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DEFORMATIONS OF COVERS, BRILL-NOETHER THEORY, AND WILD RAMIFICATION

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Abstract. In this paper, we give a simple description of the deformations of a map between two smooth curves with partially prescribed branching, in the cases that both curves are fixed, and that the source is allowed to vary. Both descriptions work equally well in the tame or wild case. We then apply this result to obtain a positive-characteristic Brill-Noether-type result for ramified maps from general curves to the projective line, which even holds for wild ramification indices. Lastly, in the special case of rational functions on the projective line, we examine what we can say as a result about families of wildly ramified maps.

1. Introduction

In studying ramified maps of curves, questions frequently arise which demand fixing ramification on the source, or branching on the target. In the case that the target curve is $\mathbb{P}^1$, the former is treated by the theory of linear series, which naturally works up to automorphism of the image, so we will refer to this as the linear series perspective. In contrast, we will refer to working with fixed branching on the target (and typically allowing the source curve itself to vary) as the branched covers perspective. Often, and particularly in the context of degeneration arguments over $\mathbb{C}$ [1], these perspectives have been considered to be more or less interchangeable. However, recently a number of fundamental differences have come to light (see for instance [3, Prop. 5.4, Rem. 8.3]), particularly in positive characteristic, and it has also proven fruitful to pass between the perspectives to take advantages of the distinct features of each, perhaps most notably in Tamagawa's [4]. In this note, we examine the deformation theory of covers with partial branching specified, and then apply it to the perspective of linear series to obtain a ramified Brill-Noether theorem in positive characteristic, in the case of one-dimensional target.

Our deformation theory result is straightforward to obtain from extremely well-known results, but does not appear to be stated in the literature. It is the following.

Theorem 1.1. *Given C, D smooth curves over a field k, and $f : C \to D$ of degree d, together with k-valued points $P_1, \dots, P_n$ of C such that f is ramified to*

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order at least e_i at each P_i for some e_i, then the space of first-order infinitesmal deformations of f together with the P_i over k, such that $f(P_i)$ remains fixed and the P_i remain ramified to order at least e_i, is parametrized by

$$H^0(C, f^*T_D(-\sum_i (e_i - \delta_i)P_i)) \oplus k^{\sum_i (1-\delta_i)}, \tag{1.1}$$

where $\delta_i = 0$ if $p | e_i$ or f is ramified to order higher than e_i at P_i and is 1 otherwise.

Furthermore, the space of first-order infinitesmal deformations of C, the P_i and f, fixing $f(P_i)$ and preserving the ramification condition at each P_i, is parametrized by

$$\mathbb{H}^1(C, T_C(-\sum_i P_i) \to f^*T_D(-\sum_i e_i P_i)) \cong k^{d(2-2g_D)-(2-2g_C)-\sum_i (e_i - 1)}, \tag{1.2}$$

where the last isomorphism requires also that f be separable.

Finally, both statements still hold when some e_i are allowed to be 0, which we interpret to put no condition on the P_i at all.

In the case that f is tame and ramified to order exactly e_i at the P_i, this is well-known. The main observation of the theorem, particularly in the first case, is that in order to obtain a useful theory, it is important to also consider the moduli of the ramification points, even when the branch points remain fixed. In the classical setting, this issue does not arise.

Next, if one considers the situation with $D = \mathbb{P}^1$, and ramification points specified on C, the perspective changes from branched covers to linear series with prescribed ramification. From this point of view, classical Brill-Noether theory gives a lower bound on the dimension of the space of maps as the P_i (and even C) are allowed to move. The deformation theory of Theorem 1.1 gives the necessary upper bound, and allows us to conclude the following Brill-Noether result, generalizing [1, Thm. 4.5] to positive characteristic in the case $r = 1$.

Theorem 1.2. *Fix d, n and $e_1, \dots, e_n$, together with n general points P_i on a general curve C of genus g. Then the space of separable maps of degree d from C to $\mathbb{P}^1$, ramified to order at least e_i at P_i, and taken modulo automorphism of the image space, is pure of dimension $2d - 2 - g - \sum_i (e_i - 1)$.*

Finally, we note that although from the perspective of branched covers, tame ramification is always well-behaved and wild ramification seems more pathological, the situation is not the same from the perspective of ramified linear series. Indeed, from this perspective a simple dimension count justifies the fact that wildly-ramified maps always come in infinite families. On the other hand, we have examples from [3, Prop. 5.4] of cases where tame ramification could only produce infinite families of separable maps with fixed ramification, and as a result of our Brill-Noether theorem, can exist only for special configurations of P_i. We make some elementary observations that, at least in certain cases when $C = D = \mathbb{P}^1$, wildly ramified linear series are in fact rather well-behaved. One such result is the following.

Theorem 1.3. *Fix d, n, m, together with n general points P_i on $\mathbb{P}^1$, and $e_1, \ldots, e_n$, with e_i wild for $i \leq m$, and e_i tame for $i > m$, and satisfying $2d - 2 = m + \sum_i (e_i - 1)$. Then the dimension of the space of separable maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^1$, ramified to order exactly e_i at P_i and unramified elsewhere, and taken modulo automorphism of the image space, is exactly m. Moreover, if $m = 1$, $e_1 = p$, and $e_i < p$ for $i > 1$, this space is non-empty if and only if the corresponding space is non-empty when one replaces $e_1 = p$ with $e_1 = p - 1$, and considers maps of degree $d - 1$.*

Note that except when explicitly stated otherwise, we make no assertions about the non-emptyness of the space of maps with given ramification. However, the last statement of Theorem 1.3 certainly produces cases of wild ramification in which for general P_i, the space of maps is non-empty of the expected dimension. This is thus better behavior than the pathological tame examples mentioned above. The subject of existence and non-existence will be taken up in [2]. Although the proof given here of Theorem 1.2 is not necessary for [2], it does provide the only purely algebraic, instrinsically characteristic-p argument for the existence and non-existence results in question.

2. Deformation theory

Let C, D be smooth curves over a field k, and $f : C \to D$ a morphism of degree $d > 0$. We begin by reviewing some standard deformation theory, so that we can use formal local analysis to obtain Theorem 1.1. It is well-known (see, e.g., [5, Appendix]) that the first-order infinitesmal deformations of f are parametrized by $H^0(C, f^*(T_D))$, deformations of a pointed curve $(C, \{P_i\})$ by $H^1(C, T_C(-\sum_i P_i))$, and deformations of $(C, \{P_i\}, f)$ by the hypercohomology group $\mathbb{H}^1(C, T_C(-\sum_i P_i) \to f^*T_D)$. By the same token, deformations of k-valued points P_i are parametrized simply by $H^0(\operatorname{Spec} k, f^*(T_C)) \cong k$ (note that this is different from the case of deforming C along with the P_i because in this case, there are no automorphisms of C to mod out by). These may be verified directly on the Cech cocycle level using the facts that T_C is the sheaf of infinitesmal automorphisms of C, and that deformations of smooth, pointed curves are always locally trivial. In the case of pointed curves, one trivializes the deformation (including of the points) locally on C, and obtains the 1-cocycle by considering the resulting transition functions, taking values in infinitesmal automorphisms; in order to give well-defined deformations of the P_i, these transition functions must vanish along them.

To prove Theorem 1.1, we therefore simply need to determine the locus inside $H^0(C, f^*(T_D)) \oplus k^n$ corresponding to maps which fix $f(P_i)$ and preserve the ramification at P_i, and similarly for $\mathbb{H}^1(C, T_C(-\sum_i P_i) \to f^*T_D)$. We accomplish this easily by formal local analysis.

Proof of Theorem 1.1. Since the first statement we are trying to prove gives a self-contained, purely local description of a subspace of $H^0(C, f^*(T_D)) \oplus k^n$, and fixing $f(P_i)$ and the ramification at P_i are likewise purely local conditions, it suffices to check agreement formally locally around each P_i and $f(P_i)$. Accordingly, let s, t be formal coordinates at P_i, $f(P_i)$ respectively. In terms of these coordinates, we then have $f(s) = \sum_{j \geq 0} a_j s^j$ for some $a_j \in k$ with $a_j = 0$ for $j < e_i$. First, a deformation of f will be of the form $\tilde{f}(s) = f(s) + \epsilon \sum_{j \geq 0} b_j s^j$, with the vanishing order of the b_j being the vanishing order of the section of $H^0(C, f^*(T_D))$ inducing the deformation. Since P_i corresponds to $s = 0$, a deformation of P_i can be written simply as ϵx for $x \in k$.

If we fix both P_i and $f(P_i)$, then requiring that ramification of order e_i be preserved is simply equivalent to requiring that $b_j = 0$ for $j < e_i$. If we fix $f(P_i)$, but allow P_i to move, the condition that $f(P_i)$ is fixed is simply $\tilde{f}(\epsilon x) = 0$, while the condition that f remain ramified to order at least e_i at P_i may be expressed by vanishing to order at least e_i of $\tilde{f}$ when expanded around $s - \epsilon x$, with the possible exception of a non-zero constant term. Thus, both conditions together are equivalent to vanishing to order at least e_i of this Taylor expansion, which yields $\tilde{f}(s) = \sum_{j \geq 0} (a_j + \epsilon(b_j + (j+1)a_{j+1}x))(s - \epsilon x)^j$. We see that for the first $e_i - 1$ terms to vanish, we need $b_j = 0$. For the e_ith term, which is $j = e_i - 1$, we need $b_{e_i - 1} + (e_i)a_{e_i}x = 0$. Now, if $e_i a_{e_i}$ is non-zero, we see that b_{e_i-1} may be chosen arbitrarily, and uniquely determines x, giving the classical case of the theorem, where $\delta = 1$. On the other hand, if $e_i a_{e_i} = 0$, then we must have $b_{e_i - 1} = 0$, but x can be arbitrary, giving the $\delta = 0$ case and completing the proof of the theorem.

Given the deformation-theoretic machinery we have already recalled, the second statement of the theorem is even easier. Indeed, in our formal-local trivialization, we assume by construction that the sections also correspond to the trivial deformation, so that we are in the case above that we have fixed both P_i and $f(P_i)$, where we already noted we simply find that $b_j = 0$ for all $j < e_i$, which is equivalent to saying that our 0-cochain of f^*T_D must vanish to order e_i at P_i, as desired. If f is separable, the map $T_C(-\sum_i P_i) \to f^*T_D(-\sum_i e_i P_i))$ is injective, with cokernel equal to the skyscraper sheaf of length $\delta_{P_i} - (e_i - 1)$ at each P_i, where δ_{P_i} is the order of the different of f at P_i. The Riemann-Hurwitz formula then gives the desired value for the dimension of the deformation space.

Finally, it is also clear from our construction that both statements work when some $e_i = 0$ and no condition is placed on the corresponding P_i. $\square$

Remark 2.1. The proof of the first part of the theorem shows why the deformation theory of maps with prescribed ramification, rather than prescribed branching, is poorly behaved: only by adding the condition that $f(P_i)$ remains fixed do we get an honest vanishing condition on the cocycle of $f^*(T_D)$ determining the deformation of f, allowing our cohomological interpretation.

3. Brill-Noether theory

In this section, we prove Theorem 1.2, using classical Brill-Noether theory together with Theorem 1.1. Given two smooth curves C, D over a finite-type k-scheme S, and integers d, n and $e_1, \dots e_n$, we have a moduli scheme $MR :=$

$MR(C, D, d, e_1, \dots e_n)$ parametrizing $(n+1)$-tuples $(f, P_1, \dots P_n)$, where $f : C \to D$ is a separable morphism of degree d, the P_i are distinct sections of C, and f is ramified to order at least e_i at P_i (and possibly elsewhere); see [3, Appendix]. This comes with natural forgetful morphisms $\mathrm{ram} : MR \to C^n$ and $\mathrm{branch} : MR \to D^n$ giving the portion of the ramification and branch loci of the map f which is mandated by the definition of MR; that is to say, the P_i and $f(P_i)$ respectively. If we fix a scheme-valued point b in D^n, $\mathrm{branch}^{-1}(b)$ is then the locus of maps $f : C \to D$ with the specified branching above each of the n points corresponding to b.

Recall that in the case that $D = \mathbb{P}^1$, MR admits a natural quotient scheme $\overline{MR}$ which parametrizes the appropriate linear series on C, together with ramification sections; that is to say, $\overline{MR}$ represents the quotient functor obtained simply by modding out by postcomposition with $\mathrm{Aut}(\mathbb{P}^1)$. This may be realized as a classical relative G^r_d scheme over the base C^n, with prescribed ramification at the corresponding n sections. Since this group action fixes ram, we have that ram factors through $\overline{MR}$.

We present the $g = 0$ case of Theorem 1.2 first, as a simpler and more direct illustration of the general idea, using the first statement of Theorem 1.1 directly. We thus specialize to the case that $C = \mathbb{P}^1$. It is not hard to see that separable maps from $\mathbb{P}^1$ to itself of degree d are parametrized by an open subscheme of the Grassmannian $\mathbb{G}(1, d)$, and a ramification condition of order e_i corresponds to a Schubert cycle of codimension $e_i - 1$; see, e.g., [3, §2]. Moreover, this description works in the relative setting, so we conclude that $\overline{MR}$ has codimension $\sum_i (e_i - 1)$ in the trivial $\mathbb{G}(1, d)$ bundle over $C^n = (\mathbb{P}^1)^n$. With these observations, we may easily prove the theorem.

Proof of Theorem 1.2, $g = 0$ case. Since $\mathbb{G}(1, d)$ is smooth of relative dimension $2d-2$ over $(\mathbb{P}^1)^n$, $\overline{MR}$ has dimension at least $n+2d-2-\sum_i(e_i-1)$, and it follows that $\dim MR = \dim \overline{MR} + \dim \mathrm{Aut}(\mathbb{P}^1) \geq n + 2d + 1 - \sum_i (e_i - 1)$. On the other hand, by Theorem 1.1 if we are given an $f \in MR$ the dimension of the tangent space of its fiber of branch is $h^0(\mathbb{P}^1, f^* T_{\mathbb{P}^1}(-\sum_i (e_i - \delta_i))) + \sum_i (1 - \delta_i)$ where $\delta_i = 0$ if $p | e_i$ or f is ramified to order higher than e_i at P_i and is 1 otherwise. Since $T_{\mathbb{P}^1} \cong \mathcal{O}(2)$, we have $f^* T_{\mathbb{P}^1}(-\sum_i (e_i - \delta_i)) \cong \mathcal{O}(2d - \sum_i (e_i - \delta_i))$. If $2d - \sum_i (e_i - \delta_i)$ is negative, Riemann-Hurwitz implies that MR is empty, and otherwise we find that our h^0 is given by $2d + 1 - \sum_i (e_i - \delta_i)$, and the dimension of our tangent space by $2d + 1 - \sum_i (e_i - 1)$. Thus MR has dimension at most $n + 2d + 1 - \sum_i (e_i - 1)$, and this must give its dimension precisely. The theorem then follows. □

We now consider the higher-genus case, assuming initially that $g \geq 2$. Instead of working over $\operatorname{Spec} k$, we let our base S be a scheme étale over the moduli stack $\mathcal{M}_{g,0}$ over k, and let C be the corresponding universal curve over S. Even in this relative setting, if we twist by a sufficiently ample divisor D on C (since $g \geq 2$, we could use high powers of the canonical sheaf), and then impose vanishing along D, we can realize $\overline{MR}$ as a closed subscheme of a Grassmannian bundle over $\mathrm{Pic}_S(C) \times_S C^n$, with the ramification conditions corresponding to relative Schubert cycles. This construction is carried out in the more general setting of limit linear series on families of curves of compact type by Eisenbud and Harris in the proof of [1, Thm. 3.3].

Proof of Theorem 1.2, $g > 0$ case. The above classical description again gives a dimensional lower bound for $\overline{MR}$, this time as $(n + 3g - 3) + (2d - 2 - g) - (\sum_i (e_i - 1))$, giving that $\dim MR \geq n + 2g + 2d - 2 - \sum_i (e_i - 1)$. On the other hand, we see that the tangent space to a fiber of MR over a point $(\mathbb{P}^1)^n$ (under the branch morphism composed with $S \to \operatorname{Spec} k$) is precisely a deformation of the corresponding curve C_0 with marked points P_i, together with the map to $\mathbb{P}^1$, fixing the $f(P_i)$ and the ramification conditions at the P_i. Since $g \geq 2$, there are no infinitesmal automorphisms to mod out by in the corresponding deformation theory problem of Theorem 1.1, so we find that the tangent space of the fiber is described by that theorem, and thus has dimension $2d-2+2g-\sum_i (e_i - 1)$. We find as before that the dimension of MR is at most, hence exactly, $n + 2d - 2 + 2g - \sum_i (e_i - 1)$, and a fiber of $\overline{MR}$ over a general point of C^n must have dimension $n+2d-2+2g-\sum_i (e_i - 1) - (n+3g-3) - 3 = 2d - 2 - g - \sum_i (e_i - 1)$, as desired.

We conclude with the $g = 1$ case (we could handle the $g = 0$ case similarly, but since we have already given a proof in that case, we will not do so). Here, we argue as when $g \geq 2$, but let S be étale over $\mathcal{M}_{1,1}$ (we may therefore use the tautological section as a relatively ample divisor for constructing $\overline{MR}$). The Brill-Noether lower bound works as before to give the relative dimension of $2d + 1 - g - \sum_i (e_i - 1)$ for MR over C^n. Because S has dimension $3g - 3 + 1$ in this case, we find we need the fiber of MR over a point $(\mathbb{P}^1)^n$ to have dimension 1 greater than before. Because we have included a choice of section in our base scheme S (i.e., because we are working over $\mathcal{M}_{1,1}$), the tangent space of this fiber at a point $(C_0, \{P_i\}_i, f)$ now includes a P_0 with $e_0 = 0$, giving the extra dimension, as desired. □

Remark 3.1. One observes in the $g = 1$ case above that even if all ramification is specified, the fiber dimension for fixed branching is 1. The reason for this is that the construction of MR doesn't see the marked point on the genus 1 curve which comes from a point of S, and still allows changing the ramification sections $P_1, \ldots, P_n$ by automorphism of the underlying curve C_0.

We remark that although the $g = 0$ case of this theorem is extremely easy in characteristic 0, the situation is more delicate in characteristic p. In particular, the intersection of the ramification Schubert cycles frequently has an excess intersection corresponding to inseparable maps of lower degree. Furthermore, examples such as $x^{p+2} + tx^p + x$ give tamely ramified situations where all non-empty fibers of ram have greater than the expected dimension; in such situations, $\mathrm{ram}(MR)$ necessarily fails to dominate $(\mathbb{P}^1)^n$ even though the expected dimension is non-negative. However, the argument of Theorem 1.2 implies that this can never happen for $\mathrm{branch}(MR)$.

Corollary 3.2. *In the situation of Theorem 1.2, but allowing the P_i and C to vary as in the proof, if $f \in MR$ is any point with any neighborhood $U \subset MR$, then U dominates $(\mathbb{P}^1)^n$ under the* branch *morphism, with all fibers smooth of dimension $2d-2+2g-\sum(e_i-1)+\epsilon_g$, where ϵ_g is the dimension of the infinitesmal automorphism space of a curve of genus g (i.e., 3 if $g=0$, 1 if $g=1$, and 0 otherwise).*

Proof. We saw in the proof of Theorem 1.2 that if MR is non-empty, it is pure of dimension $n+2g+2d-2+\epsilon_g-\sum_i(e_i-1)$, and that the tangent space at any point in any fiber of the branch morphism has dimension precisely $2d-2+2g+\epsilon_g-\sum_i(e_i-1)$. The corollary follows. □

4. Wild ramification

We conclude with some largely elementary remarks on wild ramification, in the situation that $C=D=\mathbb{P}^1$. The first observation is that by Riemann-Hurwitz in characteristic p, if any e_i are wild, in order to have separable maps with the desired ramification, we must have $2d-2>\sum_i(e_i-1)$. The codimension count of the previous section then implies that the separable maps with at least the specified ramification will necessarily form an infinite family. Thus, the fact that wildly ramified maps come in infinite families is elementary from the perspective of linear series. With the exception of one application of Theorem 1.2 to prove Theorem 1.3, all our observations will be of a completely elementary nature, but we hope they may shed some light on the behavior of wildly ramified maps.

Proof of Theorem 1.3. The primary assertion follows from Theorem 1.2 together with the claim that under our hypotheses, the locus of maps f with exactly the specified ramification is open in the locus of maps with at least the specified ramification. Indeed, having ramification exactly e_i at the points P_i is always open, since ramification can only decrease under deformation. On the other hand, by Riemann-Hurwitz a deformation cannot have additional ramification away from the P_i, since the different at each wild P_i necessarily remains at least e_i under deformation.

The second assertion will be a special case of the following proposition. □

Considering Theorem 1.2 and the preceding argument, one might be led to expect that the space of wildly ramified maps having higher different at the wild points would have higher dimension. However, this is not necessarily the case. In the argument for openness above, we use minimality of the different in a key way. Indeed, if one deforms a map with greater than minimal discriminant, a new ramification point specializing to the wild point can appear, as illustrated (indirectly) by the following two propositions.

Proposition 4.1. *Let d, n and $e_1, \dots, e_n$ be positive integers, with the e_i less than p, and $2d-2=\sum_i(e_i-1)$. Also, let $P_1, \dots, P_n$ be distinct points on $\mathbb{P}^1$. Then there exists a separable map of degree d from $\mathbb{P}^1$ to itself, ramified to order*

e_i at P_i, if and only if there exists a separable map of degree $d+p-e_1$, ramified to order e_i at P_i for $i>1$, and order p at P_1. The dimension of the space wild maps in this situation is 1 *more than the dimension of the space of tame ones.*

Proof. We may assume that $P_1=\infty$, and $f(\infty)=\infty$. Then we can go back and forth between the wild and tame cases simply by adding appropriate multiples of x^p, since the fact that $e_i<p$ for $i>1$ implies that the ramification away from ∞ will remain unchanged. We also use that the different in the wild case at ∞ is $2p-2e_1<2p$, from which one checks that if we subtract a multiple of x^p from a wild map, the degree of the numerator cannot drop below the degree of the denominator. The difference in dimension comes from the fact that the multiple of x^p added to obtain a wild map can be arbitrary. □

Proposition 4.2. *Let d,n and $e_1,\dots,e_n$ be positive integers, with $e_1=d=p$, e_i less than p for $i>1$, and $2d-2>\sum_i(e_i-1)$. Also, let $P_1,\dots,P_n$ be distinct points on $\mathbb{P}^1$. Then the space of separable maps of degree d from $\mathbb{P}^1$ to itself, ramified to order e_i at P_i and unramified elsewhere, and taken modulo automorphism of the image, is non-empty of dimension* 1*.*

Proof. Without loss of generality, we can assume $P_1=\infty$ and $f(\infty)=\infty$; then f is given by a polynomial of degree p. Since $e_i<p$ for $i>1$, the ramification conditions for $i>1$ determine the derivative of f (up to scaling). On the other hand, since $\sum_{i>1}(e_i-1)<p-1$, an f with the desired derivative always exists. The space is 1-dimensional because the x^p term may be scaled independently from the lower-order terms. □

Up until now, all of our examples have suggested that the dimension of a wildly ramified family will always be equal to the number of wildly ramified points. However, the following example shows that this is not always the case, even for families existing for general P_i.

Example 4.3. The family $\dfrac{x^{2p}+t_1x^{p+1}+t_2}{x^p+t_1x}$ for t_1,t_2 non-zero is a two-dimensional family of rational functions (modulo automorphism of the image) ramified to order p at infinity, and unramified elsewhere.

Remark 4.4. One can try to say more about the case with a single wildly ramified point by generalizing the argument of Proposition 4.1, inductively inverting as necessary and subtracting off inseparable polynomials. However, there are subtleties to be aware for this sort of argument. In particular, neither the tame ramification indices nor the dimension of the tame family obtained in this process will be determined by the ramification indices and degree of the wildly ramified map. Indeed, the maps $\dfrac{x^5(x^{10}+x^7-2x)+1}{x^{10}+x^7-2x}$ and $\dfrac{x^5(x^5(x^5+x^4-x^3+2x)+x^2+2x+1)+x^5+x^4-x^3+2x}{x^5(x^5+x^4-x^3+2x)+x^2+2x+1}$ in characteristic 5 are both of degree 15, ramified to order 5 at infinity, and simply ramified at the 6th roots of unity, but the tame functions they reduce to are $x^7 - 2x$ and $\frac{x^2+2x+1}{x^5+x^4-x^3+2x}$ respectively; the former moves in a one-dimensional family, while the latter doesn't. Nonetheless, it is interesting to note that each of the wild maps moves in a 2-dimensional family.

However, even in the tame case the situation of one index being at least p while the others are less than p is pathological, so it is not clear how much general intuition one should attempt to draw from this case. That said, it is interesting that at least for this example, it seems the dimension in the wild case is in fact more uniform than the dimension in the tame case. This suggests that an approach other than reducing to the tame case is likeliest to be productive for analyzing the dimension of families of wildly ramified maps.

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