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ON THE CUBIC MOMENT OF QUADRATIC DIRICHLET L-FUNCTIONS

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1. Introduction

An important problem in analytic number theory is to understand the m-th moment of quadratic Dirichlet L-functions

$$\sum_{\substack{|D|\le x \\ D \text{ fund. disc.}}} L\left(\frac{1}{2}, \chi_D\right)^m, \tag{1.1}$$

where χ_D is the quadratic Dirichlet character associated to $\mathbb{Q}(\sqrt{D})$ as defined in [11]. Besides its own interest, this mean value problem also plays a crucial role in the studies of such as the Lindelöf Conjecture and the folklore non-vanishing conjecture that $L\left(\frac{1}{2}, \chi_D\right) \neq 0$. See, for example, [5] [7] [12].

Jutila [7] was the first to obtain the asymptotic formulas for the cases $m = 1, 2$, and Soundararajan [12] succeeded in the cubic case. For higher moments, a good upper bound has been obtained by Heath-Brown [6] in the quartic case, but their asymptotic formulas are still out of reach. In general, motivated by the fundamental work of Katz and Sarnak [8] on symmetric types associated to families of L-functions, and by calculations of Keating and Snaith [9] based on random matrix theory, Conrey and Farmer have made the following conjecture

$$\sum_{\substack{|D|\le x \\ D \text{ fund. disc.}}} L\left(\frac{1}{2}, \chi_D\right)^m \sim x R_m(\log x) \tag{1.2}$$

for some polynomial R_m of degree $\frac{m(m+1)}{2}$; see also [2]. Many even suggest that the error term here be of order $x^{\frac{1}{2}+\epsilon}$.

Diaconu, Goldfeld and Hoffstein [4] studied this problem through the approach of multiple Dirichlet series and gave another heuristic argument of (1.2). In particular, to analyze the cubic moment, they considered the multiple Dirichlet series

$$Z(s, w) = \sum_{D \text{ fund. disc.}} \frac{L(s, \chi_D)^3}{|D|^w}, \tag{1.3}$$

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and its related functions (for notations, see Section 2)

$$Z(s,w;a,b) = \sum_{\substack{d=1 \\ d \text{ odd sq. free}}}^{\infty} \frac{L^2(s,\chi_d\chi_a)^3\chi_b(d)}{d^w}, \tag{1.4}$$

$$Z(s,w;a,b;r) = \sum_{\substack{d=1 \\ r^2|d,\, 2\nmid d}}^{\infty} \frac{L^2\left(s,\chi_a\chi_d\right)^3 P_d^a(s)}{d^w}, \tag{1.5}$$

where $a,b \in \mathbb{Z}$ with $ab = \pm 1, \pm 2$ and $P_d^a(s)$ is a Dirichlet polynomial as defined in Section 3. For s in a neighborhood of $\frac{1}{2}$, the authors showed that $Z(s,w;a,b;r)$ can be meromorphically continued up to $\Re w > 0$ and that the series expansion

$$Z(s,w;a,b) = \sum_{\substack{r=1 \\ 2\nmid r}}^{\infty} \mu(r) Z(s,w;a,b;r) \tag{1.6}$$

converges absolutely for $\Re w > \frac{4}{5}$, thus obtaining the meromorphic continuation of $Z(s,w;a,b)$, as well as $Z(s,w)$ (c.f. (4.2)), in this region. With this information, the authors were able to establish the asymptotic formula for the cubic moment of quadratic Dirichlet L-functions with the best known error term

$$\sum_{|D|\le x} L\left(\frac{1}{2},\chi_D\right)^3 = xR_3(\log x) + O(x^{\frac{47-\sqrt{265}}{36}+\epsilon}). \tag{1.7}$$

The present paper continues the study of analytic properties for $Z(s,w)$. Under certain technical assumptions, we find that $Z\left(\frac{1}{2},w\right)$ has a simple pole at $w = \frac{3}{4}$ and its residue is explicitly computed. More precisely, we have the following result.

Theorem 1. *Let $\sigma_0 \in (\frac{1}{2},\frac{3}{4})$ be a constant. Assume that, for s in a neighborhood of $\frac{1}{2}$, every $Z(s,w;a,b)$ has a meromorphic continuation to $\Re w > \sigma_0$, and that the series expansion* (1.6) *is valid in this region. Then $Z(\frac{1}{2},w)$ can be analytically continued up to $\Re w > \sigma_0$ and has a simple pole at $w = \frac{3}{4}$ with residue*

$$\operatorname*{Res}_{w=\frac{3}{4}} Z\left(\frac{1}{2},w\right) = \frac{223-253\sqrt{2}}{256} P\left(\frac{\Gamma_{\mathbb{R}}(\frac{1}{4})^4}{\Gamma_{\mathbb{R}}(\frac{3}{4})^4} + \frac{\Gamma_{\mathbb{R}}(\frac{1}{4})\Gamma_{\mathbb{R}}(\frac{5}{4})^3}{\Gamma_{\mathbb{R}}(\frac{3}{4})\Gamma_{\mathbb{R}}(\frac{7}{4})^3}\right) \approx -0.162, \tag{1.8}$$

where

$$P = \frac{\zeta(\frac{1}{2})^7}{\zeta_2(\frac{1}{2})^3} \prod_{p>2} \left(1 - \frac{14}{p^{\frac{3}{2}}} - \frac{1}{p^2} + \frac{78}{p^{\frac{5}{2}}} - \frac{84}{p^3} - \frac{58}{p^{\frac{7}{2}}} + \frac{154}{p^4} - \frac{70}{p^{\frac{9}{2}}} - \frac{49}{p^5} + \frac{64}{p^{\frac{11}{2}}} - \frac{22}{p^6} + \frac{1}{p^7}\right) \approx -0.00193. \tag{1.9}$$

The existence of the pole of $Z(\frac{1}{2},w)$ at $w = \frac{3}{4}$, together with some suitable growth conditions for $Z(\frac{1}{2},w)$ itself, would give us information about the cubic moment of quadratic Dirichlet L-functions. For example, we have

Theorem 2. *Assume that the conditions in Theorem 1 are satisfied, and that*

$$Z\left(\frac{1}{2}, \sigma_0 + it\right) \ll (2+|t|)^{r+\epsilon} \tag{1.10}$$

for some positive constant $r < 3 - 4\sigma_0$. *Then we have*

$$\sum_{\substack{|D|\le x \\ D \text{ fund. disc.}}} L\left(\frac{1}{2}, \chi_D\right)^3 = xR_3(\log x) + bx^{\frac{3}{4}} + O(x^{\frac{r+\sigma_0}{r+1}+\epsilon}), \tag{1.11}$$

where $b \approx -0.215$ *is a constant.*

Since our main interest is to study the existence of the possible "exceptional main term" $x^{\frac{3}{4}}$, in Theorem 2 we do not seek the weakest possible growth condition for (1.11). A less restrictive condition on r would, at least, lead to a similar asymptotic expansion as in (1.11) for certain weighted cubic moments. One may observe, from Propositions 1 and 2, that $Z(\frac{1}{2}, w)$ (or more precisely, the function $Z_M(\frac{1}{2}, w; a, b)$ as defined in (3.4)) satisfies certain functional equation of form $w \mapsto 1 - w$, so a application of the Phragmen-Lindelöf Principle suggests that we might be able to take $r = \epsilon$. Such an estimate is still beyond our reach, but in [4] it is shown that we can take $r = 5(1-\sigma)$ for $\sigma > \frac{4}{5}$, which is already sufficient to yield a weighted cubic moment with error terms of order $x^{\frac{4}{5}+\epsilon}$.

The asymptotic expression (1.11) can be compared with the available data [10] for $x \le 10^7$, and numerical calculations over a much larger region are expected. The appearance of the exceptional main term $x^{\frac{3}{4}}$ suggests an unexpectedly fine structure for moments of quadratic Dirichlet L-functions (1.1). It will be interesting to locate such exceptional main terms for higher moments, and we will return to this topic in a separate paper.

The paper is organized as follows. In Section 2 we set the notations. Section 3 introduces a family of multiple Dirichlet series $Z_M(s, w; a, b)$ as well as their analytic properties, especially their functional equations. These functional equations readily imply the appearance of the pole for $Z(\frac{1}{2}, w)$ at $w = \frac{3}{4}$; c.f. the remarks following Proposition 2. In Section 4, the functional equations are applied to compute the residues of $Z_M(s, w; a, b)$ along some polar lines, which in turn leads to the determination of the residue of $Z\left(\frac{1}{2}, w\right)$ at $w = \frac{3}{4}$. This proves Theorem 1. Finally, Theorem 2 is proved in Section 5.

2. Symbols and Notations

For a (positive or negative) integer ℓ, we always write $\ell = \ell_0 \ell_1^2$, where ℓ_0 is squarefree and ℓ_1 is positive. Also, we write $\delta_\ell = \mathrm{sgn}(\ell)$, and let $\widetilde{\chi}_{\ell_0} = \left(\frac{\cdot}{\ell_0}\right)$ be the quadratic Dirichlet character of conductor ℓ_0. Furthermore, we introduce some variants of the divisor function.

$$\tau_3(m) = \sum_{m_1 m_2 m_3 = m} 1, \qquad \tau_3(m; L) = \sum_{\substack{m_1 m_2 m_3 = m \\ m_1, m_2, m_3 | L}} 1, \tag{2.1}$$

$$\tau_3(m;L;\mu) = \sum_{\substack{m_1m_2m_3=m\\ m_1,m_2,m_3|L}} \mu(m_1)\mu(m_2)\mu(m_3). \tag{2.2}$$

For every integer $N \geq 1$, we let

$$\zeta_N(s) = \prod_{p|N}\left(1-\frac{1}{p^s}\right)^{-1}, \qquad \zeta^N(s) = \frac{\zeta(s)}{\zeta_N(s)} \tag{2.3}$$

denote the local and partial Riemann zeta functions at N respectively. Similarly, for every Dirichlet character χ we also write

$$L_N(s,\chi) = \prod_{p|N}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}, \qquad L^N(s,\chi) = \frac{L(s,\chi)}{L_N(s,\chi)}. \tag{2.4}$$

For $\epsilon = \pm 1$, we write

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \qquad \Gamma_\epsilon(s) = \begin{cases}\Gamma_{\mathbb{R}}(s), & \text{if } \epsilon = 1;\\ \Gamma_{\mathbb{R}}(s+1), & \text{if } \epsilon = -1.\end{cases} \tag{2.5}$$

Finally, for reference we list some functions that will be used extensively in our future discussions.

$$B_m(s) = \prod_{p|m}\left(1+\frac{3}{p^{2-2s}}-\frac{9}{p}-\frac{3}{p^{3-2s}}+\frac{3}{p^{2s}}+\frac{9}{p^2}-\frac{3}{p^{1+2s}}-\frac{1}{p^3}\right), \tag{2.6}$$

$$B^*_m(s) = \prod_{p|m}\left(\frac{3}{p^{1-s}}+\frac{1}{p^{3-3s}}-\frac{3}{p^s}-\frac{9}{p^{2-s}}+\frac{9}{p^{1+s}}+\frac{3}{p^{3-s}}-\frac{1}{p^{3s}}-\frac{3}{p^{2+s}}\right), \tag{2.7}$$

$$A_m(s,\chi) = \prod_{p|m}\left(1-\frac{\chi(p)^2}{p}+\left(p^{\frac{1-3s}{2}}-p^{\frac{3s-3}{2}}\right)\frac{\chi(p)p^{\frac{3s-3}{2}}}{\zeta_p(s)^3B_p(\frac{2-s}{2})}\right), \tag{2.8}$$

$$A^*_m(s,\chi) = \prod_{p|m}\left(\chi(p)\left(p^{\frac{1-3s}{2}}-p^{\frac{3s-3}{2}}\right)+\left(1-\frac{\chi(p)^2}{p}\right)\frac{p^{\frac{3s-3}{2}}}{\zeta_p(s)^3B_p(\frac{2-s}{2})}\right), \tag{2.9}$$

$$A_N(s,\chi;a) = A_{\frac{N}{a}}(s,\chi)A^*_a(s,\chi), \qquad E_N(s;b) = B_{\frac{N}{b}}(s)B^*_b(s), \tag{2.10}$$

$$R_M(s) = \frac{\zeta(2s)^6\zeta(6s-1)}{\zeta_M(1)\zeta_M(2s)^3\zeta_M(6s-1)}. \tag{2.11}$$

3. Functional Equations

Let M be a positive squarefree even integer and a, b its (positive or negative) coprime divisors, then [1] shows that, for every integers d, n coprime to N, there exist (unique) Dirichlet polynomials $P_d^a(s), Q_n^b(w)$ such that

$$P_{d_0}^a(s) = 1, \qquad Q_{n_0}^b(w) = \tau_3(n_0); \tag{3.1}$$

$$d_1^{3s} P_d^a(s) = d_1^{3(1-s)} P_d^a(1-s), \qquad n_1^w Q_n^b(w) = n_1^{1-w} Q_n^b(1-w); \tag{3.2}$$

$$\sum_{\substack{d=1\\(d,M)=1}}^{\infty} \frac{L^M(s,\chi_{d_0}\chi_a)^3 \chi_b(d_0) P_d^a(s)}{d^w} = \sum_{\substack{n=1\\(n,M)=1}}^{\infty} \frac{L^M(w,\widetilde{\chi}_{n_0}\chi_b)\chi_a(n_0) Q_n^b(w)}{n^s}. \tag{3.3}$$

Now define the multiple Dirichlet series

$$Z_M(s,w;a,b) = \sum_{\substack{d=1\\(d,M)=1}}^{\infty} \frac{L^M(s,\chi_{d_0}\chi_a)^3 \chi_b(d_0) P_d^a(s)}{d^w}, \tag{3.4}$$

then it is known [1] [4] that $Z_M(s,w;a,b)$ has a meromorphic continuation in s, w to everywhere, and that the only possible poles for $Z_M(\frac{1}{2}, w; a, b)$ are at $w = 0, 1, \frac{3}{4}$. Furthermore, $w = 1$ is a pole for $Z_M(s,w;a,b)$ only if $b = 1$, and this pole, if it exists, must be simple with residue

$$\operatorname*{Res}_{w=1} Z_M(s,w;a,1) = \frac{R_M(s)}{\zeta_M(2s)^3} = \frac{\zeta(2s)^6 \zeta(6s-1)}{\zeta_M(1)\zeta_M(2s)^6\zeta_M(6s-1)}, \tag{3.5}$$

where $R_M(s)$ is as defined in (2.11). The possible pole $w = \frac{3}{4}$, which must be simple if it exists, comes from the polar line $w = \frac{3-3s}{2}$ for $Z_M(s,w;a,b)$.

The basic properties of $Z_M(s,w;a,b)$ are the following functional equations, one coming from those of quadratic Dirichlet L-functions $L(s,\chi_{d_0}\chi_a)^3$ and (3.2), and the other coming from the "quadratic reciprocity law" (3.3). They were indicated in [1] [4], but for our purposes we need their exact formulations, so we list them explicitly as follows. The proofs are essentially the same as those for Propositions 4.2 and 4.3 in [4], and we omit them here.

Proposition 1. *The function $Z_M(s,w;a,b)$ satisfies a functional equation*

$$(s,w) \mapsto \left(1-s, w+3s-\frac{3}{2}\right);$$

more precisely, we have

$$\begin{aligned} Z_M(s,w;a,b) = {} & \frac{\zeta_{\frac{M}{a}}(2-2s)^3}{2|a|^{3s-\frac{3}{2}}} \frac{\Gamma_{\delta_a}(1-s)^3}{\Gamma_{\delta_a}(s)^3} \sum_{u,v|\left(\frac{M}{a}\right)^3} \frac{\tau_3(u;\frac{M}{a};\mu)\tau_3(v;\frac{M}{a})}{u^s v^{1-s}} \\ & \times \sum_{\epsilon=\pm1} D_\epsilon(s;a,u,v) Z_M\left(1-s, w+3s-\frac{3}{2}; a, \epsilon(buv)_0\right), \end{aligned} \tag{3.6}$$

where

$$D_\epsilon(s;a,u,v) = \begin{cases} \chi_a(uv)\left(1+\epsilon\frac{2^{3-6s}\chi_{-1}(uv)}{\zeta_2(2-2s)^3}\right), & \text{if } a\equiv 1 \pmod 4;\\ 2^{3-6s}\chi_a(uv)\Big(1+\epsilon\cdot\chi_{-1}(uv)\Big), & \text{if } a\equiv 2 \pmod 4;\\ \chi_{(uv)_0}(a)\left(\frac{2^{3-6s}\chi_{-1}(uv)}{\zeta_2(2-2s)^3}+\epsilon\right), & \text{if } a\equiv -1 \pmod 4.\end{cases}$$

Proposition 2. *The function* $Z_M(s,w;a,b)$ *satisfies a functional equation*

$$(s,w)\mapsto\left(s+w-\frac{1}{2},1-w\right);$$

more precisely, we have

(3.7)

$$Z_M(s,w;a,b) = \frac{\zeta_{\frac{M}{D_b b}}(2-2w)D_b(w)}{2|b|^{w-\frac{1}{2}}}\sum_{u,v\mid\frac{M}{D_b b}}\frac{\mu(u)\chi_b(uv)}{u^w v^{1-w}}\sum_{\epsilon=\pm1}\frac{\Gamma_{\delta_b\epsilon}(1-w)}{\Gamma_{\delta_b\epsilon}(w)}$$
$$\times\left(Z_M\left(s+w-\frac{1}{2},1-w;(auv)_0,b\right)+\epsilon\cdot Z_M\left(s+w-\frac{1}{2},1-w;-(auv)_0,b\right)\right),$$

where

$$D_b(w)=\begin{cases}1, & \text{if } b\equiv 1 \pmod 4;\\ 2^{1-2w}, & \text{if } b\not\equiv 1 \pmod 4,\end{cases}\qquad D_b=\begin{cases}2, & \text{if } b\equiv -1 \pmod 4;\\ 1, & \text{if } b\not\equiv -1 \pmod 4.\end{cases}$$

One may observe that these functional equations imply that $Z_M(\frac{1}{2},w;a,b)$ has a possible simple pole at $w=\frac{3}{4}$. In fact, as shown in [4], $Z_M(s,w;a,b)$ can be analytically continued to everywhere through these functional equations, so its poles can only come from those transformed during this process from the original polar line $w=1$. Now (3.5) shows that $Z_M(s,w;a,b)$ has a simple pole at $(s,w)=(\frac{1}{4},1)$. The first functional equation (3.6) transforms this simple pole to $(s,w)=(\frac{3}{4},\frac{1}{4})$, and then the second functional equation (3.7) takes it to $(s,w)=(\frac{1}{2},\frac{3}{4})$. It is easy to check that this is the only origin for the possible pole at $w=\frac{3}{4}$ for $Z_M(\frac{1}{2},w;a,b)$, so the only question that remains is whether $w=\frac{3}{4}$ is indeed a pole, i.e., whether the residue of $Z_M(\frac{1}{2},w;a,b)$ at $w=\frac{3}{4}$ is nonzero.

4. Computation of Residues

In this section, we apply the above functional equations to compute the residue of $Z_M(s,w;a,b)$ at $w=\frac{3-3s}{2}$; in particular, this gives the residue of $Z_M\left(\frac{1}{2},w;a,b\right)$, and so that of $Z\left(\frac{1}{2},w\right)$, at $w=\frac{3}{4}$.

To begin with, we apply (3.5) to Proposition 1.

Proposition 3. *We have*

$$\operatorname*{Res}_{w=\frac{5}{2}-3s} Z_M(s,w;a,b) = \frac{1}{2|a|^{3s-\frac{3}{2}}}\frac{\Gamma_{\delta_a}(1-s)^3}{\Gamma_{\delta_a}(s)^3}\frac{R_M(1-s)E_{\frac{M}{a}}(s;b)D_b(s;a)}{\zeta_a(2-2s)^3},$$

where $R_M(s)$ is as defined in (2.11) *and*

$$D_b(s;a) = \begin{cases} \chi_a(|b|)\left(1+\delta_b \frac{2^{3-6s}\chi_{-1}(|b|)}{B_2(s)\zeta_2(2-2s)^3}\right), & \text{if } a \equiv 1 \pmod 4; \\ 2^{3-6s}\chi_a(|b|)\left(1+\delta_b\chi_{-1}(|b|)\right), & \text{if } a \equiv 2 \pmod 4; \\ \chi_a(|b|)\left(\delta_b\chi_{-1}(|b|)+\frac{2^{3-6s}}{B_2(s)\zeta_2(2-2s)^3}\right), & \text{if } a \equiv -1 \pmod 4, 2\nmid b; \\ \delta_b\chi_{-a}(\frac{|b|}{2})\chi_2(a), & \text{if } a \equiv -1 \pmod 4, 2|b. \end{cases}$$

Proof. First assume that $a \equiv -1 \pmod 4$, then (3.6) gives

$$Z_M(s,w;a,b) = \frac{\zeta_{\frac{M}{a}}(2-2s)^3}{2|a|^{3s-\frac{3}{2}}} \frac{\Gamma_{\delta_a}(1-s)^3}{\Gamma_{\delta_a}(s)^3} \sum_{u,v|(\frac{M}{2a})^3} \frac{\tau_3(u;\frac{M}{2a};\mu)\tau_3(v;\frac{M}{2a})\chi_{(uv)_0}(a)}{u^s v^{1-s}}$$

$$\times\left(\sum_{\epsilon=\pm1}\left(\frac{2^{3-6s}\chi_{-1}(uv)}{\zeta_2(2-2s)^3}+\epsilon B_2(s)\right) Z_M\left(1-s, w+3s-\frac{3}{2}; a, \epsilon(buv)_0\right)\right.$$

$$\left. + \chi_2(a)B_2^*(s)\sum_{\epsilon=\pm1}\epsilon\cdot Z_M\left(1-s,w+3s-\frac{3}{2};a,2\epsilon(buv)_0\right)\right).$$

If $2 \nmid b$, then we have

$$\operatorname*{Res}_{w=\frac{5}{2}-3s} Z_M(s,w;a,b) = \frac{\chi_a(|b|)}{2|a|^{3s-\frac{3}{2}}} \frac{\Gamma_{\delta_a}(1-s)^3}{\Gamma_{\delta_a}(s)^3} \frac{R_M(1-s)}{\zeta_a(2-2s)^3}\left(\frac{2^{3-6s}}{\zeta_2(2-2s)^3}+\delta_b\chi_{-1}(|b|)B_2(s)\right)$$

$$\times \sum_{\substack{u,v|(\frac{M}{2a})^3 \\ |buv|=\square}} \frac{\tau_3(u;\frac{M}{2a};\mu)\tau_3(v;\frac{M}{2a})}{u^s v^{1-s}}$$

$$= \frac{\chi_a(|b|)}{2|a|^{3s-\frac{3}{2}}} \frac{\Gamma_{\delta_a}(1-s)^3}{\Gamma_{\delta_a}(s)^3} \frac{R_M(1-s)E_{\frac{M}{2a}}(s;b)}{\zeta_a(2-2s)^3}\left(\frac{2^{3-6s}}{\zeta_2(2-2s)^3}+\delta_b\chi_{-1}(|b|)B_2(s)\right)$$

$$= \frac{\chi_a(|b|)}{2|a|^{3s-\frac{3}{2}}} \frac{\Gamma_{\delta_a}(1-s)^3}{\Gamma_{\delta_a}(s)^3} \frac{R_M(1-s)E_{\frac{M}{a}}(s;b)}{\zeta_a(2-2s)^3}\left(\frac{2^{3-6s}}{\zeta_2(2-2s)^3 B_2(s)}+\delta_b\chi_{-1}(|b|)\right).$$

This proves the proposition in the present case. The other cases can be studied similarly, so we omit the details. □

To determine the residues along the polar line $w = \frac{3-3s}{2}$, we apply Proposition 2 upon Proposition 3.

Proposition 4. *We have*

$$\operatorname*{Res}_{w=\frac{3-3s}{2}} Z_M(s,w;a,b) = \frac{\zeta_{\frac{M}{b}}(3s-1)}{8|b|^{\frac{2-3s}{2}}} R_M\left(\frac{s}{2}\right) E_M\left(\frac{2-s}{2};b\right) C(s;a,b), \tag{4.1}$$

where, for $\delta, \epsilon = \pm1$, we write

$$\Gamma_\delta^\epsilon(s) = \frac{\Gamma_{\delta\epsilon}(\frac{s}{2})^3}{\Gamma_{\delta\epsilon}(\frac{2-s}{2})^3}\left(\frac{\Gamma_+(\frac{3s-1}{2})}{\Gamma_+(\frac{3-3s}{2})}+\epsilon\frac{\Gamma_-(\frac{3s-1}{2})}{\Gamma_-(\frac{3-3s}{2})}\right), \qquad G_\delta(s) = \Gamma_\delta^+(s)+\Gamma_\delta^-(s)$$

and $C(s;a,b)$ *equals to*

$$\begin{cases} \chi_a(|b|)A_{\frac{M}{2b}}(s,1;a)G_{\delta_a}(s)\left(\frac{2^{3s-3}+2^{3s-4}-2^{6s-5}}{\zeta_2(s)^3B_2(1-\frac{s}{2})}+\frac{1}{2}\right), & \text{if } b\equiv 1 \pmod 4,\ 2\nmid a;\\ \chi_a(|b|)A_{\frac{M}{2b}}(s,1;\frac{a}{2})G_{\delta_a}(s)\left(2^{\frac{1-3s}{2}}-2^{\frac{3s-3}{2}}+\frac{2^{\frac{3s-5}{2}}}{B_2(1-\frac{s}{2})\zeta_2(s)^3}\right) & \text{if } b\equiv 1 \pmod 4,\ 2|a;\\ \frac{\chi_b(|a|)}{2^{3-3s}}G_{\delta_a}(s)\sum_{\epsilon=\pm1}\chi_\epsilon(a)(1+\epsilon\delta_b)A_{\frac{M}{b}}(s,\chi_\epsilon;a), & \text{if } b\equiv 2 \pmod 4;\\ \frac{\chi_a(|b|)\chi_{-1}(a)G_{\delta_a}(s)A_{\frac{M}{2b}}(s,1;a)}{2^{2-3s}\zeta_2(3s-1)}\left(1-\frac{2^{3s-3}}{\zeta_2(s)^3B_2(1-\frac{s}{2})}\right), & \text{if } b\equiv -1 \pmod 4. \end{cases}$$

Proof. If $b\equiv 1 \pmod 4$, we may check that $\chi_{(auv)_0}(|b|)\chi_b(uv)=\chi_a(|b|)$ and $\chi_{-1}(|b|)=\delta_b$, so combining Propositions 2 and 3 we have

$$\begin{aligned}\operatorname*{Res}_{w=\frac{3-3s}{2}} & Z_M(s,w;a,b)\\ &= \frac{\zeta_{\frac{M}{b}}(3s-1)}{8|b|^{\frac{2-3s}{2}}}\sum_{u,v|\frac{M}{b}}\frac{\mu(u)\chi_b(uv)}{u^{\frac{3-3s}{2}}v^{\frac{3s-1}{2}}}\frac{R_M(\frac{s}{2})E_{\frac{M}{(auv)_0}}(1-\frac{s}{2};b)}{|(auv)_0|^{\frac{3-3s}{2}}\zeta_{(auv)_0}(s)^3}\sum_{\epsilon=\pm1}\frac{\Gamma_{\delta_b\epsilon}(\frac{3s-1}{2})}{\Gamma_{\delta_b\epsilon}(\frac{3-3s}{2})}\\ &\times\left(\frac{\Gamma_{\delta_a}(\frac{s}{2})^3D_b(1-\frac{s}{2};(auv)_0)}{\Gamma_{\delta_a}(1-\frac{s}{2})^3}+\epsilon\frac{\Gamma_{-\delta_a}(\frac{s}{2})^3D_b(1-\frac{s}{2};-(auv)_0)}{\Gamma_{-\delta_a}(1-\frac{s}{2})^3}\right)\end{aligned}$$

whence in (4.1) we have

$$\begin{aligned}C(s;a,b) = &\sum_{u,v|\frac{M}{b}}\frac{\mu(u)\chi_b(uv)}{u^{\frac{3-3s}{2}}v^{\frac{3s-1}{2}}}\frac{|(auv)_0|^{\frac{3s-3}{2}}}{B_{(auv)_0}(1-\frac{s}{2})\zeta_{(auv)_0}(s)^3}\\ &\times\left(\Gamma^+_{\delta_a}(s)D_b\left(1-\frac{s}{2};(auv)_0\right)+\delta_b\Gamma^-_{\delta_a}(s)D_b\left(1-\frac{s}{2};-(auv)_0\right)\right).\end{aligned}$$

In case $2\nmid a$, this gives

$$\begin{aligned}C(s;a,b) &= \sum_{\substack{u,v|\frac{M}{b}\\ 2|(auv)_0}}+\sum_{\substack{u,v|\frac{M}{b}\\ 2\nmid(auv)_0}} = \sum_{\substack{u,v|\frac{M}{b}\\ 2\|uv}}+\frac{1}{2}\sum_{u,v|\frac{M}{2b}}\\ &= \sum_{u,v|\frac{M}{2b}}\frac{\chi_a(|b|)\mu(u)|(auv)_0|^{\frac{3s-3}{2}}G_{\delta_a}(s)}{u^{\frac{3-3s}{2}}v^{\frac{3s-1}{2}}B_{(auv)_0}(1-\frac{s}{2})\zeta_{(auv)_0}(s)^3}\left(\frac{2^{3s-3}+2^{3s-4}-2^{6s-5}}{\zeta_2(s)^3B_2(1-\frac{s}{2})}+\frac{1}{2}\right)\\ &= \chi_a(|b|)A_{\frac{M}{2b}}(s,1;a)G_{\delta_a}(s)\left(\frac{2^{3s-3}+2^{3s-4}-2^{6s-5}}{\zeta_2(s)^3B_2(1-\frac{s}{2})}+\frac{1}{2}\right).\end{aligned}$$

Hence the proposition is proved in the case that $b\equiv 1 \pmod 4$ and $2\nmid a$. The other cases can be proved similarly. □

To compute the residue of $Z(\frac{1}{2},w)$ at $w=\frac{3}{4}$, the last ingredient is the relation between $Z_M(s,w;a,b)$ and $Z(s,w)$.

Proposition 5. *Let $Z(s,w;a,b)$ be as defined in* (1.4). *Then we have*

(4.2)
$$\begin{aligned}Z(s,w) &= \frac{(1-\frac{1}{2^s})^{-3}+(1+\frac{1}{2^s})^{-3}+2^{1-2w}}{4}\Big(Z(s,w;1,1)+Z(s,w;-1,1)\Big)\\ &+\frac{(1-\frac{1}{2^s})^{-3}+(1+\frac{1}{2^s})^{-3}-2^{1-2w}}{4}\Big(Z(s,w;1,-1)-Z(s,w;-1,-1)\Big)\\ &+\frac{(1-\frac{1}{2^s})^{-3}-(1+\frac{1}{2^s})^{-3}}{4}\Big(Z(s,w;1,2)+Z(s,w;1,-2)+Z(s,w;-1,2)\\ &-Z(s,w;-1,-2)\Big)+2^{-3w}\Big(Z(s,w;2,1)+Z(s,w;-2,1)\Big).\end{aligned}$$

Proof. By definition (1.3), we have

$$Z(s,w)=\sum_{D \text{ fund. disc.}}\left(1-\frac{\chi_D(2)}{2^s}\right)^{-3}\frac{L^2(s,\chi_D)^3}{|D|^w}.$$

Now (4.2) follows easily from a case-by-case study for the sums over fundamental discriminants D with given sign and given congruence class modulo 8. □

Proposition 6. *Let $a,b\in\mathbb{Z}$ with $ab=\pm1,\pm2$, and assume that the conditions in Theorem 1 are satisfied. Then*

(4.3)
$$\begin{aligned}\operatorname*{Res}_{w=\frac{3}{4}} Z\left(\frac{1}{2},w;a,b\right) &= \sum_{\substack{r=1\\2\nmid r}}^{\infty}\mu(r)\sum_{\ell\mid r}\mu(\ell)\sum_{\delta\mid\ell}\frac{\chi_b(\delta)\zeta_{\frac{\ell}{\delta}}(1)^3}{\delta^{\frac{3}{4}}}\sum_{m\mid(\frac{\ell}{\delta})^3}\frac{\tau_3(m;\frac{\ell}{\delta})\chi_{a\delta}(m_0)}{m^{\frac{1}{2}}}\\ &\times\sum_{\epsilon=\pm1}\frac{1+\epsilon\chi_{-1}(m_0)}{2}\operatorname*{Res}_{w=\frac{3}{4}} Z_{2\ell}\left(\frac{1}{2},w;a\delta,\epsilon b m_0\right).\end{aligned}$$

Proof. By our assumption (1.6) in Theorem 1, we have

$$\operatorname*{Res}_{w=\frac{3}{4}} Z\left(\frac{1}{2},w;a,b\right)=\sum_{\substack{r=1\\2\nmid r}}^{\infty}\mu(r)\operatorname*{Res}_{w=\frac{3}{4}} Z\left(\frac{1}{2},w;a,b;r\right). \tag{4.4}$$

Furthermore, we have [4]

$$\begin{aligned}Z(s,w;a,b;r) &= \sum_{\ell\mid r}\mu(\ell)\zeta_\ell(2s)^3\sum_{\delta\mid\ell}\frac{\chi_b(\delta)}{\delta^w\zeta_\delta(2s)^3}\sum_{m\mid(\frac{\ell}{\delta})^3}\frac{\tau_3(m;\frac{\ell}{\delta})\chi_{a\delta}(m_0)}{m^s}\\ &\times\sum_{\epsilon=\pm1}\frac{1+\epsilon\chi_{-1}(m_0)}{2}Z_{2\ell}(s,w;a\delta,\epsilon b m_0).\end{aligned}$$

Combining this with (4.4) gives the proposition. □

Finally, we are ready to explicitly compute the residue of $Z(\frac{1}{2},w)$ at $w=\frac{3}{4}$.

Proof of Theorem 1. The decomposition (4.2) readily gives the meromorphic continuation for $Z(\frac{1}{2}, w)$ up to $\Re w > \sigma_0$, so we focus on the computation of its residue at $w = \frac{3}{4}$.

Theoretically, we can directly compute this residue via Proposition 4 by taking $s = \frac{1}{2}$ there and then applying the sieving equalities (4.2), (4.3). To simplify this process, we observe that many terms given in Proposition 4 contribute trivially to the residue of $Z(\frac{1}{2}, w)$ at $w = \frac{3}{4}$. More precisely, by Propositions 4 and 6, the residue of $Z\left(\frac{1}{2}, w; a, b\right)$, and so that of $Z\left(\frac{1}{2}, w\right)$, at $w = \frac{3}{4}$ is a linear combination of terms of form

$$R(\chi, \chi', \chi'') = \lim_{\epsilon \to 0} \sum_{\substack{r=1 \\ 2 \nmid r}}^{\infty} \frac{\mu(r)}{r^{\epsilon}} \sum_{\ell | r} \frac{\mu(\ell)\zeta_\ell(1)^2}{\zeta_\ell(\frac{1}{2})^4} \sum_{\delta | \ell} \frac{\chi(\delta)}{\delta^{\frac{3}{4}} \zeta_\delta(1)^3} \sum_{m | (\frac{\ell}{\delta})^3} \frac{\tau_3(m; \frac{\ell}{\delta})\chi'(m)}{m^{\frac{1}{2}} m_0^{\frac{1}{4}}}$$
$$\times \zeta_{\frac{\ell}{m_0}}\left(\frac{1}{2}\right) E_\ell\left(\frac{3}{4}; m_0\right) A_{\frac{\ell}{m_0}}\left(\frac{1}{2}, \chi''; \delta\right),$$

where χ, χ', χ'' are some quadratic characters of conductors dividing 8, and the factor $\frac{1}{r^\epsilon}$ is inserted as the convergence factor. Now it is easy to see that

$$R(\chi, \chi', \chi'') = \lim_{\epsilon \to 0} \prod_{p \neq 2} \left(1 + \frac{9\chi'(p) + \chi''(p) + \chi(p)\chi''(p) - 11}{p^{1+\epsilon}} + O\left(\frac{1}{p^{\frac{3}{2}}}\right)\right)$$
$$= c \lim_{\epsilon \to 0} \frac{L(1+\epsilon, \chi')^9 L(1+\epsilon, \chi'') L(1+\epsilon, \chi\chi'')}{\zeta(1+\epsilon)^{11}},$$

where $c > 0$ is some constant. Hence $R(\chi, \chi', \chi'') = 0$ unless $\chi = \chi' = \chi'' = 1$. In fact, the constant P defined in (1.9) is just $\frac{\zeta(\frac{1}{2})^7}{\zeta_2(\frac{1}{2})^3} R(1,1,1)$. With this at hand, we can apply Propositions 4 and 6 to compute the residue of $Z(\frac{1}{2}, w; a, b)$, safely discarding those terms which will introduce nontrivial characters to the summation. A direct computation, combined with (4.2), shows that

$$\operatorname*{Res}_{w=\frac{3}{4}} Z\left(\frac{1}{2}, w\right) = \frac{223\sqrt{2} - 253}{256} P \left(\frac{\Gamma_{\mathbb{R}}(\frac{1}{4})^4}{\Gamma_{\mathbb{R}}(\frac{3}{4})^4} + \frac{\Gamma_{\mathbb{R}}(\frac{1}{4})\Gamma_{\mathbb{R}}(\frac{5}{4})^3}{\Gamma_{\mathbb{R}}(\frac{3}{4})\Gamma_{\mathbb{R}}(\frac{7}{4})^3}\right) \approx -0.162.$$

This completes the proof. □

5. Proof of Theorem 2

Finally, we are ready to prove our Theorem 2.

Proof of Theorem 2. By [4], we see that $Z(\frac{1}{2}, w) \ll 1$ if $\Re w > 1$, so

$$\sum_{\substack{|D| \leq x \\ D \text{ fund. disc.}}} L\left(\frac{1}{2}, \chi_D\right)^3 = \frac{1}{2\pi i} \int_{1+\epsilon-iT}^{1+\epsilon+iT} \frac{Z\left(\frac{1}{2}, w\right) x^w}{w} \, \mathrm{d}w + O\left(\frac{x^{1+\epsilon}}{T}\right).$$

Now shift the integration line from $\Re w = 1 + \epsilon$ to $\Re w = \sigma_0 + \epsilon$. It is well-known [4] that $Z\left(\frac{1}{2}, w\right)$ has a pole at $w = 1$ of order 7, and we have shown that it also has a simple pole at $w = \frac{3}{4}$, so our growth condition (1.10) gives

$$\sum_{\substack{|D|\le x \\ D \text{ fund. disc.}}} L\left(\frac{1}{2}, \chi_D\right)^3$$
$$= xR_3(\log x) + \frac{4x^{\frac{3}{4}}}{3} \operatorname*{Res}_{w=\frac{3}{4}} Z\left(\frac{1}{2}, w\right) + O\left(T^{r+\epsilon} x^{\sigma_0+\epsilon} + \frac{x^{1+\epsilon}}{T}\right).$$

where R_3 is a polynomial of degree 6. Now if we take $T = x^{\frac{1-\sigma_0}{r+1}}$, then this gives

$$\sum_{\substack{|D|\le x \\ D \text{ fund. disc.}}} L\left(\frac{1}{2}, \chi_D\right)^3 = xR_3(\log x) + bx^{\frac{3}{4}} + O\left(x^{\frac{r+\sigma_0}{r+1}+\epsilon}\right), \tag{5.1}$$

where

$$b = \frac{4}{3} \operatorname*{Res}_{w=\frac{3}{4}} Z\left(\frac{1}{2}, w\right) \approx -0.215.$$

Note that our assumption on r guarantees $\frac{r+\sigma_0}{r+1} < \frac{3}{4}$. This completes the proof of the theorem. □

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