda).$$

The principal symbol of $S(\lambda)$ is given in (1.2) and the renormalization $\widetilde{S}(\lambda)$ of $S(\lambda)$ defined in (1.3) is Fredholm with index 0, consequently we are in the framework of Section 2.

Using Lemmas 3.1 and (3.9), we then obtain the

Lemma 3.2. *Let $\lambda_0\in\{\Re(\lambda)<\frac{n}{2}\}$ a pole of $S(\lambda)$. Then $\lambda_0\in\mathcal{R}$ and, following the notations of Lemma 3.1, we have near λ_0*

$$S(\lambda) = (2\lambda-n)\,{}^t\Phi^\sharp(\lambda)F_1(\lambda)\left(\sum_{j=1}^{m}(z(\lambda)-z(\lambda_0))^{k_j}P_j\right)F_2(\lambda)\Phi^\sharp(\lambda)+H^\sharp(\lambda) \tag{3.11}$$

with $H^\sharp(\lambda)\in\mathcal{H}ol(V_{\lambda_0},\mathcal{L}(L^2(\partial\bar{X})))$ and $\Phi^\sharp(\lambda)\in\mathcal{H}ol(V_{\lambda_0},\mathcal{L}(L^2(\partial\bar{X}),\mathbb{C}^q))$.

Proof. the fact that $\lambda_0\in\mathcal{R}$ is straightforward since if $r(\lambda)$ was holomorphic one would have $s(\lambda)$ holomorphic in view of (3.8). Now, $\widetilde{R}(\lambda)$ being defined in (3.5), we saw in Lemma 3.1 that the polar part of $\widetilde{R}(\lambda)$ at λ_0 has a Schwartz kernel $\Xi_{\lambda_0}(\widetilde{r}(\lambda))$ satisfying

$$\Xi_{\lambda_0}(\widetilde{r}(\lambda))\in(xx')^{\frac{n}{2}}C^\infty(\bar{X}\times\bar{X},\Gamma_0^{\frac{1}{2}}). \tag{3.12}$$

Let

$$\Phi(\lambda) := \sum_{i=0}^{p-1} \frac{(\lambda-\lambda_0)^i}{i!} \frac{d^i}{d\lambda^i}(\Phi x^{-\lambda+\frac{n}{2}})|_{\lambda=\lambda_0}$$

where, as before, p is the order of the pole of $R(\lambda)$ at λ_0. Then $\Phi(\lambda)$: $x^{2\epsilon}L^2(X) \to \mathbb{C}^q$ is given by

$$\Phi(\lambda)f = \left(\sum_{i=0}^{p-1} \frac{(\lambda_0-\lambda)^i}{i!} \langle \log^i(x) x^{-\lambda_0+\frac{n}{2}}\psi_l, f\rangle\right)_{l=1,\dots,q}.$$

Lemma 3.1 implies that

$$\Xi_{\lambda_0}(\widetilde{R}(\lambda)) = \Xi_{\lambda_0}\left({}^t\Phi(\lambda)F_1(\lambda)\Big(\sum_{j=1}^{m}(z(\lambda)-z(\lambda_0))^{k_j}P_j\Big)F_2(\lambda)\Phi(\lambda)\right). \tag{3.13}$$

Let $C := \sum_{j=-p}^{-1} \mathrm{Im}(B_j)$ with B_j the operators defined in (3.6) and let Π_C be the orthogonal projection of $x^{-2\epsilon}L^2(X)$ onto C. We multiply (3.13) on the left by Π_C and on the right by ${}^t\Pi_C$, and using that $\Xi_{\lambda_0}(\widetilde{R}(\lambda))$ is symmetric (since ${}^tR(\lambda) = R(\lambda)$) we deduce that (3.13) remains true if $\Phi(\lambda)$ is replaced by

$$\begin{cases} x^{2\epsilon}L^2(X) & \to & \mathbb{C}^q \\ f & \to & \left(\sum_{j=0}^{p-1} \frac{(\lambda_0-\lambda)^j}{j!}\langle \Pi_C(\log^j(x)x^{-\lambda_0+\frac{n}{2}}\psi_l), f\rangle\right)_{l=1,\dots,q} \end{cases}$$

so that the logarithmic terms disappear. Finally, we can use the representation of $S(\lambda)$ by its Schwartz kernel (3.9) and we obtain

$$\Xi_{\lambda_0}(S(\lambda)) = \Xi_{\lambda_0}\left((2\lambda-n){}^t\Phi^\sharp(\lambda)F_1(\lambda)\Big(\sum_{j=1}^{m}(z(\lambda)-z(\lambda_0))^{k_j}P_j\Big)F_2(\lambda)\Phi^\sharp(\lambda)\right),$$

with

$$\Phi^\sharp(\lambda) : \begin{cases} L^2(\partial\bar{X}) & \to & \mathbb{C}^q \\ f & \to & \left(\sum_{j=0}^{p-1} \frac{(\lambda_0-\lambda)^j}{j!}\langle \Pi_C(\log^j(x)x^{-\lambda_0+\frac{n}{2}}\psi_l)|_{\partial\bar{X}}, f\rangle\right)_{l=1,\dots,q} \end{cases},$$

the proof is achieved. □

From this lemma, we deduce:

Corollary 3.3. *If $\lambda_0 \in \{\Re(\lambda) < \frac{n}{2}\}$ is a pole of $S(\lambda)$, it is a pole of $R(\lambda)$ such that*

$$m_{\lambda_0}(z'(\lambda)R(\lambda)) \geq N_{\lambda_0}\Big(c(n-\lambda)\widetilde{S}(n-\lambda)\Big).$$

Proof. first, (3.11) implies that

$$c(\lambda)\widetilde{S}(\lambda) = F_3(\lambda)\Big(\sum_{j=1}^{m}(z(\lambda)-z(\lambda_0))^{k_j}P_j\Big)F_4(\lambda) + \widetilde{H}^\sharp(\lambda),$$

$$F_3(\lambda) := (2\lambda-n)\Lambda^{-\lambda+\frac{n}{2}}\,{}^t\Phi^\sharp(\lambda)F_1(\lambda), \quad F_4(\lambda) := F_2(\lambda)\Phi^\sharp(\lambda)\Lambda^{-\lambda+\frac{n}{2}},$$

$$\widetilde{H}^\sharp(\lambda) := (2\lambda - n)\Lambda^{-\lambda+\frac{n}{2}} H^\sharp(\lambda)\Lambda^{-\lambda+\frac{n}{2}}.$$

Note that we can take $k_1 \leq \cdots \leq k_m < 0$ and set $(\varphi_0^{(j)})_{j=1,\ldots,M}$ a canonical system of eigenvectors of $c(n-\lambda)\widetilde{S}(n-\lambda)$ at λ_0 with $r_1 \geq \cdots \geq r_M$ the associated partial null multiplicities (this canonical system exists and is deduced from the one of $\widetilde{S}(n-\lambda)$). Let us show that $M \leq m$ and, by induction, that $r_j \leq -k_j$ for all $j = 1, \ldots, M$.

If $\varphi^{(j)}(\lambda)$ is a root function of $c(n-\lambda)\widetilde{S}(n-\lambda)$ at λ_0 corresponding to $\varphi_0^{(j)}$, there exists a holomorphic function $\phi^{(j)}(\lambda)$ such that

$$c(n-\lambda)\widetilde{S}(n-\lambda)\varphi^{(j)}(\lambda) = (z(\lambda) - z(\lambda_0))^{r_j}\phi^{(j)}(\lambda)$$

with $\phi^{(j)}(\lambda_0) \neq 0$, hence when λ approaches λ_0 in the following identity

$$\varphi^{(j)}(\lambda) = \sum_{l=1}^{m} (z(\lambda) - z(\lambda_0))^{r_j+k_l} F_3(\lambda) P_l F_4(\lambda)\phi^{(j)}(\lambda) + (z(\lambda) - z(\lambda_0))^{r_j}\widetilde{H}^\sharp(\lambda)\phi^{(j)}(\lambda),$$

we find that $r_1 \leq -k_1$ and $\varphi_0^{(j)}$ is in the vector space

$$E_j := \sum_{l, r_j \leq -k_l} \mathrm{Im}(F_3(\lambda_0) P_l F_4(\lambda_0)).$$

Moreover, the order on $(r_j)_{j=1,\ldots,M}$ implies that $E_j \subset E_M$ for $j = 1, \ldots, M$ but $\dim E_M \leq m$ since $\mathrm{rank}(P_l) = 1$, thus we necessarily have $M \leq m$, $(\varphi_0^{(j)})_j$ being independent by assumption. Now let $j \leq M$ and suppose that $r_i \leq -k_i$ for all $i \leq j$. We first note that $E_j \subset E_{j+1}$ since $r_{j+1} \leq r_j$. If $r_{j+1} > -k_{j+1}$, we would have $\dim E_{j+1} \leq j$ but E_{j+1} contains the linearly independent vectors $\varphi_0^{(1)}, \ldots, \varphi_0^{(j+1)}$, so a contradiction. One concludes that $r_{j+1} \leq -k_{j+1}$ and

$$N_{\lambda_0}\Big(c(n-\lambda)\widetilde{S}(n-\lambda)\Big) = \sum_{j=1}^{M} r_j \leq -\sum_{l=1}^{m} k_l = q = m_{\lambda_0}(z'(\lambda)R(\lambda)),$$

the corollary is proved. □

Lemma 3.4. *Let $\lambda_0 \in \{\Re(\lambda) < \frac{n}{2}\}$ be a pole of $R(\lambda)$ of finite multiplicity. If $\lambda_0(n-\lambda_0) \notin \sigma_{pp}(\Delta_g)$ or $\lambda_0 \notin \frac{1}{2}(n-\mathbb{N})$, then λ_0 is a pole of $S(\lambda)$ such that*

$$m_{\lambda_0}(z'(\lambda)R(\lambda)) \leq N_{\lambda_0}\Big(c(n-\lambda)\widetilde{S}(n-\lambda)\Big).$$

Proof. we first suppose that λ_0 is not a pole of $c(\lambda)$ (i.e. $\lambda_0 \notin \frac{n}{2} - \mathbb{N}$). From Gohberg-Sigal theory, one can factorize $\widetilde{S}(\lambda)$ near λ_0 as in (2.2)

$$c(\lambda)\widetilde{S}(\lambda) = U_1(\lambda)\left(P_0 + \sum_{l=1}^{m} (\lambda - \lambda_0)^{k_l} P_l\right) U_2(\lambda) \tag{3.14}$$

with $U_1(\lambda)$, $U_2(\lambda)$ some holomorphically invertible operators near λ_0 and

$$P_iP_j = \delta_{ij}P_j, \quad \mathrm{rank}(P_l) = 1 \text{ for } l = 1, \dots, m, \quad 1 = \sum_{j=0}^{m} P_j, \quad k_l \in \mathbb{Z}^*.$$

Take the Green equation between the resolvent and the scattering operator (see for instance [15, Th. 5.3])

$$R(\lambda) - R(n-\lambda) = (2\lambda - n){}^tE(n-\lambda)\Lambda^{\lambda-\frac{n}{2}}c(\lambda)\widetilde{S}(\lambda)\Lambda^{\lambda-\frac{n}{2}}E(n-\lambda) \tag{3.15}$$

on $\mathcal{L}(x^NL^2(X), x^{-N}L^2(X))$ with $\frac{n}{2} - N < |\Re(\lambda)| < \frac{n}{2}$ and $E(\lambda)$ the transpose of the Eisenstein operator, its Schwartz kernel being

$$e(\lambda) := \widetilde{\beta}_* \left(\beta^*(x^{-\lambda+\frac{n}{2}}r(\lambda))|_{\mathcal{T}}\right).$$

We can suppose that $k_1 \leq \cdots \leq k_m$ and set $p := \max(0, -k_1)$. We consider the following Laurent expansions at λ_0

(3.16)

$$\begin{aligned} (n-2\lambda)R(n-\lambda) &= \textstyle\sum_{i=-1}^{p} R_i(\lambda-\lambda_0)^i + O((\lambda-\lambda_0)^{p+1}),\\ (2\lambda-n)U_2(\lambda)\Lambda^{\lambda-\frac{n}{2}}E(n-\lambda) &= \textstyle\sum_{i=-1}^{p} E_i^{(2)}(\lambda-\lambda_0)^i + O((\lambda-\lambda_0)^{p+1}),\\ (n-2\lambda){}^tE(n-\lambda)\Lambda^{\lambda-\frac{n}{2}}U_1(\lambda) &= \textstyle\sum_{i=-1}^{p} E_i^{(1)}(\lambda-\lambda_0)^i + O((\lambda-\lambda_0)^{p+1}),\end{aligned}$$

where R_{-1} and $E_{-1}^{(j)}$ are not 0 if and only if $\lambda_0(n-\lambda_0) \in \sigma_{pp}(\Delta_g)$, and in this case

$$\begin{aligned} R_{-1} &= -\textstyle\sum_{i=1}^{k} \phi_i \otimes \phi_i,\\ E_{-1}^{(2)} &= \textstyle\sum_{i=1}^{k} U_2(\lambda_0)\Lambda^{\lambda_0-\frac{n}{2}}(x^{\lambda_0-\frac{n}{2}}\phi_i)|_{\partial\bar{X}} \otimes \phi_i,\\ E_{-1}^{(1)} &= -\textstyle\sum_{i=1}^{k} \phi_i \otimes {}^tU_1(\lambda_0)\Lambda^{\lambda_0-\frac{n}{2}}(x^{\lambda_0-\frac{n}{2}}\phi_i)|_{\partial\bar{X}},\end{aligned} \tag{3.17}$$

with $\phi_i \in x^{n-\lambda_0}C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}})$ the normalized eigenfunctions of Δ_g for the eigenvalue $\lambda_0(n-\lambda_0)$. From (3.14), (3.15) and (3.16) we obtain

(3.18)

$$A := \mathrm{Res}_{\lambda_0}((n-2\lambda)R(\lambda)) = R_{-1} + \sum_{\substack{j+i+k_l=-1\\ k_l\geq 0}} E_i^{(1)}P_lE_j^{(2)} + \sum_{\substack{j+i+k_l=-1\\ k_l<0}} E_i^{(1)}P_lE_j^{(2)}$$

where by convention $k_l = 0 \iff l = 0$. We set $V := \mathrm{Im}(A_1) + \mathrm{Im}(A_2)$ with

$$A_1 := R_{-1} + E_{-1}^{(1)}P_0E_0^{(2)} + E_{-1}^{(1)}\left(\sum_{k_l=1} P_l\right)E_{-1}^{(2)},$$

$$A_2 := E_0^{(1)}P_0E_{-1}^{(2)} + \sum_{\substack{j+i+k_l=-1\\ k_l<0}} E_i^{(1)}P_lE_j^{(2)}.$$

Remark from (3.17) that

$$\mathrm{Im}(A_1) \subset x^{n-\lambda_0}C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}), \quad (\Delta_g - \lambda_0(n-\lambda_0))A_1 = 0$$

and in view of Lemma 3.1 we know that there exists $p \in \mathbb{N}$ such that

$$\mathrm{Im}(A) \subset \sum_{j=0}^{p-1} x^{\lambda_0} \log^j(x) C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}), \quad (\Delta_g - \lambda_0(n-\lambda_0))^p A = 0$$

thus we can argue that

$$\forall u \in V, \quad (\Delta_g - \lambda_0(n-\lambda_0))^p u = 0.$$

Note that if $\lambda_0 \notin \frac{1}{2}(n - \mathbb{N})$, we clearly have

$$x^{n-\lambda_0} C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}) \cap \sum_{j=0}^{p-1} x^{\lambda_0} \log^j(x) C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}) \subset \dot{C}^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}),$$

therefore, if V_1, V_2 are defined by

$$V_1 = V \cap x^{n-\lambda_0} C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}), \quad V_2 = V \cap \sum_{j=0}^{p-1} x^{\lambda_0} \log^j(x) C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}),$$

we deduce from the unique continuation principle proved by Mazzeo [11] that

$$V_1 \cap V_2 \subset \dot{C}^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}}) \cap \ker_{L^2}(\Delta_g - \lambda_0(n-\lambda_0))^p = 0.$$

Hence, we can split $V = V_1 \oplus V_2 \oplus V_3$ with V_3 a direct complement of $V_1 \oplus V_2$ in V. Let Π_{V_2} be the projection of V onto V_2 parallel to $V_1 \oplus V_3$, Π_V the orthogonal projection of $x^{-N}L^2(X)$ onto V and ι_V the inclusion of V into $x^{-N}L^2(X)$. We multiply (3.18) on the left by $\Pi'_{V_2} := \iota_V \Pi_{V_2} \Pi_V$ and on the right by ${}^t\Pi'_{V_2}$ to obtain

$$A = \sum_{\substack{j+i+k_l=-1 \\ k_l<0}} \Pi'_{V_2} E_i^{(1)} P_l E_j^{(2)} {}^t\Pi'_{V_2}$$

by construction of V_2 and using the symmetry ${}^tA = A$ (since ${}^tR(\lambda) = R(\lambda)$). Now remark that

$$\sum_{\substack{j+i+k_l=-1 \\ k_l<0}} \Pi'_{V_2} E_i^{(1)} P_l E_j^{(2)} {}^t\Pi'_{V_2} = \sum_{k_l<0} \sum_{i=0}^{-k_l-1} \Pi'_{V_2} E_i^{(1)} P_l E_{-k_l-1-i}^{(2)} {}^t\Pi'_{V_2}$$

and the rank of this operator is bounded by $-\sum_{k_l<0} k_l = N_{\lambda_0}(c(n-\lambda)\widetilde{S}(n-\lambda))$ since $\mathrm{rank}(P_l) = 1$. The lemma is proved when $\lambda_0 \notin \frac{n}{2} - \mathbb{N}$.

On the other hand if $\lambda_0 \in \frac{n}{2} - \mathbb{N}$ and $\lambda_0(n-\lambda_0) \notin \sigma_{pp}(\Delta_g)$, we have $R_{-1} = 0$, $E_{-1}^{(1)} = 0$ and $E_{-1}^{(2)} = 0$ in (3.16). Therefore, the same proof works if we replace (3.14) and (3.18) by

$$c(\lambda)\widetilde{S}(\lambda) = U_1(\lambda) \left((\lambda - \lambda_0) P_0 + \sum_{l=1}^{m} (\lambda - \lambda_0)^{k_l+1} P_l \right) U_2(\lambda),$$

$$\mathrm{Res}_{\lambda_0}((n-2\lambda)R(\lambda)) = \sum_{\substack{j+i+k_l=-2\\ k_l<-1}} E_i^{(1)} P_l E_j^{(2)}$$

the first one being obtained from Gohberg-Sigal factorization (2.2) of $\widetilde{S}(\lambda)$ at λ_0. Now observe that the rank of

$$\sum_{\substack{j+i+k_l=-2\\ k_l<-1}} \Pi'_{V_2} E_i^{(1)} P_l E_j^{(2)\,t}\Pi'_{V_2} = \sum_{k_l<-1} \sum_{i=0}^{-k_l-2} \Pi'_{V_2} E_i^{(1)} P_l E_{-k_l-2-i}^{(2)\,t}\Pi'_{V_2}$$

is bounded by

$$-\sum_{k_l<-1}(k_l+1) = -\sum_{k_l<0}(k_l+1) = N_{\lambda_0}(\widetilde{S}(n-\lambda)) - \dim\ker_{\lambda_0}\widetilde{S}(n-\lambda).$$

But using (2.5) with $M(\lambda) := \widetilde{S}(n-\lambda)$ and the fact that $c(n-\lambda)$ has a first-order pole at $\lambda_0 \in \frac{n}{2} - \mathbb{N}$ we see that

$$N_{\lambda_0}\Big(c(n-\lambda)\widetilde{S}(n-\lambda)\Big) = N_{\lambda_0}(\widetilde{S}(n-\lambda)) - \dim\ker_{\lambda_0}\widetilde{S}(n-\lambda) \tag{3.19}$$

and the proof is complete. □

Proof of Theorem 1.1. we combine the results of Corollary 3.3 and Lemma 3.4 to deduce that

$$m(\lambda_0) = m_{\lambda_0}(z'(\lambda)R(\lambda)) = N_{\lambda_0}\Big(c(n-\lambda)\widetilde{S}(n-\lambda)\Big) \tag{3.20}$$

if $\Re(\lambda_0) < \frac{n}{2}$ and $\lambda_0 \notin \{\lambda\in\mathbb{C}; \lambda(n-\lambda)\in\sigma_{pp}(\Delta_g)\} \cap \frac{1}{2}(n-\mathbb{N})$. Using (2.4) with $M(\lambda) := \widetilde{S}(\lambda)$ we obtain

(3.21)

$$N_{\lambda_0}(\widetilde{S}(n-\lambda)) = N_{\lambda_0}(\widetilde{S}(\lambda)) - \mathrm{Tr}(\mathrm{Res}_{\lambda_0}(\widetilde{S}'(\lambda)\widetilde{S}^{-1}(\lambda))) = N_{\lambda_0}(\widetilde{S}(\lambda)) + \nu(\lambda_0).$$

If $\lambda_0 \notin \frac{n}{2} - \mathbb{N}$, then $c(\lambda), c(n-\lambda)$ are holomorphic at λ_0 and $N_{\lambda_0}(c(n-\lambda)\widetilde{S}(n-\lambda)) = N_{\lambda_0}(\widetilde{S}(n-\lambda))$, thus (3.20) and (3.21) leads to

$$m(\lambda_0) = N_{\lambda_0}(\widetilde{S}(\lambda)) + \nu(\lambda_0).$$

Now if $\lambda_0 \in \frac{n}{2} - \mathbb{N}$, $c(n-\lambda)$ has a first-order pole at λ_0 hence (3.19), (3.20) and (3.21) give

$$m(\lambda_0) = N_{\lambda_0}(\widetilde{S}(\lambda)) + \nu(\lambda_0) - \dim\ker_{\lambda_0}\widetilde{S}(n-\lambda).$$

with, in this case,

$$\dim\ker_{\lambda_0}\widetilde{S}(n-\lambda) = \dim\ker\widetilde{S}(n-\lambda_0) = \dim\ker\mathrm{Res}_{n-\lambda_0}S(\lambda)$$

since $\widetilde{S}(\lambda)$ is holomorphic at $\lambda_0 \in \frac{n}{2} + \mathbb{N}$ and $c(\lambda)$ has first-order poles at the same points. To achieve the proof of the Theorem, it remains to show that $m(n-\lambda_0) = N_{\lambda_0}(\widetilde{S}(\lambda))$ if $\Re(\lambda_0) < \frac{n}{2}$ and $\lambda_0 \notin \frac{1}{2}(n-\mathbb{N})$. Whereas the case $\lambda_0(n-\lambda_0) \notin \sigma_{pp}(\Delta_g)$ is clear since $\widetilde{S}(\lambda)^{-1} = \widetilde{S}(n-\lambda)$ is holomorphic near λ_0

and $m(n-\lambda_0)=0$, the case $\lambda_0(n-\lambda_0)\in\sigma_{pp}(\Delta_g)$ needs a little more care. In view of (3.2) and (3.8), $\widetilde{S}(\lambda)$ has the following polar part at $n-\lambda_0$

$$C(\lambda_0)(\lambda-n+\lambda_0)^{-1}\sum_{j=1}^{k}\Lambda^{\lambda_0-\frac{n}{2}}\phi_j^\sharp\otimes\Lambda^{\lambda_0-\frac{n}{2}}\phi_j^\sharp$$

with $C(\lambda_0)\neq 0$ if $\lambda_0\notin\frac{n}{2}-\mathbb{N}$, $k=m(n-\lambda_0)$ and $\phi_j^\sharp:=x^{\lambda_0-\frac{n}{2}}\phi_j|_{\partial\bar{X}}$ (where $(\phi_j)_j$ is an orthonormal basis of $\ker_{L^2}(\Delta_g-\lambda_0(n-\lambda_0))$ as in (3.2)). It is not difficult to see that $(\phi_j^\sharp)_j$ are independent, otherwise there would exist a non-zero solution $u\in x^{n-\lambda_0+1}C^\infty(\bar{X},\Gamma_0^{\frac{1}{2}})$ of $(\Delta_g-\lambda_0(n-\lambda_0))u=0$ and a Taylor expansion of this equation at $x=0$ proves that $u\in\dot{C}^\infty(\bar{X},\Gamma_0^{\frac{1}{2}})$, which is excluded according to Mazzeo's result [11]. Since the pole is a first-order pole, the factorization of $\widetilde{S}(\lambda)$ as in (2.2) near $n-\lambda_0$ is clear for the $k_l<0$: we have $m=k$ and $k_l=-1$ for $l=1,\dots,k$. Using (2.3) and $\widetilde{S}(\lambda)^{-1}=\widetilde{S}(n-\lambda)$, one then obtain that the partial null multiplicities of $\widetilde{S}(\lambda)$ at λ_0 are $\{-k_1,\dots,-k_k\}$ which proves that $m(n-\lambda_0)=N_{\lambda_0}(\widetilde{S}(\lambda))$ and the theorem. □

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