

BIRATIONAL CHARACTERIZATION OF PRODUCTS OF CURVES OF GENUS 2

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ABSTRACT. Let X be a variety of maximal Albanese dimension. In this paper we prove that if $\chi(\omega_X) = 1$ then $q(X) \leq 2 \dim X$ and if $q(X) = 2 \dim X$, then X is birational to a product of curves of genus 2.

1. Introduction

The purpose of this paper is to show how a result of Beauville in the classification theory of irregular surfaces of general type can be generalized to higher dimensional varieties of maximal Albanese dimension.

It is well known that for a surface X of general type one has $\chi(\omega_X) > 0$. Therefore surfaces with $\chi(\omega_X) = 1$ can be regarded as the “first case in the classification” and for this reason they have always been of interest to people working on the theory of surfaces. The case in which the irregularity $q(X)$ is equal to 0 has been investigated for a long time by several authors, but a complete classification seems at the moment out of reach, due to the existence of a great number of different examples. On the other hand, the following result of Beauville ([Be]) suggests that the classification should be easier when the irregularity $q(X)$ is large:

If X is a surface of general type with $\chi(\omega_X) = 1$ then $q(X) \leq 4$ and if $q(X) = 4$, then X is birational to the product of two curves of genus 2.

In fact, more recently, Hacon and Pardini [HP1] and Pirola [Pi] have classified surfaces of general type with $\chi(\omega_X) = 1$ and $q(X) = 3$, showing that they belong to two families. Surfaces of general type with $\chi(\omega_X) = 1$ and $q(X) = 1, 2$ have proven to be much harder to understand and they are still an active topic of research.

It is a natural question to try and generalize the above results to higher dimension. In general, if $\dim X \geq 3$, one only expects results of a qualitative nature. Indeed, in Section 4 we give a construction of threefolds of general type with $\chi(\omega_X) = 1$ and arbitrarily large irregularity, thus showing that it is not possible to generalize Beauville’s statement verbatim. Therefore, the situation appears to be substantially more complicated than that of surfaces. However, Green and Lazarsfeld have made the surprising remark that varieties of

Received May 13, 2004.

2000 *Mathematics Subject Classification*. 14J40, 14E99.

maximal Albanese dimension seem to behave similarly to surfaces. For example, they prove the inequality $\chi(\omega_X) \geq 0$ for a variety X of maximal Albanese dimension (cf. [GL1]). Another example of this analogy is the fact, proven by Chen and Hacon ([ChH1]), that for a variety of general type and maximal Albanese dimension, the 6-canonical map is always birational.

In this note we give an additional instance of the analogy between surfaces and varieties of maximal Albanese dimension, by proving the following generalization of the above mentioned result of Beauville:

Theorem. *Let X be a smooth projective variety of maximal Albanese dimension. If $\chi(\omega_X) = 1$, then $q(X) \leq 2 \dim X$. If in addition $q(X) = 2 \dim X$, then X is birational to a product of curves of genus 2.*

The proof of the Theorem above relies on the generic vanishing theorems of Green and Lazarsfeld [GL1], [GL2] and their generalizations [Si], [ClH], [ChH2], [Hac]. It would be interesting to see if these techniques can be successfully applied in the case when $\dim X < q(X) < 2 \dim X$. However, already the case $\chi(\omega_X) = 1$ and $q(X) = 2 \dim X - 1$ seems considerably harder. As we have already mentioned, there is a complete classification in dimension 2, but the number of examples grows with the dimension and one does not have clear picture of the situation.

Notation and conventions. We work over the field of complex numbers; all varieties are projective. We identify Cartier divisors and line bundles on a smooth variety, and we use the additive and multiplicative notation interchangeably. If X is a smooth projective variety, we let K_X be a canonical divisor, so that $\omega_X = \mathcal{O}_X(K_X)$, we denote by $\kappa(X)$ the Kodaira dimension and by $q(X) := h^1(\mathcal{O}_X) = h^0(\Omega_X^1)$ the *irregularity*. We denote by $a: X \rightarrow A(X)$ the Albanese map and by $\text{Pic}^0(X)$ the dual abelian variety to $A(X)$, which parameterizes all topologically trivial line bundles on X . $\text{Pic}^0(X)_{\text{tors}}$ denotes the set of torsion points in $\text{Pic}^0(X)$. If the Albanese morphism is generically finite onto its image, we say that X is of *maximal Albanese dimension*. If $\kappa(X) = \dim X$, we say that X is of *general type*.

2. Preliminaries.

The main ingredients in the proof of Theorem 3.1 are the relative vanishing theorems due Kollár and a generalized version of the generic vanishing theorems of Green and Lazarsfeld. We recall here these results in the form which is more convenient for our later use. Next we deduce two statements about the holomorphic Euler–Poincaré characteristic of a variety of maximal Albanese dimension fibered over a curve.

Theorem 2.1. *Let X, Y, Z be projective varieties, X smooth, let $f: X \rightarrow Y$, $g: Y \rightarrow Z$ be surjective morphisms and let $P_0 \in \text{Pic}^0(X)$ be a torsion point. Then:*

- a) $R^i g_* R^j f_*(\omega_X \otimes P_0)$ is torsion free;

- b) $R^i g_* R^j f_*(\omega_X \otimes P_0) = 0$ if $i > \dim Y - \dim Z$;
- c) $R^k (g \circ f)_*(\omega_X \otimes P_0) \cong \sum_i R^i g_* R^{k-i} f_*(\omega_X \otimes P_0)$ (\cong denotes equality in the derived category).

Proof. If $P_0 = 0$ this is part of [Ko, Thm. 3.4].

If $P_0 \neq 0$, then let d be the order of P_0 in $\text{Pic}^0(X)$, let $p: \tilde{X} \rightarrow X$ be the étale \mathbb{Z}_d -cover given by P_0 and let $\tilde{f} := f \circ p$. We have $p_* \omega_{\tilde{X}} = \bigoplus_{m=0}^{d-1} \omega_X(mP_0)$, hence $R^j f_*(\omega_X \otimes P_0)$ is a direct summand of $R^j \tilde{f}_* \omega_{\tilde{X}}$, and statements a) and b) for f follow by the corresponding statements for \tilde{f} .

Consider now statement c). Since the morphism p is finite, one has:

$$(2.1) \quad R^k (g \circ \tilde{f})_*(\omega_{\tilde{X}}) = \bigoplus_{m=0}^{d-1} R^k (g \circ f)_*(\omega_X \otimes mP_0)$$

and the summands on the right hand side are the eigensheaves for the action of the Galois group $G \cong \mathbb{Z}_d$ of $p: \tilde{X} \rightarrow X$.

On the other hand, by applying the case $P_0 = 0$ of c) one gets:

$$(2.2) \quad \begin{aligned} R^k (g \circ \tilde{f})_*(\omega_{\tilde{X}}) &\cong \sum_i R^i g_* R^{k-i} \tilde{f}_*(\omega_{\tilde{X}}) = \\ &= \sum_i \bigoplus_{m=0}^{d-1} R^i g_* R^{k-i} f_*(\omega_X \otimes mP_0). \end{aligned}$$

The result now follows by comparing the formulae 2.1 and 2.2, provided that the identification in 2.2 is compatible with the G -action. This can indeed be achieved by choosing the smooth ample divisor H in the proof of [Ko, Thm. 3.1] to be G -invariant. \square

Let A be an abelian variety and $a: X \rightarrow A$ a morphism. Given a sheaf \mathcal{F} on X , we define the cohomological support loci $V^i(A, \mathcal{F}) := \{P \in \text{Pic}^0(A) \mid h^i(\mathcal{F} \otimes a^* P) > 0\}$. When $a: X \rightarrow A(X)$ is the Albanese map, we write simply $V^i(\mathcal{F})$.

Theorem 2.2 (Generic Vanishing Theorem).

Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$ be morphisms of smooth projective varieties and let $a: Z \rightarrow A$ be a morphism to an abelian variety. Let $P_0 \in \text{Pic}^0(X)_{\text{tors}}$ and set $\mathcal{F} := R^k g_* R^j f_*(\omega_X \otimes P_0)$. Then:

- a) $V^i(A, \mathcal{F})$ has codimension $\geq i - (\dim Z - \dim a(Z))$;
- b) Every irreducible component of $V^i(A, \mathcal{F})$ is a translate of a sub-torus of $\text{Pic}^0(A)$ by a torsion point.
The same also holds for the irreducible components of $V_m^i(A, \mathcal{F}) := \{P \in \text{Pic}^0(A) \mid h^i(\mathcal{F} \otimes a^* P) \geq m\}$;
- c) Let T be an irreducible component of $V^i(A, \mathcal{F})$, let $P \in T$ be a point such that $V^i(A, \mathcal{F})$ is smooth at P , and let $v \in H^1(A, \mathcal{O}_A) \cong T_P \text{Pic}^0(A)$. If v is not tangent to T , then the sequence

$$H^{i-1}(Z, \mathcal{F} \otimes a^* P) \xrightarrow{\cup v} H^i(Z, \mathcal{F} \otimes a^* P) \xrightarrow{\cup v} H^{i+1}(Z, \mathcal{F} \otimes a^* P)$$

is exact. Moreover, if P is a general point of T and v is tangent to T then both maps vanish;

d) If Z has maximal Albanese dimension, then there are inclusions:

$$V^0(A, \mathcal{F}) \supseteq V^1(A, \mathcal{F}) \supseteq \cdots \supseteq V^n(A, \mathcal{F}).$$

Proof. Assume first that $X = Y = Z$ and that f and g are the identity map. If A is the Albanese variety and $P_0 = 0$, then this is the classical generic vanishing theorem, which is due to Green–Lazarsfeld ([GL1], [GL2]), apart from the statement in b) that the components of $V^i(A, \mathcal{F})$ are translates by a torsion point, which is due to Simpson ([Si]). The extension to the case of a morphism $a: X \rightarrow A$ into any abelian variety is straightforward (cf. [EL, Rem. 1.6]). To prove the statement in the case $P_0 \neq 0$, one considers the étale \mathbb{Z}_d -cover $p: \tilde{X} \rightarrow X$ associated to P_0 as in the proof of Theorem 2.1. Then for $P \in \text{Pic}^0(A)$ one has:

$$h^i(\tilde{X}, \omega_{\tilde{X}} \otimes (a \circ p)^* P) = \sum_{j=0}^{d-1} h^i(X, \omega_X \otimes jP_0 \otimes a^* P).$$

Since $h^i(X, \omega_X \otimes jP_0 \otimes a^* P)$ is an upper semicontinuous function of $P \in \text{Pic}^0(A)$, it is easy to show that if T is a component of $V_m^i(A, \omega_X \otimes P_0)$ then T is also a component of $V_{m'}^i(A, \omega_{\tilde{X}})$ for some m' . Hence statements a), b), c), d) follow from the corresponding statements for the loci $V_m^i(A, \omega_{\tilde{X}})$.

Consider now the general case. By Theorem 2.1, c), for $P \in \text{Pic}^0(A)$ one has:

$$\begin{aligned} H^i(X, \omega_X \otimes P_0 \otimes (a \circ g \circ f)^* P) &= \oplus_b H^b(Z, R^{i-b}(g \circ f)_*(\omega_X \otimes P_0) \otimes a^* P) = \\ &= \oplus_{b,c} H^b(Z, R^c g_* R^{i-b-c} f_*(\omega_X \otimes P_0) \otimes a^* P). \end{aligned}$$

In view of this equality and of the semicontinuity of the function

$$h^b(Z, R^c g_* R^{i-b-c} f_*(\omega_X \otimes P_0) \otimes a^* P),$$

statements b), c) follow by the analogous statements in the case $f = g = \text{Id}$ by the same argument as above. For the statements a), d), we refer to [CIH] §10, [Hac] Corollary 4.2 and [ChH2] §2. \square

Corollary 2.3. *Let X be a smooth variety of maximal Albanese dimension, let $f: X \rightarrow Y$ be a surjective map and let F be a general fiber of f . If $P_0 \in \text{Pic}^0(X)_{\text{tors}}$ is general, then $f_*(\omega_X \otimes P_0)$ is torsion free of rank $\chi(\omega_F)$ and $R^i f_*(\omega_X \otimes P_0) = 0$ for $i > 0$.*

Proof. Since X is of maximal Albanese dimension, the restriction of the Albanese map $F \rightarrow A(X)$ is generically finite onto its image. Since P_0 is general, by Theorem 2.2 we have $h^i(\omega_F \otimes P_0) = 0$ for $i > 0$ and $h^0(\omega_F \otimes P_0) = \chi(\omega_F \otimes P_0) = \chi(\omega_F)$. It follows that for $i > 0$, the sheaf $R^i f_*(\omega_X \otimes P_0)$ is supported on a proper closed subset of Y and that $f_*(\omega_X \otimes P_0)$ has rank equal to $\chi(\omega_F)$. The claim now follows since by Theorem 2.1 the sheaves $R^i f_*(\omega_X \otimes P_0)$ are torsion free for every $i \geq 0$. \square

Theorem 2.4. *Let X be a smooth variety of maximal Albanese dimension and let $f: X \rightarrow B$ be a map onto a smooth curve of genus ≥ 2 and let F be a general fiber of f . Then for general $P_0 \in \text{Pic}^0(X)_{\text{tors}}$, one has:*

$$\chi(\omega_X) = \chi(\omega_F)\chi(\omega_B) + \deg(f_*(\omega_{X/B} \otimes P_0)) \geq \chi(\omega_F)\chi(\omega_B).$$

Proof. Let $P_0 \in \text{Pic}^0(X)$ be a general torsion point and $Q \in \text{Pic}^0(B)$ be a general point. Then Theorem 2.2 implies:

$$\chi(\omega_X) = \chi(\omega_X \otimes P_0 \otimes f^*Q) = h^0(\omega_X \otimes P_0 \otimes f^*Q) = h^0(f_*(\omega_X \otimes P_0) \otimes Q).$$

In turn, generic vanishing for the vector bundle $f_*(\omega_X \otimes P_0)$ on B (cf. Theorem 2.2) implies:

$$h^0(f_*(\omega_X \otimes P_0) \otimes Q) = \chi(f_*(\omega_X \otimes P_0) \otimes Q) = \chi(f_*(\omega_X \otimes P_0)).$$

Let F be a general fiber of f . By Corollary 2.3, $f_*(\omega_X \otimes P_0)$ is a vector bundle on B of rank $\chi(\omega_F)$. Now, since $f_*(\omega_X \otimes P_0) = \omega_B \otimes f_*(\omega_{X/B} \otimes P_0)$, Riemann-Roch for vector bundles on a curve gives

$$\begin{aligned} \chi(f_*(\omega_X \otimes P_0)) &= \deg f_*(\omega_X \otimes P_0) + (1 - g(B))\chi(\omega_F) = \\ &= \deg(f_*(\omega_{X/B} \otimes P_0)) + (g(B) - 1)\chi(\omega_F). \end{aligned}$$

To finish the proof it is enough to show that $\deg(f_*(\omega_{X/B} \otimes P_0)) \geq 0$ for $P_0 \in \text{Pic}^0(X)_{\text{tors}}$. If $P_0 = 0$, this follows from [V] Satz V. The analogous statement for any torsion point P_0 follows from the case $P_0 = 0$ by the same argument as in the proof of Theorem 2.1. \square

Remark: It is not the case that for $f: X \rightarrow Y$ a surjective morphism of varieties of maximal Albanese dimension, one has $\chi(\omega_X) \geq \chi(\omega_Y)\chi(\omega_F)$.

Consider in fact for $i = 1, 2, 3$ curves C_i with elliptic involutions σ_i . Then there is a morphism

$$X := (C_1 \times C_2 \times C_3)/\mathbb{Z}_2 \rightarrow (C_2 \times C_3)/\mathbb{Z}_2 =: Y$$

where the groups \mathbb{Z}_2 act diagonally. If $g(C_i) \geq 2$ for all i , it follows that X is birational to a smooth projective variety of general type and $\chi(\omega_X) = 0$ (cf. [EL] or the computation in Example 4.1). However Y is a surface of general type and hence $\chi(\omega_Y) > 0$.

Proposition 2.5. *Let X be a smooth variety of maximal Albanese dimension, $f: X \rightarrow Y$ a surjective morphism and F a general fiber of f . If $\chi(\omega_F) = 0$, then $\chi(\omega_X) = 0$.*

Proof. Let $P_0 \in \text{Pic}^0(X)_{\text{tors}}$ be a general point. By Corollary 2.3, we have $R^i f_*(\omega_X \otimes P_0) = 0$ for every $i \geq 0$. It follows:

$$\chi(\omega_X) = \chi(\omega_X \otimes P_0) = \sum_i (-1)^i \chi(R^i f_*(\omega_X \otimes P_0)) = 0.$$

\square

3. The theorem

This section is devoted to the proof of the following theorem:

Theorem 3.1. *Let X be a smooth projective variety of maximal Albanese dimension. If $\chi(\omega_X) = 1$, then $q(X) \leq 2 \dim X$. If in addition $q(X) = 2 \dim X$, then X is birational to a product of curves of genus 2.*

Our proof uses induction on the dimension of X . The inductive step consists in showing that X admits a fibration $f: X \rightarrow C$ onto a curve of genus ≥ 2 and that the fibers of f satisfy the assumptions of Theorem 3.1. The existence of the fibration f is in turn established by induction, exploiting the geometry of the cohomological support loci (cf. Section 2) of a suitable sheaf. We start by giving an upper bound on the codimension of such a locus.

Lemma 3.2. *Let $f: X \rightarrow Y$ be a morphism of smooth varieties of maximal Albanese dimension, let $P_0 \in \text{Pic}^0(X)_{\text{tors}}$ be general, and let T be a component of $V^1(f_*(\omega_X \otimes P_0))$. Assume that $\chi(\omega_X) = 1$.*

If $T \not\subset V^i(f_(\omega_X \otimes P_0))$ then $\text{codim} T \leq i$.*

Proof. We write $\mathcal{F} := f_*(\omega_X \otimes P_0)$. Since P_0 is general, by Corollary 2.3 we have $R^i f_*(\omega_X \otimes P_0) = 0$ for $i > 0$, and thus $\chi(\mathcal{F}) = \chi(\omega_X \otimes P_0) = 1$.

Let $P \in T$ be a general point, so that P is smooth for $V^1(\mathcal{F})$ and $P \notin V^i(\mathcal{F})$. Assume by contradiction that T has codimension $\geq i + 1$ and let $W \subseteq H^1(\mathcal{O}_X)$ be a linear subspace of dimension $i + 1$ transversal to T at P . Writing $\mathbb{P} := \mathbb{P}(W)$, the derivative complexes of Thm. 2.2, d) fit together to give an exact complex of vector bundles (cf. [EL], proof of Theorem 3):

$$(3.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}}(-d) \rightarrow H^0(\mathcal{F} \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-i) \rightarrow \dots \rightarrow H^{i-1}(\mathcal{F} \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-1) \rightarrow 0.$$

Notice that we have $d \geq i$ by the exactness of the complex. Considering the hypercohomology spectral sequence associated to the above complex, one sees that $H^i(\mathcal{O}_{\mathbb{P}}(-d)) = 0$, hence $d = i$. Now twisting (3.1) by $\mathcal{O}_{\mathbb{P}}(-1)$ and considering the corresponding spectral sequence, one obtains an isomorphism $H^i(\mathcal{O}_{\mathbb{P}}(-i-1)) \cong H^0(\mathcal{F} \otimes P) \otimes H^i(\mathcal{O}_{\mathbb{P}}(-i-1))$. It follows that $h^0(\mathcal{F} \otimes P) = 1$ and that the map $H^1(\mathcal{F} \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-i+1) \rightarrow H^2(\mathcal{F} \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-i+2)$ is injective. Hence the following complex is exact:

$$(3.2) \quad 0 \rightarrow H^1(\mathcal{F} \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-i-1) \rightarrow \dots \rightarrow H^{i-1}(\mathcal{F} \otimes P) \otimes \mathcal{O}_{\mathbb{P}}(-3) \rightarrow 0$$

By applying the previous argument again, we get $H^1(\mathcal{F} \otimes P) = 0$, a contradiction. \square

Lemma 3.3. *Let X, Y be smooth varieties, with Y of maximal Albanese dimension, and let $f: X \rightarrow Y$ be a surjective morphism. Let $P_0 \in \text{Pic}^0(X)$ be a torsion point, let T be a component of $V^i(f_*(\omega_X \otimes P_0))$ and let $g: Y \rightarrow S := \text{Pic}^0(T)$ be the induced map. Then:*

$$\dim g(Y) \leq \dim Y - i.$$

Proof. Since $f_*(\omega_X \otimes P_0)$ satisfies Theorem 2.2, we have $T = Q_0 + T_0$, where $Q_0 \in \text{Pic}^0(Y)$ is a torsion point and $T_0 \subset \text{Pic}^0(Y)$ is a subtorus. For $P \in T_0$ general, for $j \geq 0$ and for $s > 0$, by Theorem 2.2 one has:

$$H^s(R^j g_*(f_*(\omega_X \otimes P_0) \otimes Q_0) \otimes P) = H^s(R^j g_*(f_*(\omega_X \otimes P_0 \otimes f^* Q_0)) \otimes P) = 0.$$

Hence for $P \in T_0$ general, the Leray spectral sequence gives:

$$H^i(f_*(\omega_X \otimes P_0) \otimes P \otimes Q_0) = H^0(R^i g_*(f_*(\omega_X \otimes P_0 \otimes f^* Q_0)) \otimes P).$$

Now assume by contradiction that the relative dimension of g is strictly smaller than i . Then by Kollár's relative vanishing (cf. Theorem 2.1) we have

$$R^i g_*(f_*(\omega_X \otimes P_0 \otimes f^* Q_0)) = 0,$$

hence $H^i(f_*(\omega_X \otimes P_0) \otimes P \otimes Q_0) = 0$, contradicting the assumption that $T = T_0 + Q_0$ is contained in $V^i(\mathcal{F})$. \square

Proposition 3.4. *Let Y be a smooth variety of maximal Albanese dimension such that $q(Y) \geq 2\dim Y > 0$. Assume that there exists a surjective map $f: X \rightarrow Y$, with X a smooth variety of maximal Albanese dimension such that $\chi(\omega_X) = 1$.*

Then there exists a surjective map $\psi: Y \rightarrow C$, with C a smooth curve of genus ≥ 2 .

Proof. We prove the statement by induction on the dimension of Y , the case $\dim Y = 1$ being obvious. So assume $\dim Y \geq 2$.

Let $P_0 \in \text{Pic}^0(X)$ be a general torsion point and set $\mathcal{F} := f_*(\omega_X \otimes P_0)$. Since P_0 is general, we have $R^i f_*(\omega_X \otimes P_0) = 0$ for $i > 0$ by Corollary 2.3. Therefore, we have $H^k(Y, \mathcal{F} \otimes P) = H^k(X, \omega_X \otimes P_0 \otimes f^* P)$ for every $P \in \text{Pic}^0(Y)$ and for every k . In particular, $\chi(\mathcal{F}) = \chi(\omega_X \otimes P_0) = 1$.

Assume that $\dim V^1(\mathcal{F}) \geq 0$, let $T \subseteq V^1(\mathcal{F})$ be a component and let $i \geq 1$ be such that $T \subseteq V^i(\mathcal{F})$ and $T \not\subseteq V^{i+1}(\mathcal{F})$. By Theorem 2.2, T is a proper subset of $\text{Pic}^0(Y)$. By Lemma 3.2 we have $\dim T \geq 2\dim Y - i - 1 \geq \dim Y - 1 > 0$. Let $g: Y \rightarrow S := \text{Pic}^0(T)$ be the induced map and let Z be a nonsingular model of $g(Y)$. Since $g(Y)$ generates S and $\dim S = \dim T > 0$, it follows that $\dim Z > 0$ and Z has maximal Albanese dimension. Hence we have

$$q(Z) \geq \dim S \geq q(Y) - i - 1.$$

By Lemma 3.3, we have

$$\dim Z = \dim g(Y) \leq \dim Y - i.$$

Since $q(Y) \geq 2\dim Y$, it follows that

$$q(Z) \geq 2\dim Y - i - 1 \geq 2\dim Z + i - 1 \geq 2\dim Z.$$

Hence by the inductive hypothesis there exists a smooth curve C of genus at least 2 and a surjective morphism $h: Z \rightarrow C$. We let $\psi: Y \rightarrow C$ be the map obtained by composing the rational map $Y \rightarrow Z$ with h . In principle ψ is just a rational map, but it is not difficult to show that ψ is indeed regular. In fact,

let $Y' \rightarrow Y$ be a birational modification such that ψ induces a regular map $\psi': Y' \rightarrow C$ and let $\psi'_*: A(Y) \rightarrow J(C)$ be the induced morphism. Then ψ is the composition of the Albanese map $Y \rightarrow A(Y) = A(Y')$ with ψ'_* , hence it is a morphism.

To finish the proof we have to rule out the possibility that $V^1(\mathcal{F}) = \emptyset$. Assume that this is the case. Let $a: Y \rightarrow A := A(Y)$ be the Albanese morphism and set $\mathcal{F}' := a_*\mathcal{F}$. The map a is generically finite, hence by Theorem 2.1 we have $R^i a_*\mathcal{F} = 0$ for all $i > 0$. It follows that $h^i(\mathcal{F} \otimes a^*P) = h^i(\mathcal{F}' \otimes P)$ for all $i \geq 0$ and $P \in \text{Pic}^0(Y)$. Since Y has maximal Albanese dimension, by Theorem 2.2 the assumption $V^1(\mathcal{F}) = \emptyset$ implies $h^i(\mathcal{F}' \otimes P) = 0$ for all $i > 0$ and any $P \in \text{Pic}^0(Y)$, and therefore $h^0(\mathcal{F}' \otimes P) = \chi(\mathcal{F}' \otimes P) = 1$ for all $P \in \text{Pic}^0(Y)$. Let $g = q(Y)$, $\hat{S}: D(A) \rightarrow D(\text{Pic}^0(Y))$ be the Fourier–Mukai transform on A and $S: D(\text{Pic}^0(Y)) \rightarrow D(A)$ the Fourier–Mukai transform on $\text{Pic}^0(Y)$. The sheaf \mathcal{F}' satisfies the Index Theorem with index 0 (cf. [M]), so that $L := \hat{S}(\mathcal{F}')$ is a line bundle and $S^g L = (-1_A)^*\mathcal{F}'$. In particular, there exists $P \in \text{Pic}^0(Y)$ such that $h^g(L \otimes P) \neq 0$. By [LB, §3.5], this implies that the line bundle L is negative semidefinite. Hence there exist an abelian variety A' , a surjective morphism $\text{Pic}^0(Y) \rightarrow A'$ and a polarization M on A' such that L is algebraically equivalent to the pull back of M^\vee on $\text{Pic}^0(Y)$. It follows that the support of $S^g(L)$ is a translate of $\text{Pic}^0(A') \subset \text{Pic}^0(\text{Pic}^0(A)) = A$. On the other hand, by Theorem 2.1 the support of \mathcal{F}' is equal to $a(Y)$. Since $a(Y)$ is a proper subvariety that generates A , we have obtained a contradiction and the proof is complete. \square

We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. We proceed by induction on $\dim X$. By Proposition 3.4 (with $f = \text{id}$), there exists a surjective morphism $\psi: X \rightarrow C$ where C is a smooth curve of genus at least 2. Up to replacing ψ by the first term of its Stein factorization, we may assume that ψ has connected fibers. Let F be a general fiber. Since X is of maximal Albanese dimension, F is also of maximal Albanese dimension. By Proposition 2.5, we have $\chi(\omega_F) > 0$. By Theorem 2.4, one has

$$1 = \chi(\omega_X) \geq \chi(\omega_F)\chi(\omega_C) > 0$$

and so $\chi(\omega_F) = \chi(\omega_C) = 1$. The inductive assumption gives $q(F) \leq 2 \dim F = 2(\dim X - 1)$ and therefore one has $q(X) \leq q(F) + g(C) \leq 2 \dim X$.

Assume now that $q(X) = 2 \dim X = 2n$. In this case one has $q(F) \geq q(X) - 2 \geq 2 \dim F$, hence by induction F is birational to a product $C_1 \times \dots \times C_{n-1}$ of curves of genus 2. We need to show that the fibration ψ is birationally a product.

The 3-canonical map of F is a birational morphism whose image is isomorphic to $C_1 \times \dots \times C_{n-1}$. Hence, by replacing X with a desingularization of the relative 3-canonical image, we may assume that the general fiber F is isomorphic to a product $C_1 \times \dots \times C_{n-1}$. Next we show that the decomposition

$F = C_1 \times \dots \times C_{n-1}$ is unique, up to permuting the C_i 's. Notice that the curves of the form

$$x_1 \times \dots \times x_{i-1} \times C_i \times x_{i+1} \times \dots \times x_{n-1}$$

have trivial normal bundle on F . Let now C be a curve of genus 2 on F and assume that C is not as above. Then at least $k > 1$ of the projections $p_i: F \rightarrow C_i$ restrict to non constant maps, and thus they give isomorphisms $C \rightarrow C_i$. Up to a permutation of the indices, we may assume that $p_i|_C$ is constant for $i > k$ and it is an isomorphism otherwise. Hence we have $C_i = C$ for $i = 1, \dots, k$ and there exist x_{k+1}, \dots, x_{n-1} such that $C \subset C \times \dots \times C \times x_{k+1} \times \dots \times x_n \subset F$. Moreover C can be identified with the strict diagonal inside $C \times \dots \times C$. It follows that the normal bundle of C is isomorphic to the sum of $k - 1$ copies of ω_C^{-1} and of $n - k$ copies of \mathcal{O}_C . Hence the curves

$$x_1 \times \dots \times x_{i-1} \times C_i \times x_{i+1} \times \dots \times x_{n-1}$$

are the only curves of genus 2 on F which have trivial normal bundle.

It follows that every automorphism of F either preserves these systems of curves or permutes them (we have $C_i = C_{\sigma(i)}$ in this case, where $\sigma \in S_{n-1}$ is the corresponding permutation). Using this remark it is easy to check also that the action of $\text{Aut}(F)$ on $H^0(\Omega_F^1)$ is faithful.

Denote by P the cokernel of the injection $J(C) \rightarrow \text{Pic}^0(X)$. Arguing as in the proof of the Lemme on page 345 of [Be], one shows that there exists a nonempty open subset $C_0 \subset C$ such that, denoting by $\psi_0: X_0 \rightarrow C_0$ the restriction of ψ , there is an isomorphism $\text{Pic}^0(X_0/C_0) \xrightarrow{\sim} C_0 \times P$ over C_0 . By the discussion above, for every $t \in C_0$ there is a canonical (up to the order) splitting $\text{Pic}^0(F_t) = J(C_{1,t}) \times \dots \times J(C_{n-1,t})$, where F_t denotes, as usual, the fiber over t . Since the abelian subvarieties of a fixed abelian variety are a discrete set, it follows that the induced decomposition $P = J_1 \times \dots \times J_{n-1}$ is independent of F_t .

Take a finite covering $C' \rightarrow C$ such that the induced fibration $X' \rightarrow C'$ has a section (we may assume X' smooth, by possibly replacing it with a desingularization). Then we can consider the composition of the relative Albanese map $A(X'/C') \rightarrow C' \times P^\vee$ with the projection $C' \times P^\vee \rightarrow C' \times J_1$. Let Y be a smooth projective model of the image of this map. By construction, Y is a surface fibered in curves of genus 2 and has relative irregularity ≥ 2 . By the Lemme on page 345 of [Be], Y is birationally a product and, in particular, the curve C_1 does not vary in moduli as t varies in C . The analogous statement holds of course for C_2, \dots, C_{n-1} , hence two general fibers F of $f: X \rightarrow C$ are isomorphic.

Since the automorphism group of F is finite, one can argue as in [Se] and prove that X is birational to a quotient $(F \times B)/G$, where G is a finite group that acts faithfully on F and B and diagonally on $F \times B$, and such that $B/G = C$. One has $2n = q(X) = q(B/G) + q(F/G)$, namely $q(F/G) = 2(n - 1)$. Since, as we have remarked above, the representation of $\text{Aut}(F)$ on $H^0(\Omega_F^1)$ is faithful, it follows that the group G is trivial and that X is birational to $C_1 \times \dots \times C_{n-1} \times C$.

□

4. Examples

We describe here two families of varieties with $\chi(\omega_X) = 1$ and unbounded irregularity.

The first example shows that in Beauville's result the assumption that “ X be a surface of general type” cannot be weakened to $\kappa(X) \geq 1$. The second example shows that in dimension ≥ 3 the statement of Theorem 3.1 does not hold under the weaker assumption that X be of general type, hence the analog of Beauville's result is not true in higher dimension.

Example 4.1. We construct a family of properly elliptic surfaces Y with $\chi(\omega_Y) = 1$ and arbitrary irregularity $g \geq 2$.

Let B be a hyperelliptic curve of genus $g \geq 2$ and let M be a line bundle of degree 1 on B such that the linear series $|2M|$ is the g_2^1 of B . Set $\mathbb{F} := \mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(2M))$, and let $p: \mathbb{F} \rightarrow B$ be the corresponding projection. Let σ_∞ be the section at “infinity” and set $\sigma_0 := \sigma_\infty + 2p^*M$. One has:

$$\sigma_\infty^2 = -2, \quad \sigma_0^2 = 2, \quad \sigma_0\sigma_\infty = 0, \quad K_{\mathbb{F}} = -2\sigma_\infty + p^*(K_B - 2M).$$

It is easy to check that $|\sigma_0|$ is a free linear system of dimension 2 on \mathbb{F} .

Let $D_0 \in |3\sigma_0|$ be a smooth divisor, set $L := 2\sigma_\infty + 3p^*M$, $D = D_0 + \sigma_\infty$ and let $\pi: Y \rightarrow \mathbb{F}$ be the double cover associated to the relation $2L \equiv D$. The surface Y is smooth, since D is smooth, and p induces an elliptic fibration $f: Y \rightarrow B$. Notice that the surface Y is properly elliptic, since $g \geq 2$.

The standard formulae for double covers give:

$$\chi(Y) = 2\chi(\mathbb{F}) + L(K_{\mathbb{F}} + L)/2 = 1,$$

$$p_g(Y) = h^0(\mathbb{F}, K_{\mathbb{F}} + L) = h^0(B, K_B + M) = g.$$

It follows that $q(Y) = g$ and $f: Y \rightarrow B$ is the Albanese map.

Notice that if C_1, \dots, C_{n-2} are curves of genus 2, then the variety $X := Y \times C_1 \times \dots \times C_{n-2}$ has dimension n and it satisfies:

$$\chi(\omega_X) = 1, \quad q(X) = g + 2n - 4, \quad \kappa(X) = n - 1.$$

Example 4.2. We construct a family of 3-folds Y of general type with $\chi(\omega_Y) = 1$ and arbitrarily large irregularity. Let D be a smooth plane quartic and let S be the second symmetric product of D . The surface S is smooth minimal of general type with $q(S) = p_g(S) = 3$. Let σ be the involution of S that maps a pair $\{P, Q\} \in S$ to the pair $\{P', Q'\}$ such that $P + Q + P' + Q'$ is the section of D with a line in \mathbb{P}^2 . It is well known (cf. for instance [CPT, Ex. 3.3]) that the fixed locus of σ consists of 28 isolated fixed points and that the quotient surface $\Sigma := S/\sigma$ is a nodal surface of general type with $p_g(\Sigma) = 3$ and $q(\Sigma) = 0$.

Let C be a smooth curve of genus $g \geq 2$, let $C' \rightarrow C$ be a smooth double cover branched on $2m$ points and let τ be the corresponding involution of C' . Set $Z := (S \times C')/(\sigma \times \tau)$. The variety Z is of general type, since it dominates $\Sigma \times C$, and it has terminal singularities at the images of the $56m$ fixed points

of $\sigma \times \tau$. A desingularization Y of Z can be obtained as follows: blow up X at the fixed points of $\sigma \times \tau$ to obtain a 3-fold \hat{X} such that $\sigma \times \tau$ induces an involution of \hat{X} whose fixed locus is a divisor and then take Y to be the quotient of \hat{X} by this involution. This description shows that for every $i \geq 0$ one can identify $H^0(Y, \Omega_Y^i)$ with the subspace of $H^0(S \times C', \Omega_{S \times C'}^i)$ invariant under the involution. We write $H^0(C', \omega_{C'}) = H^0(C, \omega_C) \oplus V$, where V is the subspace of antiinvariant forms, which has dimension $g+m-1$. Then we have identifications:

$$\begin{aligned} H^0(\omega_Y) &= H^0(\omega_C) \otimes H^0(\omega_S), & H^0(\Omega_Y^1) &= H^0(\omega_C), \\ H^0(\Omega_Y^2) &= H^0(\Omega_S^1) \otimes V \oplus H^0(\omega_S) \end{aligned}$$

It follows that $\chi(\omega_Y) = g-3m-1$, $q(Y) = g$ and, in particular the map $Y \rightarrow C$ is the Albanese map. For $g = 3m+2$ we have the required examples.

As in the previous example, one can obtain smooth n -dimensional varieties of general type with $\chi(\omega_Y) = 1$ and with arbitrarily large irregularity by taking the product of the above 3-folds with $n-3$ curves of genus 2.

Acknowledgments

The first author was partially supported by NSA research grant no: MDA904-03-1-0101 and by a grant from the Sloan Foundation. The second author was partially supported by P.R.I.N. 2002 “Geometria sulle varietà algebriche” of Italian M.I.U.R. and is a member of G.N.S.A.G.A. of C.N.R.

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