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ON THE DRINFELD GENERATORS OF $\mathfrak{grt}_1(\mathbf{k})$ AND Γ-FUNCTIONS FOR ASSOCIATORS

Benjamin Enriquez

Abstract. We prove that the Drinfeld generators of $\mathfrak{grt}_1(\mathbf{k})$ linearly span the image of this Lie algebra in the abelianization of the commutator of the free Lie algebra with two generators. We show that this result implies Γ-function formulas for arbitrary associators.

Introduction and main results

0.1. Results on $\mathfrak{grt}_1(\mathbf{k})$. Let A, B be free noncommutative variables and $\mathbf{k}$ be a field with char$(\mathbf{k}) = 0$. Let $\mathfrak{f}_2(A, B)$ be the free Lie algebra generated by A, B. The Lie algebra $\mathfrak{grt}_1(\mathbf{k})$ is defined in [Dr] as the set of all $\psi \in \mathfrak{f}_2(A, B)$, such that

$$\psi(B, A) = -\psi(A, B), \tag{1}$$

$$\psi(A, B) + \psi(B, C) + \psi(C, A) = 0 \text{ if } C = -A - B, \tag{2}$$

$$\psi^{12,3,4} - \psi^{1,23,4} + \psi^{1,2,34} = \psi^{2,3,4} + \psi^{1,2,3}. \tag{3}$$

The last relation takes place in the Lie algebra $\mathfrak{t}_4$, defined as follows. When $n \geq 2$, $\mathfrak{t}_n$ is the Lie algebra with generators t_{ij}, $i \neq j \in \{1, \dots, n\}$ and relations $t_{ij} = t_{ji}$ if $i \neq j$, $[t_{ij} + t_{ik}, t_{jk}] = 0$ if i, j, k are distinct, and $[t_{ij}, t_{kl}] = 0$ if i, j, k, l are all distinct. If $I_1, \dots, I_n$ are disjoint subsets of $\{1, \dots, m\}$, then the Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_m$, $\psi \mapsto d^{I_1,\dots,I_n}(\psi) = \psi^{I_1,\dots,I_n}$ is defined by $t_{ij} \mapsto \sum_{\alpha \in I_i, \beta \in I_j} t_{\alpha\beta}$. Then $\mathfrak{t}_3$ is the direct sum of its center $\mathbf{k}(t_{12}+t_{13}+t_{23})$ and the free Lie algebra generated by t_{12}, t_{23}, and we use the identifications $A = t_{12}$, $B = t_{23}$.

$\mathfrak{grt}_1(\mathbf{k})$ is also equipped with a graded Lie algebra structure (it is not a Lie subalgebra of $\mathfrak{f}_2(A, B)$).

Define $\mathfrak{p} \subset \mathfrak{f}_2(A, B)$ as the commutator subalgebra. If we assign degrees 1 to A and B, then $\mathfrak{p}$ is the sum of all the components of $\mathfrak{f}_2(A, B)$ of degree > 1. It follows from Lazard elimination (see, e.g., [Reu]) that $\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$ is an abelian Lie algebra, linearly spanned by the classes $p_{k\ell}$ of $\mathrm{ad}(A)^{k-1}\,\mathrm{ad}(B)^{\ell-1}([A, B])$, where $k, \ell \geq 1$. We have $\mathfrak{grt}_1(\mathbf{k}) \subset \mathfrak{p}$; $\mathfrak{grt}_1(\mathbf{k})$ is a graded subspace of $\mathfrak{p}$.

In [Dr], Drinfeld constructed a family of elements $\sigma_n \in \mathfrak{grt}_1(\mathbf{k})$ $(n = 3, 5, 7, \dots)$, such that the image of the class $[\sigma_n]$ of σ_n in $\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$ under the isomorphism

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$i : \mathfrak{p}/[\mathfrak{p},\mathfrak{p}] \simeq (\overline{A}\ \overline{B}) \subset \mathbf{k}[\overline{A},\overline{B}]$, $p_{k\ell} \mapsto \overline{A}^k\overline{B}^\ell$ is

$$i([\sigma_n]) = (\overline{A}+\overline{B})^n - \overline{A}^n - \overline{B}^n.$$

We will prove:

Theorem 0.1. *Assume that $\psi \in \mathfrak{grt}_1(\mathbf{k})$ is homogeneous of degree n. If n is odd and ≥ 3, then the image $[\psi]$ of ψ in $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ is proportional to $[\sigma_n]$. Otherwise, this image is zero.*

Let us summarize the proof. Condition (3) may be written as $d(\psi) = 0$, where $d : \mathfrak{t}_3 \to \mathfrak{t}_4$ is a certain linear map. We show that it restricts to a linear map $d' + d'' : \mathfrak{f}_2 \to \mathfrak{f}_3$, which is itself the restriction of a linear map $d' + \Pi_4 \circ d'' : F_2 \to F_3$. Here $\mathfrak{f}_k$ (resp., F_k) is the Lie (resp., associative) algebra with k generators. We also know that $\psi \in \mathfrak{p}$. We introduce ideals $I_2 \subset F_2$, $I_3 \subset F_3$ of "ill-ordered" elements. We have vector space isomorphisms of the corresponding quotients with symmetric algebras, namely $F_2/I_2 \simeq \mathbf{k}[\overline{X},\overline{Y}]$ and $F_3/I_3 \simeq \mathbf{k}[\overline{x},\overline{y},\overline{z}]$. The key result is that $(d' + \Pi_4 \circ d'')(I_2) \subset I_3$, so that we get a map $\overline{d' + \Pi_4 \circ d''} : \mathbf{k}[\overline{X},\overline{Y}] \to \mathbf{k}[\overline{x},\overline{y},\overline{z}]$, which we explicitly compute. Moreover, $[\mathfrak{p},\mathfrak{p}] \subset I_2$, hence we get a map $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}] \to F_2/I_2$, which is injective (its image is the ideal $(\overline{X}\ \overline{Y})$). This allows to partially determine $[\psi] \in \mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$. Condition (2) imposes an additional condition on $[\psi]$. These conditions imply that $[\psi]$ is as in Theorem 0.1.

Remark 0.2. Actually, we show that the conclusion of this theorem is valid if we assume that ψ only satisfies (2) and (3).

Remark 0.3. The maps $\psi \mapsto \psi^{12,3,4}$, $\psi \mapsto \psi^{1,23,4}$, etc., extend to algebra morphisms $U(\mathfrak{f}_2(A,B)) \to U(\mathfrak{t}_4)$. Similarly to [EH], one can show that $\{\psi \in U(\mathfrak{f}_2(A,B)) | \psi$ satisfies (1), (2) and (3)$\} = \{\alpha^{12,3} - \alpha^{1,23} - \alpha^{2,3} + \alpha^{1,2} | \alpha \in t_{12}\mathbf{k}[t_{12}]\} \oplus \mathfrak{grt}_1(\mathbf{k})$.

0.2. Γ-functions for associators. Let $\widehat{F}_2$ be the degree completion of the algebra $U(\mathfrak{f}_2(A,B))$ (A and B have degree 1).

If $\lambda \in \mathbf{k}^\times$, then $\mathbf{Assoc}_\lambda(\mathbf{k})$ is defined as the set of all $\Phi \in \widehat{F}_2^\times$, such that

$$\Delta(\Phi) = \Phi \otimes \Phi,$$

$$\Phi(A,B)e^{\lambda A/2}\Phi(C,A)e^{\lambda C/2}\Phi(B,C)e^{\lambda B/2} = 1 \text{ if } C = -A-B,$$

$$\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} = \Phi^{1,2,34}\Phi^{12,3,4}$$

In particular, Φ has the form $\Phi = \exp(\varphi)$, with $\varphi \in \widehat{\mathfrak{f}}_2(A,B)$ (the degree completion of $\mathfrak{f}_2(A,B)$).

We also set $\mathbf{Assoc}(\mathbf{k}) = \{(\lambda,\Phi) | \lambda \in \mathbf{k}^\times, \Phi \in \mathbf{Assoc}_\lambda(\mathbf{k})\}$.

If X is any element in $\widehat{F}_2$, then there is a unique pair (X_A, X_B) of elements of $\widehat{F}_2$, such that $X = \varepsilon(X)1 + X_A A + X_B B$ (here ε is the counit map of $\widehat{F}_2$). We denote by $X \mapsto X^{\mathrm{ab}}$ the abelianization morphism $\widehat{F}_2 \to \mathbf{k}[[\overline{A},\overline{B}]]$, defined as the unique continuous algebra morphism such that $A \mapsto \overline{A}$, $B \mapsto \overline{B}$.

Recall the formula $\zeta(n) = (2\pi i)^n r_n$ for n even, where r_n is a rational number (we have $r_n = -B_n/(2n!)$, where B_n is the Bernoulli number defined by $u/(e^u - 1) = \sum_{k\geq 0} B_k u^k/k!$).

Corollary 0.4. *Let $\lambda \in \mathbf{k}^\times$ and $\Phi \in \mathbf{Assoc}_\lambda(\mathbf{k})$, then there exists a unique sequence $(\zeta_\Phi(n))_{n\geq 2}$ of elements of $\mathbf{k}$, such that*

$$(1 + \Phi_B B)^{\mathrm{ab}} = \frac{\Gamma_\Phi(\overline{A} + \overline{B})}{\Gamma_\Phi(\overline{A})\Gamma_\Phi(\overline{B})}, \tag{4}$$

where Γ_Φ is the invertible formal series $\Gamma_\Phi(u) = \exp(-\sum_{n\geq 2} \zeta_\Phi(n)u^n/n)$. We have $\zeta_\Phi(n) = \lambda^n r_n$ for n even.

This result is contained in an unpublished paper by Deligne and Terasoma. Our proof relies on Theorem 0.1 and the torsor structure of $\mathbf{Assoc}(\mathbf{k})$.

1. Proof of Theorem 0.1

According to [Dr], the Lie algebras $\mathfrak{t}_n$ have the following properties. The elements t_{in}, $i = 1, \dots, n-1$ generate a free subalgebra $\mathfrak{f}_{n-1} \subset \mathfrak{t}_n$. The Lie subalgebra of $\mathfrak{t}_n$ generated by the t_{ij}, $i \neq j \in \{1, \dots, n-1\}$ is isomorphic to $\mathfrak{t}_{n-1}$. We have $\mathfrak{t}_n = \mathfrak{f}_{n-1} \oplus \mathfrak{t}_{n-1}$; this is a semidirect product as $\mathfrak{t}_{n-1}$ may be viewed as a Lie algebra of derivations of $\mathfrak{f}_{n-1}$.

Let us set $T_n = U(\mathfrak{t}_n)$. The Lie algebra morphisms $\psi \mapsto d^{I_1,\dots,I_n}(\psi) = \psi^{I_1,\dots,I_n}$ extend to algebra morphisms $T_n \to T_m$, which we denote in the same way.

We set $d = -d^{2,3,4} + d^{12,3,4} - d^{1,23,4} + d^{1,2,34} - d^{1,2,3}$. So $d = d' + d''$, where $d' = -d^{2,3,4} + d^{12,3,4} - d^{1,23,4}$ and $d'' = d^{1,2,34} - d^{1,2,3}$. Then d, d', d'' are linear maps $T_3 \to T_4$, which restrict to linear maps $\mathfrak{t}_3 \to \mathfrak{t}_4$ (which we denote the same way).

Lemma 1.1. *The linear maps d, d' and d'' map $\mathfrak{f}_2 \subset \mathfrak{t}_3$ to $\mathfrak{f}_3 \subset \mathfrak{t}_4$.*

Proof. There is a unique Lie algebra morphism $\varepsilon_4 : \mathfrak{t}_4 \to \mathfrak{t}_3$, with $\varepsilon_4(t_{ij}) = t_{ij}$ for $i < j < 4$ and $\varepsilon_4(t_{i4}) = 0$ for $i < 4$. Then $\mathrm{Ker}(\varepsilon_4) = \mathfrak{f}_3$.

We have $\varepsilon_4 \circ d'' = 0$, hence $d''(\mathfrak{f}_2) \subset \mathrm{Ker}(\varepsilon_4) = \mathfrak{f}_3$. On the other hand, the Lie algebra morphisms $d^{2,3,4}, d^{12,3,4}$ and $d^{1,23,4} : \mathfrak{t}_3 \to \mathfrak{t}_4$ are such that $(t_{13}, t_{23}) \mapsto (t_{24}, t_{34})$, $(t_{13}, t_{23}) \mapsto (t_{14} + t_{24}, t_{34})$, $(t_{13}, t_{23}) \mapsto (t_{14}, t_{24} + t_{34})$, so these morphisms take the generators of $\mathfrak{f}_2$ to $\mathfrak{f}_3$, so they induce Lie algebra morphisms $\mathfrak{f}_2 \to \mathfrak{f}_3$. Therefore $d'(\mathfrak{f}_2) \subset \mathfrak{f}_3$. It follows that $d(\mathfrak{f}_2) \subset \mathfrak{f}_3$. □

We set $F_{n-1} := U(\mathfrak{f}_{n-1})$, $T_n := U(\mathfrak{t}_n)$. Then the tensor product of inclusions followed by multiplication induces a linear isomorphism $F_{n-1} \otimes T_{n-1} \stackrel{\simeq}{\to} T_n$. We denote by $\Pi_n : T_n \to F_{n-1}$ the composition $T_n \stackrel{\simeq}{\to} F_{n-1} \otimes T_{n-1} \stackrel{\mathrm{id}\otimes\varepsilon}{\to} F_{n-1}$, where $\varepsilon : T_{n-1} = U(\mathfrak{t}_{n-1}) \to \mathbf{k}$ is the counit map.

Lemma 1.2. *$d' : T_3 \to T_4$ is such that $d'(F_2) \subset F_3$. On the other hand, the composition $F_2 \subset T_3 \stackrel{d''}{\to} T_4 \stackrel{\Pi_4}{\to} F_3$ is a linear map $F_2 \to F_3$ extending $d'' : \mathfrak{f}_2 \to \mathfrak{f}_3$.*

So we have commuting diagrams

$$\begin{array}{ccc} F_2 & \stackrel{d'}{\to} & F_3 \\ \cup & & \cup \\ \mathfrak{f}_2 & \stackrel{d'}{\to} & \mathfrak{f}_3 \end{array} \quad \text{and} \quad \begin{array}{ccccc} F_2 & \stackrel{d''}{\to} & T_4 & \stackrel{\Pi_4}{\to} & F_3 \\ \cup & & & & \cup \\ \mathfrak{f}_2 & & \stackrel{d''}{\to} & & \mathfrak{f}_3 \end{array}$$

Proof. We have seen that the Lie algebra morphisms $d^{2,3,4}, d^{12,3,4}$ and $d^{1,23,4}$: $\mathfrak{t}_3 \to \mathfrak{t}_4$ restrict to Lie algebra morphisms $\mathfrak{f}_2 \to \mathfrak{f}_3$. It follows that their extensions to algebra morphisms $T_3 \to T_4$ restrict to algebra morphisms $F_2 \to F_3$. As d' is a linear combination of these morphisms, it follows that $d'(F_2) \subset F_3$.

Let ψ be an element of $\mathfrak{f}_2$. We have seen that $d''(\psi) \in \mathfrak{f}_3 \subset F_3$. For any $x \in F_3$, we have $\Pi_4(x) = x$. Therefore $\Pi_4(d''(\psi)) = d''(\psi)$. □

It follows that

$$\operatorname{Ker}(d : \mathfrak{f}_2 \to \mathfrak{f}_3) = \operatorname{Ker}(d' + \Pi_4 \circ d'' : F_2 \to F_3) \cap \mathfrak{f}_2. \tag{5}$$

We now define vector subspaces $I_2 \subset F_2$ and $I_3 \subset F_3$ as follows.

Set $X := t_{13}$, $Y := t_{23}$ (elements of F_2). Then $F_2 = \mathbf{k}\langle X, Y\rangle$. A basis of F_2 is the set of all words in X, Y. We define I_2 to be the linear span of all words of the form $WXW'YW''$, where W, W', W'' are words in X, Y. So I_2 is spanned by the non-lexicographically ordered words, where the order is $Y < X$.

Set $x := t_{14}$, $y := t_{24}$, $z := t_{34}$ (elements of F_3). Then $F_3 = \mathbf{k}\langle x, y, z\rangle$. A basis of F_3 is the set of all words in x, y, z. We define I_3 to be the linear span of all words of the form $wxw'yw''$, $wxw'zw''$ or $wyw'zw''$, where w, w', w'' are words in x, y, z. So I_3 is spanned by the non-lexicographically ordered words, where the order is $z < y < x$.

Lemma 1.3. *We have linear isomorphisms $F_2/I_2 \simeq \mathbf{k}[\overline{X}, \overline{Y}]$ and $F_3/I_3 \simeq \mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$, where $\overline{X}, \overline{Y}$ on one hand, $\overline{x}, \overline{y}, \overline{z}$ on the other hand are free commutative variables.*

Proof. Let $V_2 \subset F_2$ be the subspace with basis $Y^b X^a$, where $a, b \geq 0$. Then $V_2 \oplus I_2 = F_2$, so we have an isomorphism $F_2/I_2 \simeq V_2$. We then compose this isomorphism with $V_2 \to \mathbf{k}[\overline{X}, \overline{Y}]$, $Y^b X^a \mapsto \overline{X}^a \overline{Y}^b$.

Let $V_3 \subset F_3$ be the subspace with basis $z^c y^b x^a$, where $a, b, c \geq 0$. Then $V_3 \oplus I_3 = F_3$, so we have an isomorphism $F_3/I_3 \simeq V_3$. We then compose this isomorphism with $V_3 \to \mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$, $z^c y^b x^a \mapsto \overline{x}^a \overline{y}^b \overline{z}^c$. □

Remark 1.4. Even though I_α is a two-sided ideal of F_α ($\alpha = 2, 3$), the isomorphisms of Lemma 1.3 are not algebra isomorphisms. Indeed, the algebras F_α/I_α are noncommutative and have zero divisors. □

Lemma 1.5. *Define $\pi : F_2 \to F_3$ as the composition $F_2 \subset T_3 \stackrel{d^{1,2,34}}{\to} T_4 \stackrel{\Pi_4}{\to} F_3$. Then $\Pi_4 \circ d''(W) = \pi(W) - \varepsilon(W)1$ for any $W \in F_2$ (here $\varepsilon : F_2 = U(\mathfrak{f}_2) \to \mathbf{k}$ is the counit map).*

Let τ_{13}, τ_{23} be the derivations of F_3 defined by $\tau_{13} : x \mapsto [x, z]$, $y \mapsto 0$, $z \mapsto [z, x]$ and $\tau_{23} : x \mapsto 0$, $y \mapsto [y, z]$, $z \mapsto [z, y]$.

Then we have, for any $W \in F_2$,

$$\pi(XW) = x\pi(W) + \tau_{13}(\pi(W)), \quad \pi(YW) = y\pi(W) + \tau_{23}(\pi(W)). \tag{6}$$

Proof. If $W \in T_3$, then $\Pi_4 \circ d^{1,2,3}(W) = \varepsilon(W)1$, where $\varepsilon : T_3 = U(\mathfrak{t}_3) \to \mathbf{k}$ is the counit map. So if $W \in F_2$, we have $\Pi_4 \circ d''(W) = \Pi_4 \circ d^{1,2,34}(W) - \varepsilon(W)1$.

Let us now prove formulas (6). Let $W \in F_2$, then $d^{1,2,34}(W) = \pi(W) + \sum_i w_i t_i$, where $w_i \in F_3$ and $t_i \in \mathrm{Ker}\,(\varepsilon : T_3 = U(\mathfrak{t}_3) \to \mathbf{k})$ (here ε is the counit map). So

$$\begin{aligned} d^{1,2,34}(W) &= (t_{13} + t_{14})(\pi(W) + \sum_i w_i t_i) \\ &= x(\pi(W) + \sum_i w_i t_i) + [t_{13}, \pi(W)] + \pi(W)t_{13} + \sum_i [t_{13}, w_i]t_i + \sum_i w_i(t_{13}t_i). \end{aligned}$$

We have $[t_{13}, w] = \tau_{13}(w)$ for any $w \in F_3$, so this is the sum of $x\pi(W) + \tau_{13}(\pi(W))$ and $\sum_i xw_i t_i + \pi(W)t_{13} + \sum_i \tau_{13}(w_i)t_i + \sum_i w_i(t_{13}t_i)$. The first term belongs to F_3 and the second term belongs to $F_3\mathrm{Ker}\,(\varepsilon : T_3 \to \mathbf{k})$, so the image of their sum by Π_4 is the first term, i.e., $x\pi(W) + \tau_{13}(\pi(W))$. This proves the first identity of (6). The second identity is proved in the same way. □

Proposition 1.6. *We have $d'(I_2) \subset I_3$ and $\Pi_4 \circ d''(I_2) \subset I_3$.*

Proof. Let W_1, W_2, W_3 be words in X, Y. Then

$$d^{2,3,4}(W_1XW_2YW_3) = d^{2,3,4}(W_1)yd^{2,3,4}(W_2)zd^{2,3,4}(W_3),$$

which decomposes as a sum of words of the form $w_1yw_2zw_3$, where w_i are words on x, y, z. So $d^{2,3,4}(W_1XW_2YW_3) \in I_3$, which shows that $d^{2,3,4}(I_2) \subset I_3$.

In the same way,

$$d^{12,3,4}(W_1XW_2YW_3) = d^{12,3,4}(W_1)(x+y)d^{12,3,4}(W_2)zd^{12,3,4}(W_3)$$

belongs to I_3, so $d^{12,3,4}(I_2) \subset I_3$.

Similarly,

$$d^{1,23,4}(W_1XW_2YW_3) = d^{12,3,4}(W_1)xd^{12,3,4}(W_2)(y+z)d^{12,3,4}(W_3)$$

belongs to I_3, so $d^{1,23,4}(I_2) \subset I_3$.

Since $d' = -d^{2,3,4} + d^{12,3,4} - d^{1,23,4}$, it follows that $d'(I_2) \subset I_3$.

Let us now prove that $\Pi_4 \circ d''(I_2) \subset I_3$.

If $W \in F_2$, we have $\Pi_4 \circ d''(W) = \pi(W) - \varepsilon(W)1$. Since $\varepsilon(W) = 0$ if $W \in I_2$, we have to prove that $\pi(I_2) \subset I_3$.

We denote by $I_2[n]$ the degree n part of I_2 (here X, Y have degree 1). We will prove by induction on n that $\pi(I_2[n]) \subset I_3$.

When $n = 2$, $I_2[n]$ is spanned by XY. Then $\pi(XY) = \Pi_4((t_{13}+x)(t_{23}+y)) = xy + \tau_{13}(y) = xy$ belongs to I_3.

Let $n \geq 3$ and assume that we have proved that $\pi(I_2[n-1]) \subset I_3$. Let us prove that $\pi(I_2[n]) \subset I_3$.

$I_2[n]$ is spanned by the words $YW_1XW_2YW_3$, where W_1, W_2, W_3 are words in X, Y of total length $n - 3$, and by the words XW_1YW_2, where W_1, W_2 are words of total length $n - 2$. We should prove that π takes these words to I_3.

Let us study the image of the first family of words. Set $W := W_1XW_2YW_3$. Then $W \in I_2[n-1]$, and according to (6),

$$\pi(YW) = y\pi(W) + \tau_{23}(\pi(W)). \tag{7}$$

Now the induction hypothesis implies that $\pi(W) \in I_3$. Since I_3 is a two-sided ideal of F_3, $y\pi(W) \in I_3$.

On the other hand, let us prove that if $w \in I_3$, then $\tau_{23}(w) \in I_3$. If w is a word of the form $w_1xw_2yw_3$, then

$$\tau_{23}(w) = \tau_{23}(w_1)xw_2yw_3 + w_1x\tau_{23}(w_2)yw_3 + w_1xw_2[y,z]w_3 + w_1xw_2y\tau_{23}(w_3) \in I_3.$$

If w has the form $w_1xw_2zw_3$, then

$$\tau_{23}(w) = \tau_{23}(w_1)xw_2zw_3 + w_1x\tau_{23}(w_2)zw_3 + w_1xw_2[z,y]w_3 + w_1xw_2z\tau_{23}(w_3) \in I_3.$$

If w has the form $w_1yw_2zw_3$, then

$$\begin{aligned}\tau_{23}(w) \quad = \quad & \tau_{23}(w_1)yw_2zw_3 + w_1[y,z]w_2zw_3 + w_1y\tau_{23}(w_2)zw_3 + w_1yw_2[z,y]w_3 \\ & + w_1yw_2z\tau_{23}(w_3) \in I_3.\end{aligned}$$

By linearity, it follows that if $w \in I_3$, then $\tau_{23}(w) \in I_3$.

In particular, since $\pi(W) \in I_3$, $\tau_{23}(\pi(W)) \in I_3$. Therefore (7) implies that $\pi(YW) \in I_3$.

Let us now study the image of the second family of words, i.e., $\pi(XW_1YW_2)$, where the total length of W_1 and W_2 is $n-2$. Let us first show:

Lemma 1.7. *$\pi(W_1YW_2)$ has positive valuation*[1] *in y.*

Proof of Lemma. Let $\varepsilon_y : F_3 \to F_2 = \mathbf{k}\langle t_{13}, t_{23}\rangle$ be the morphism defined by $x \mapsto t_{13}$, $y \mapsto 0$, $z \mapsto t_{23}$. We want to show that $\varepsilon_y \circ \pi(W_1YW_2) = 0$.

We have a commutative diagram

$$\begin{array}{ccc} T_4 & \stackrel{\varepsilon_2}{\to} & T_3 \\ {\scriptstyle \Pi_4}\downarrow & & \downarrow{\scriptstyle \Pi_3} \\ F_3 & \stackrel{\varepsilon_y}{\to} & F_2 \end{array}$$

Here $\varepsilon_2 : T_4 \to T_3$ is the morphism induced by the Lie algebra morphism $\mathfrak{t}_4 \to \mathfrak{t}_3$, $t_{ij} \mapsto 0$ if i or $j = 2$, $t_{13} \mapsto t_{12}$, $t_{14} \mapsto t_{13}$, $t_{34} \mapsto t_{23}$. Indeed,

$$P(t_{14}, t_{24}, t_{34})Q(t_{12}, t_{13}, t_{23}) \stackrel{\varepsilon_2}{\to} P(t_{13}, 0, t_{23})Q(0, t_{12}, 0) \stackrel{\Pi_3}{\to} P(t_{13}, 0, t_{23})Q(0,0,0),$$

whereas

$$P(t_{14}, t_{24}, t_{34})Q(t_{12}, t_{13}, t_{23}) \stackrel{\Pi_4}{\to} P(t_{14}, t_{24}, t_{34})Q(0,0,0) \stackrel{\varepsilon_y}{\to} P(t_{13}, 0, t_{23})Q(0,0,0).$$

It follows that $\varepsilon_y \circ \pi(W_1YW_2) = \Pi_3 \circ \varepsilon_2 \circ d^{1,2,34}(W_1YW_2)$. Now $\varepsilon_2 \circ d^{1,2,34}(W_1YW_2) = \varepsilon_2\big(d^{1,2,34}(W_1)(t_{23}+t_{24})d^{1,2,34}(W_2)\big) = 0$. □

Lemma 1.8. *If w is a word in x, y, z of positive degree in y, then*

$$xw + \tau_{13}(w) \in I_3. \tag{8}$$

[1] The valuation in x_i of a nonzero element of a free algebra $k\langle x_1, \dots, x_n\rangle$ is the smallest degree in x_i of a word appearing with a nontrivial coefficient in its decomposition; the valuation of 0 is $+\infty$.

Proof of Lemma. The word xw contains xy as a subword, hence $xw \in I_3$. Let us now write w as a product $w'yw''$, where w', w'' are words. We have $\tau_{13}(w) = \tau_{13}(w')yw'' + w'y\tau_{13}(w'')$. In general, if w''' is a word, then $\tau_{13}(w''')$ has positive valuation both in x and z. Since $\tau_{13}(w')$ (resp., $\tau_{13}(w'')$) has positive valuation in x (resp., in z), $\tau_{13}(w')yw''$ (resp., $w'y\tau_{13}(w'')$) contains xy (resp., yz) as a subword. It follows that $\tau_{13}(w) \in I_3$. This implies (8). □

End of proof of Proposition. Now (6), Lemma 1.7 and Lemma 1.8 applied to $w := \pi(W_1 Y W_2)$ imply that $\pi(X W_1 Y W_2) \in I_3$. □

It follows that d' and $\Pi_4 \circ d''$ induce maps $F_2/I_2 \to F_3/I_3$, which we compute explicitly.

Lemma 1.9. *The maps $\overline{d}', \overline{d}'' : \mathbf{k}[\overline{X}, \overline{Y}] \to \mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$ induced by d' and $\Pi_4 \circ d''$ are given by $\overline{d}' = -\overline{d}^{2,3,4} + \overline{d}^{12,3,4} - \overline{d}^{1,23,4}$, where*

$$\overline{d}^{2,3,4} : f(\overline{X}, \overline{Y}) \mapsto f(\overline{y}, \overline{z}),$$

$$\overline{d}^{12,3,4} : f(\overline{X}, \overline{Y}) \mapsto \frac{\overline{x} f(\overline{x}, \overline{z}) - \overline{y} f(\overline{y}, \overline{z})}{\overline{x} - \overline{y}},$$

$$\overline{d}^{1,23,4} : f(\overline{X}, \overline{Y}) \mapsto \frac{\overline{y} f(\overline{x}, \overline{y}) - \overline{z} f(\overline{x}, \overline{z})}{\overline{y} - \overline{z}},$$

and by

$$\overline{d}'' : 1 \mapsto 0,\ \overline{X} f(\overline{X}) \mapsto \overline{x} f(\overline{x} - \overline{z}),\ \overline{Y} f(\overline{Y}) \mapsto \overline{y} f(\overline{y} - \overline{z}),$$
$$\overline{X}\ \overline{Y} f(\overline{X}, \overline{Y}) \mapsto \overline{x}\ \overline{y} f(\overline{x} - \overline{z}, \overline{y} - \overline{z}).$$

Here $f(\overline{X}), f(\overline{Y})$ (resp., $f(\overline{X}, \overline{Y})$) are arbitrary 1-variable (resp., 2-variable) polynomials.

Proof. The maps $d^{2,3,4}, d^{12,3,4}$ and $d^{1,23,4}$ all take I_2 to I_3, so they induce maps $\overline{d}^{2,3,4}, \overline{d}^{12,3,4}$ and $\overline{d}^{1,23,4} : F_2/I_2 \to F_3/I_3$. We then have $\overline{d}' = -\overline{d}^{2,3,4} + \overline{d}^{12,3,4} - \overline{d}^{1,23,4}$.

Let us compute $\overline{d}^{2,3,4}$. We have $d^{2,3,4}(Y^b X^a) = z^b y^a$, whose image in F_3/I_3 is $\overline{z}^b \overline{y}^a$. This implies the formula for $\overline{d}^{2,3,4}$.

Let us compute $\overline{d}^{12,3,4}$. We have $d^{12,3,4}(Y^b X^a) = z^b (x+y)^a$, whose projection on V_3 along I_3 is $z^b(y^a + y^{a-1}x + \cdots + x^a)$. The image of this element in $\mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$ is

$$(\overline{x}^a + \overline{x}^{a-1}\overline{y} + \cdots + \overline{y}^a)\overline{z}^b = \frac{\overline{x}^{a+1} - \overline{y}^{a+1}}{\overline{x} - \overline{y}} \overline{z}^b.$$

The formula for $\overline{d}^{12,3,4}$ follows by linearity.

One computes $\overline{d}^{1,23,4}$ in the same way. We have $d^{1,23,4}(Y^b X^a) = (y+z)^b x^a$, whose projection on V_3 along I_3 is is $(z^b + z^{b-1}y + \cdots + y^b)x^a$. The image of this element in $\mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$ is

$$\overline{x}^a(\overline{y}^b + \overline{y}^{b-1}\overline{z} + \cdots + \overline{z}^b) = \overline{x}^a \frac{\overline{y}^{b+1} - \overline{z}^{b+1}}{\overline{y} - \overline{z}}.$$

The formula for $\overline{d}^{1,23,4}$ follows by linearity.

Let us now compute $\overline{d}''$. Clearly $\overline{d}''(1) = 0$. Let $n > 0$ and let us compute $\overline{d}''(\overline{X}^n)$. This is the image in F_3/I_3 of $\Pi_4 \circ d^{1,2,34}(X^n) = \Pi_4((t_{13}+t_{14})^n)$. We have

$$(t_{13}+t_{14})^n = ((t_{14}+t_{34}+t_{13}) - t_{34})^n = \sum_{k=0}^{n} (-1)^k C_n^k (t_{34})^k (t_{14}+t_{34}+t_{13})^{n-k},$$

since $[t_{14}+t_{34}+t_{13}, t_{34}] = 0$. Now $[t_{13}, t_{14}+t_{34}] = 0$, hence

$$(t_{34})^k (t_{14}+t_{34}+t_{13})^{n-k} = \sum_{\alpha=0}^{n-k} C_{n-k}^{\alpha} (t_{34})^k (t_{14}+t_{34})^{n-k-\alpha} (t_{13})^{\alpha},$$

which is mapped by Π_4 to $(t_{34})^k (t_{14}+t_{34})^{n-k} = z^k (x+z)^{n-k}$. The projection of this element on V_3 along I_3 is $z^k(z^{n-k} + z^{n-k-1}x + \cdots + x^{n-k})$, whose image in $\mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$ is $\overline{z}^k(\overline{x}^{n-k+1} - \overline{z}^{n-k+1})/(\overline{x} - \overline{z})$. So

$$\overline{d}''(\overline{X}^n) = \sum_{k=0}^{n} (-1)^k C_n^k \frac{\overline{x}^{n-k+1} - \overline{z}^{n-k+1}}{\overline{x} - \overline{z}} \overline{z}^k = \overline{x}(\overline{x} - \overline{z})^{n-1}.$$

This implies the formula for $\overline{d}''(Xf(X))$. The formula for $\overline{d}''(\overline{Y} f(\overline{Y}))$ is proved in the same way.

Let us now prove by induction on $k+\ell$ that when $k, \ell > 0$,

$$\overline{d}''(\overline{X}^k \overline{Y}^\ell) = \overline{x}\ \overline{y} (\overline{x} - \overline{z})^{k-1} (\overline{y} - \overline{z})^{\ell-1}. \tag{9}$$

When $k = \ell = 1$, $d^{1,2,34}(YX) = (t_{23}t_{13})^{1,2,34} \stackrel{\Pi_4}{\to} t_{24}t_{14} + [t_{23}, t_{14}] = yx$ hence $\overline{d}''(\overline{X}\ \overline{Y}) = \overline{x}\ \overline{y}$, which proves (9) in this case.

Assume that (9) holds for $k+\ell < n$ and let us prove it for $k+\ell = n$ ($k, \ell > 0$).

When $\ell = 1$, we have $\overline{d}''(\overline{X}^k) = \overline{x}(\overline{x} - \overline{z})^{k-1}$, therefore

$$\pi(X^k) = \sum_{\alpha=0}^{k-1} (-1)^{\alpha} C_{k-1}^{\alpha} z^{\alpha} x^{k-\alpha} + \xi, \quad \text{where} \quad \xi \in I_3. \tag{10}$$

Then (6) implies that

$$\pi(YX^k) = y\pi(X^k) + \sum_{\alpha=0}^{k-1} (-1)^{\alpha} C_{k-1}^{\alpha} \tau_{23}(z^{\alpha} x^{k-\alpha}) + \tau_{23}(\xi).$$

Now (10) implies that $y\pi(X^k) \in yx^k + I_3$. The projection of $\tau_{23}(z^{\alpha} x^{k-\alpha})$ on V_3 along I_3 is $z^{\alpha} y x^{k-\alpha}$ if $\alpha \neq 0$, 0 otherwise.

Lemma 1.10. *$\tau_{23}(I_3) \subset I_3$.*

Proof of Lemma. If w, w' and w'' are any words in x, y, z, then

$$\tau_{23}(wxw'yw'') = \tau_{23}(w)xw'yw'' + wx\tau_{23}(w')yw'' + wxw'[y,z]w'' + wxw'y\tau_{23}(w'')$$

belongs to I_3. One proves similarly that $\tau_{23}(wxw'zw'')$ and $\tau_{23}(wyw'zw'')$ belong to I_3. □

So $\overline{d}''(\overline{X}^k\overline{Y}) = \overline{x}^k\overline{y} + \sum_{\alpha=1}^{k-1}(-1)^\alpha C_{k-1}^\alpha \overline{x}^{k-\alpha}\overline{y}\ \overline{z}^\alpha = \overline{x}\ \overline{y}(\overline{x}-\overline{z})^{k-1}$, which proves (9) in this case.

When $\ell > 1$, we use (9) for $(k, \ell-1)$. This gives

$$\pi(Y^{\ell-1}X^k) = \sum_{\alpha=0}^{k=1}\sum_{\beta=0}^{\ell-2}(-1)^{\alpha+\beta}C_{k-1}^\alpha C_{\ell-1}^\beta z^{\alpha+\beta}y^{\ell-1-\beta}x^{k-\alpha} + \eta,$$

where $\eta \in I_3$.

Then (6) implies that

$$\begin{aligned}\pi(Y^\ell X^k) &= y\pi(Y^{\ell-1}X^k)\\ &+ \sum_{\alpha=0}^{k-1}\sum_{\beta=0}^{\ell-2}(-1)^{\alpha+\beta}C_{k-1}^\alpha C_{\ell-1}^\beta \tau_{23}(z^{\alpha+\beta}y^{\ell-1-\beta}x^{k-\alpha}) + \tau_{23}(\eta).\end{aligned}$$

All the terms in the expansion of $y\pi(Y^{\ell-1}X^k)$ belong to I_3, except the terms corresponding to $\alpha = \beta = 0$, so $y\pi(Y^{\ell-1}X^k) \in y^\ell x^k + I_3$.

If $a, b, c \geq 0$, then the projection of $\tau_{23}(z^c y^b x^a)$ on V_3 along I_3 is $z^c(y-z)y^b x^a$ if $b \neq 0$ and $c \neq 0$; it is $z^c y^{b+1} x^a$ if $c \neq 0$ and $b = 0$; it is $-z^{c+1}y^b x^a$ if $c = 0$ and $b \neq 0$; and it is 0 if $b = c = 0$.

Lemma 1.10 implies that $\tau_{23}(\eta) \in I_3$. Then the projection of $\pi(Y^\ell X^k)$ on V_3 along I_3 is

$$\begin{aligned}&y^\ell x^k - zy^{\ell-1}x^k + \\ &\sum_{(\alpha,\beta)\in(\{0,\dots,k-1\}\times\{0,\dots,\ell-2\})-\{(0,0)\}}(-1)^{\alpha+\beta}C_{k-1}^\alpha C_{\ell-2}^\beta z^{\alpha+\beta}(y-z)y^{\ell-1-\beta}x^{k-\alpha}\\ &= \sum_{\alpha=0}^{k-1}\sum_{\beta=0}^{\ell-2}(-1)^{\alpha+\beta}C_{k-1}^\alpha C_{\ell-2}^\beta z^{\alpha+\beta}(y-z)y^{\ell-1-\beta}x^{k-\alpha}.\end{aligned}$$

So $\overline{d}''(\overline{X}^k\overline{Y}^\ell) = (\overline{y}-\overline{z})\overline{x}\ \overline{y}(\overline{x}-\overline{z})^{k-1}(\overline{y}-\overline{z})^{\ell-2} = \overline{x}\ \overline{y}(\overline{x}-\overline{z})^{k-1}(\overline{y}-\overline{z})^{\ell-1}$, which proves (9) in this case. This proves the induction. □

Lemma 1.11. *Define $\overline{d} = \overline{d}' + \overline{d}'' : \mathbf{k}[\overline{X}, \overline{Y}] \to \mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$. The kernel of $\overline{d}_{|(\overline{X}\ \overline{Y})}$: $(\overline{X}\ \overline{Y}) \to \mathbf{k}[\overline{x}, \overline{y}, \overline{z}]$ is equal to the linear span of the $\overline{X}\big(\overline{X}^n - \overline{Y}^n - (\overline{X}-\overline{Y})^n\big)/(\overline{X}-\overline{Y})$, where $n \geq 2$.*

Proof. The map $\overline{d}_{|(\overline{X}\ \overline{Y})}$ takes $f(\overline{X}, \overline{Y})$ to

$$\begin{aligned}&-f(\overline{y},\overline{z}) + \frac{\overline{x}f(\overline{x},\overline{z}) - \overline{y}f(\overline{y},\overline{z})}{\overline{x}-\overline{y}} - \frac{\overline{y}f(\overline{x},\overline{y}) - \overline{z}f(\overline{x},\overline{z})}{\overline{y}-\overline{z}} + \frac{\overline{x}\ \overline{y}f(\overline{x}-\overline{z},\overline{y}-\overline{z})}{(\overline{x}-\overline{z})(\overline{y}-\overline{z})}\\ &= \frac{\overline{x}\ \overline{y}}{(\overline{x}-\overline{y})(\overline{y}-\overline{z})}\big(g(\overline{x}-\overline{z},\overline{y}-\overline{z}) + g(\overline{x},\overline{z}) - g(\overline{y},\overline{z}) - g(\overline{x},\overline{y})\big),\end{aligned}$$

where $g(\overline{x},\overline{y}) = \frac{\overline{x}-\overline{y}}{\overline{x}}f(\overline{x},\overline{y})$.

So $f(\overline{X},\overline{Y}) \in \mathrm{Ker}\,(\overline{d}) \cap (\overline{X}\ \overline{Y})$ iff $f(\overline{X},\overline{Y}) \in (\overline{X}\ \overline{Y})$ and

$$g(\overline{x}-\overline{z},\overline{y}-\overline{z}) + g(\overline{x},\overline{z}) - g(\overline{y},\overline{z}) - g(\overline{x},\overline{y}) = 0. \tag{11}$$

Let us solve (11), where $g(\overline{x},\overline{y}) \in \mathbf{k}[\overline{x},\overline{y}]$. By linearity, we may assume that g is homogeneous; let n be its degree. If $n = 0$, we get $g =$ a constant polynomial. Assume that $n > 0$. Applying $(\partial/\partial\overline{z})_{|\overline{z}=0}$ to (11), we get

$$(\frac{\partial}{\partial\overline{x}} + \frac{\partial}{\partial\overline{y}})g(\overline{x},\overline{y}) = c(\overline{x}^{n-1} - \overline{y}^{n-1}),$$

for some $c \in \mathbf{k}$. This gives $g(\overline{X},\overline{Y}) = h(\overline{X}-\overline{Y}) + c(\overline{X}^n - \overline{Y}^n)/n$, where $h(\overline{X}) \in \mathbf{k}[\overline{X}]$ has degree n, so $g(\overline{X},\overline{Y}) = c(\overline{X}^n - \overline{Y}^n)/n + c'(\overline{X}-\overline{Y})^n$, for some $c' \in \mathbf{k}$.

Substituting this is (11), we get $c' = -c/n$, so the set of solutions of degree n of (11) is the linear span of $g(\overline{X},\overline{Y}) = \overline{X}^n - \overline{Y}^n - (\overline{X}-\overline{Y})^n$, where $n \geq 0$.

It follows that $f(\overline{X},\overline{Y}) \in \mathrm{Ker}(\overline{d}) \cap (\overline{X}\,\overline{Y})$ iff f is a linear span of the $\overline{X}\big(\overline{X}^n - \overline{Y}^n - (\overline{X}-\overline{Y})^n\big)/(\overline{X}-\overline{Y})$, $n \geq 0$ and belongs to $(\overline{X}\,\overline{Y})$. This means that f is a linear span of the same elements, where $n \geq 2$. □

Let us now prove Theorem 0.1. Let $\psi \in \mathfrak{t}_3$ be a solution of (2) and (3), homogeneous of degree n. One checks that if $n = 1$, then $\psi = 0$; let us assume that $n \geq 2$. Recall that $\mathfrak{f}_2 \subset \mathfrak{t}_3$ is the Lie subalgebra generated by $X = t_{13}$ and $Y = t_{23}$, and that $\mathfrak{t}_3 = \mathbf{k}\cdot(t_{12}+t_{13}+t_{23}) \oplus \mathfrak{f}_2$. Since this is a graded decomposition, we have $\psi \in \mathfrak{f}_2'$, where $\mathfrak{f}_2' = [\mathfrak{f}_2,\mathfrak{f}_2]$ is the degree ≥ 2 part of $\mathfrak{f}_2$ (it coincides with $\mathfrak{p}$ defined in the Introduction, since it coincides with $\mathfrak{t}_3' = [\mathfrak{t}_3,\mathfrak{t}_3]$).

Let us set $P_{k\ell} = \mathrm{ad}(X)^{k-1}\,\mathrm{ad}(Y)^{\ell-1}([X,Y])$ (here $k,\ell \geq 1$). Then $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ is an abelian Lie algebra with basis $[P_{k\ell}]$, $k,\ell \geq 1$. We have therefore $\psi = \sum_{k,\ell\geq 1,k+\ell=n} a_{k\ell}P_{k\ell} + \psi'$, where $\psi' \in [\mathfrak{p},\mathfrak{p}]$. We set

$$a(\overline{X},\overline{Y}) := \sum_{k,\ell\geq 1,k+\ell=n} a_{k\ell}\overline{X}^k\overline{Y}^\ell \in \mathbf{k}[\overline{X},\overline{Y}].$$

Lemma 1.12. *The image of $P_{k\ell}$ in $F_2/I_2 \simeq \mathbf{k}[\overline{X},\overline{Y}]$ is $(-1)^k\overline{X}^k\overline{Y}^\ell$. We have $[\mathfrak{p},\mathfrak{p}] \subset I_2$.*

Proof. The first statement follows from the expansion of $P_{k\ell}$. Let us denote by $(F_2)_{>0} \subset F_2$ the subspace of all elements of positive valuation both in X and in Y. Then $\mathfrak{p} \subset (F_2)_{>0}$. So $[\mathfrak{p},\mathfrak{p}] \subset ((F_2)_{>0})^2 \subset I_2$, which proves the second statement. □

Let us denote by $\overline{\psi}$ the image of ψ in $F_2/I_2 \simeq \mathbf{k}[\overline{X},\overline{Y}]$. Then

$$\overline{\psi} = \sum_{k,\ell\geq 1,k+\ell=n} (-1)^k a_{k\ell}\overline{X}^k\overline{Y}^\ell = a(-\overline{X},\overline{Y}).$$

The image of $(F_2)_{>0}$ by the projection map $F_2 \to F_2/I_2 = \mathbf{k}[\overline{X},\overline{Y}]$ is the ideal $(\overline{X}\,\overline{Y})$. Since $\psi \in \mathfrak{p}$, we have $\overline{\psi} \in (\overline{X}\,\overline{Y})$.

On the other hand, we have $\overline{d}(\overline{\psi}) = 0$. It then follows from Lemma 1.11 that for some $\lambda \in \mathbf{k}$, we have $a(-\overline{X},\overline{Y}) = \lambda\overline{X}\big(\overline{X}^n - \overline{Y}^n - (\overline{X}-\overline{Y})^n\big)/(\overline{X}-\overline{Y})$, i.e.,

$$a(\overline{X},\overline{Y}) = (-1)^{n+1}\lambda\overline{X}\frac{(\overline{X}+\overline{Y})^n - \overline{X}^n + (-1)^n\overline{Y}^n}{\overline{X}+\overline{Y}}.$$

Recall that $p_{k\ell} = \mathrm{ad}(A)^{k-1}\,\mathrm{ad}(B)^{\ell-1}([A,B])$, where $A = t_{12}$ and $B = t_{23}$. Let $b_{k\ell}$ $(k,\ell > 0,\ k+\ell = n)$ be the coefficients such that $\psi \in \sum_{k,\ell\geq 1,k+\ell=n} b_{k\ell}p_{k\ell} + [\mathfrak{p},\mathfrak{p}]$. We set $b(\overline{A},\overline{B}) = \sum_{k,\ell\geq 1,k+\ell=n} b_{k\ell}\overline{A}^k\overline{B}^\ell \in \mathbf{k}[\overline{A},\overline{B}]$. Then $b(\overline{A},\overline{B})$ is the image of the class $[\psi]$ of ψ in $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ under $i : \mathfrak{p}/[\mathfrak{p},\mathfrak{p}] \simeq (\overline{A}\ \overline{B})$ defined in the Introduction.

In general, the polynomials $a(\overline{X},\overline{Y})$ and $b(\overline{A},\overline{B})$ are related by

$$b(\overline{A},\overline{B}) = \frac{\overline{A}}{\overline{A}+\overline{B}}a(-\overline{A}-\overline{B},\overline{B}),$$

so in our case

$$b(\overline{A},\overline{B}) = -\lambda\big(\overline{A}^n + \overline{B}^n - (\overline{A}+\overline{B})^n\big).$$

Now the image of condition (2) in $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ is that $b(\overline{A},\overline{B})$ satisfies

$$\frac{b(\overline{A},\overline{B})}{\overline{A}\ \overline{B}} + \frac{b(\overline{B},\overline{C})}{\overline{B}\ \overline{C}} + \frac{b(\overline{C},\overline{A})}{\overline{C}\ \overline{A}} = 0,$$

where $\overline{C} = -\overline{A} - \overline{B}$.

Now $\overline{C}b(\overline{A},\overline{B}) + \overline{A}b(\overline{B},\overline{C}) + \overline{B}b(\overline{C},\overline{A}) = \lambda(1+(-1)^n)(\overline{A}^{n+1} + \overline{B}^{n+1} + \overline{C}^{n+1})$.

It follows that if n is even, then the image of (2) implies $\lambda = 0$, therefore the image $[\psi]$ of ψ in $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$ is zero, and that if n is odd, then the image of (2) is automatically satisfied, so that $b(\overline{A},\overline{B})$ is proportional to $(\overline{A}+\overline{B})^n - \overline{A}^n - \overline{B}^n$, i.e., $[\psi]$ is proportional to $[\sigma_n]$. This ends the proof of Theorem 0.1. □

2. Proof of Corollary 0.4

Recall that $\mathbf{Assoc}(\mathbf{k})$ is a torsor under the right action of a group $\mathrm{GRT}(\mathbf{k})$. We will first prove:

Proposition 2.1. *Set* $\mathbf{Assoc}^*(\mathbf{k}) = \{(\lambda,\Phi) \in \mathbf{Assoc}(\mathbf{k}) | (4)$ *holds with* $\zeta_\Phi(n) = \lambda^n r_n$ *for* n *even*$\}$. *Then* $\mathbf{Assoc}^*(\mathbf{k})$ *is stable under the action of* $\mathrm{GRT}(\mathbf{k})$ *on* $\mathbf{Assoc}(\mathbf{k})$. *Therefore* $\mathbf{Assoc}^*(\mathbf{k})$ *is either* $\emptyset$ *or* $\mathbf{Assoc}(\mathbf{k})$.

Proof of Proposition 2.1. Let $(\lambda,\Phi) \in \mathbf{Assoc}^*(\mathbf{k})$ and let $g \in \mathrm{GRT}(\mathbf{k})$. We should prove that $(\lambda',\Phi') := (\lambda,\Phi) * g$ satisfies (4) with $\zeta_{\Phi'}(n) = \lambda'^n r_n$ for n even.

Recall that $\mathrm{GRT}(\mathbf{k})$ is the semidirect product $\mathrm{GRT}_1(\mathbf{k}) \rtimes \mathbf{k}^\times$, where $\mathrm{GRT}_1(\mathbf{k})$ is the prounipotent group exponentiating $\mathfrak{grt}_1(\mathbf{k})$, and the action of $\mathbf{k}^\times$ on $\mathrm{GRT}_1(\mathbf{k})$ is the exponential of its action on $\mathfrak{grt}_1(\mathbf{k})$ induced by the grading. So it suffices to check that $\Phi * g$ satisfies (4) when $g \in \mathbf{k}^\times$, and when $g \in \mathrm{GRT}_1(\mathbf{k})$.

If $g = \mu \in \mathbf{k}^\times$, then $(\lambda',\Phi') = (\lambda\mu, \Phi(\mu A,\mu B))$, therefore (λ',Φ') satisfies (4) with $\zeta_{\Phi'}(n) = \mu^n\zeta_\Phi(n)$. In particular, for n even, $\zeta_{\Phi'}(n) = \mu^n\lambda^n r_n = \lambda'^n r_n$ for n even.

If $g \in \mathrm{GRT}_1(\mathbf{k})$, then $(\lambda,\Phi) * g = (\lambda, \Phi * g)$; we will show that $\Phi * g$ satisfies (4) with $\zeta_{\Phi * g}(n) = \lambda^n r_n$ for n even. Set $g = \exp(\psi)$, where $\psi \in \mathfrak{grt}_1(\mathbf{k})$, and $\Phi_t := \Phi * \exp(t\psi)$. According to Theorem 0.1, there exist scalars $\mu_n \in \mathbf{k}$ (n odd ≥ 3) such that $[\psi] = \sum_{n\ \mathrm{odd}, n\geq 3} \mu_n[\sigma_n]$. Since $\psi \in \mathfrak{f}_2(A,B)$, this means that $(\psi_B B)^{\mathrm{ab}} = \sum_{n\ \mathrm{odd}, n\geq 3} \mu_n((\overline{A}+\overline{B})^n - \overline{A}^n - \overline{B}^n)$.

Let ε be a formal variable with $\varepsilon^2 = 0$. Then $\Phi_{t+\varepsilon} = \Phi_t + \varepsilon\big(\Phi_t\psi + D_\psi(\Phi_t)\big)$, where D_ψ is the derivation of $\mathfrak{f}_2(A,B)$ such that $D_\psi(A) = [\psi, A]$, $D_\psi(B) = 0$.

Using the decompositions $\psi = \psi_A A + \psi_B B$, $\Phi_t = 1 + (\Phi_t)_A A + (\Phi_t)_B B$, we get

$$\Phi_{t+\varepsilon} = 1 + (\Phi_t)_A A + (\Phi_t)_B B + \varepsilon\Big(\Phi_t\psi_A A + \Phi_t\psi_B B + D_\psi((\Phi_t)_A)A$$
$$+ D_\psi((\Phi_t)_B)B + (\Phi_t)_A\big(\psi A - A(\psi_A A + \psi_B B)\big)\Big),$$

so

$$(\Phi_{t+\varepsilon})_B B = (\Phi_t)_B B + \varepsilon\big(\Phi_t\psi_B B + D_\psi((\Phi_t)_B)B - (\Phi_t)_A A\psi_B B\big).$$

Let us apply the abelianization to this formula. Since $\Phi_t \in \exp(\widehat{\mathfrak{f}}_2(A,B))$, we have $\Phi_t^{\mathrm{ab}} = 1$ and so $((\Phi_t)_A A + (\Phi_t)_B B)^{\mathrm{ab}} = 0$. Therefore

$$(d/dt)(((\Phi_t)_B B)^{\mathrm{ab}}) = (\psi_B B)^{\mathrm{ab}}\big(1 - ((\Phi_t)_A A)^{\mathrm{ab}}\big) = (\psi_B B)^{\mathrm{ab}}\big(1 + ((\Phi_t)_B B)^{\mathrm{ab}}\big).$$

Therefore $1 + ((\Phi_t)_B B)^{\mathrm{ab}} = \big(1 + (\Phi_B B)^{\mathrm{ab}}\big)\exp\big(t(\psi_B B)^{\mathrm{ab}}\big)$, and with $t = 1$ this gives $1 + ((\Phi * g)_B B)^{\mathrm{ab}} = \big(1 + (\Phi_B B)^{\mathrm{ab}}\big)\exp\big((\psi_B B)^{\mathrm{ab}}\big)$.

Since Φ satisfies (4), we get

$$1 + ((\Phi * g)_B B)^{\mathrm{ab}} = \Gamma_{\Phi*g}(\overline{A} + \overline{B})/(\Gamma_{\Phi*g}(\overline{A})\Gamma_{\Phi*g}(\overline{B})),$$

where $\Gamma_{\Phi*g}(s) = \Gamma_\Phi(s)\exp(\sum_{n\ \mathrm{odd}, n\geq 3}\mu_n s^n)$, i.e., $\Phi * g$ satisfies (4) with the $\zeta_\Phi(n)$ replaced by $\zeta_{\Phi*g}(n) := \zeta_\Phi(n) - n\mu_n$ for n odd ≥ 3, and $\zeta_{\Phi*g}(n) := \zeta_\Phi(n) = \lambda^n r_n$ for n even ≥ 2. □

Let us now prove Corollary 0.4. Proposition 2.1 implies that $\mathbf{Assoc}^*(\mathbf{k})$ is either $\emptyset$ or $\mathbf{Assoc}(\mathbf{k})$.

Let $\mathbf{k}$ and $\mathbf{k}'$ be fields of characteristic 0 with $\mathbf{k} \subset \mathbf{k}'$.

Lemma 2.2. $\mathbf{Assoc}^*(\mathbf{k}') \cap \mathbf{Assoc}(\mathbf{k}) = \mathbf{Assoc}^*(\mathbf{k})$.

Proof of Lemma. If $\Phi \in \mathbf{Assoc}^*(\mathbf{k}')$, then $\log((1 + \Phi_B B)^{\mathrm{ab}}) = \gamma(\overline{A} + \overline{B}) - \gamma(\overline{A}) - \gamma(\overline{B})$, for some $\gamma(t) = \sum_{n\geq 1}\gamma_n t^n \in t\mathbf{k}'[[t]]$. If now $\Phi \in \mathbf{Assoc}(\mathbf{k})$ and $\varpi : \mathbf{k}' \to \mathbf{k}$ is any $\mathbf{k}$-linear map with $\varpi(1) = 1$, then $\varpi(\log((1 + \Phi_B B)^{\mathrm{ab}})) = 0$, hence $\log((1 + \Phi_B B)^{\mathrm{ab}}) = \varpi(\gamma)(\overline{A} + \overline{B}) - \varpi(\gamma)(\overline{A}) - \varpi(\gamma)(\overline{B})$, where $\varpi(\gamma)(t) = \sum_{n\geq 1}\varpi(\gamma_n)t^n \in t\mathbf{k}[[t]]$, with $\varpi(\gamma_n) = \lambda^n r_n$ for n even. So $\Phi \in \mathbf{Assoc}^*(\mathbf{k})$. Hence $\mathbf{Assoc}^*(\mathbf{k}') \cap \mathbf{Assoc}(\mathbf{k}) \subset \mathbf{Assoc}^*(\mathbf{k})$. The inverse inclusion is obvious. □

End of proof of Proposition 2.1. It follows that if $\mathbf{Assoc}^*(\mathbf{k}') = \mathbf{Assoc}(\mathbf{k}')$, then $\mathbf{Assoc}^*(\mathbf{k}) = \mathbf{Assoc}(\mathbf{k})$. On the other hand, if $\mathbf{k} \subset \mathbf{k}'$ and $\mathbf{Assoc}^*(\mathbf{k}) = \mathbf{Assoc}(\mathbf{k})$, then $\mathbf{Assoc}^*(\mathbf{k}') = \mathbf{Assoc}(\mathbf{k}')$: indeed, Proposition 5.3 of [Dr] implies that $\mathbf{Assoc}(\mathbf{k}) \neq \emptyset$, so $\mathbf{Assoc}^*(\mathbf{k}) \neq \emptyset$; we have obviously $\mathbf{Assoc}^*(\mathbf{k}) \subset \mathbf{Assoc}^*(\mathbf{k}')$, hence $\mathbf{Assoc}^*(\mathbf{k}') \neq \emptyset$; then Proposition 2.1 implies the equality $\mathbf{Assoc}^*(\mathbf{k}') = \mathbf{Assoc}(\mathbf{k}')$. It follows that if for some $\mathbf{k}$, $\mathbf{Assoc}^*(\mathbf{k}) \neq \emptyset$, then $\mathbf{Assoc}^*(\mathbf{k}) = \mathbf{Assoc}(\mathbf{k})$ for any $\mathbf{k}$. We will now prove that $\mathbf{Assoc}^*(\mathbb{C}) \neq \emptyset$.

Let Φ_{KZ} be the Knizhnik-Zamolodchikov associator defined as in [Dr] as the renormalized holonomy from 0 to 1 of the differential equation $G'(z) = (\frac{A}{z} +$

$\frac{B}{z-1})G(z)$. Then $(2\pi i, \Phi_{\rm KZ}) \in \mathbf{Assoc}(\mathbb{C})$ satisfies (4) with $\zeta_\Phi(n) = \zeta(n)$ for any $n \geq 2$. Indeed, in [Dr], (2.15), it is proved that

$$[\log \Phi_{\rm KZ}] = \exp\big(\sum_{n\geq 2} \frac{\zeta(n)}{n}(\overline{A}^n + \overline{B}^n - (\overline{A}+\overline{B})^n)\big) - 1.$$

Then $(\Phi_{\rm KZ})_A = \frac{\Phi_{\rm KZ}-1}{\log \Phi_{\rm KZ}}(\log \Phi_{\rm KZ})_A$, $(\Phi_{\rm KZ})_B = \frac{\Phi_{\rm KZ}-1}{\log \Phi_{\rm KZ}}(\log \Phi_{\rm KZ})_B$, and since $\log \Phi_{\rm KZ} \in \mathfrak{p}$, we get $(\Phi_{\rm KZ})^{\rm ab}_B = (\log \Phi_{\rm KZ})^{\rm ab}_B = [\log \Phi_{\rm KZ}]/\overline{B}$. So

$$1 + ((\Phi_{\rm KZ})_B B)^{\rm ab} = \frac{\Gamma_{\rm mod}(\overline{A}+\overline{B})}{\Gamma_{\rm mod}(\overline{A})\Gamma_{\rm mod}(\overline{B})},$$

where $\Gamma_{\rm mod}(u) = \exp(\sum_{n\geq 2} -\frac{\zeta(n)}{n}u^n)$ is related to the Γ-function by $\Gamma_{\rm mod}(u) = e^{\gamma u}/(-u\Gamma(-u))$, where γ is the Euler-Mascheroni constant. It follows that $(\Phi_{\rm KZ}, 2\pi i) \in \mathbf{Assoc}^*(\mathbb{C})$, therefore for any $\mathbf{k}$, $\mathbf{Assoc}^*(\mathbf{k}) = \mathbf{Assoc}(\mathbf{k})$. □

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IRMA (CNRS), rue René Descartes, F-67084 Strasbourg, France
E-mail address: enriquez@math.u-strasbg.fr