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REDUCING AND TOROIDAL DEHN FILLINGS ON 3-MANIFOLDS BOUNDED BY TWO TORI

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Abstract. We show that if M is a simple 3-manifold bounded by two tori such that $M(r_1)$ is reducible and $M(r_2)$ is toroidal, then $\Delta(r_1, r_2) \leq 2$, answering a question raised by Gordon. To do this, we first prove that there exists only one simple 3-manifold having two Dehn fillings of distance 3 apart one of which yields a reducible manifold and the other yields a 3-manifold containing a Klein bottle.

1. Introduction

Let M be a compact connected orientable 3-manifold with a torus boundary component $\partial_0 M$ and r a *slope*, the isotopy class of an essential simple closed curve, on $\partial_0 M$. The manifold obtained by r-*Dehn filling* is defined to be $M(r) = M \cup J$, where J is a solid torus glued to M along $\partial_0 M$ so that r bounds a disk in J.

Following [22], we say that M is *simple* if it contains no essential sphere, torus, disk or annulus. For two slopes r_1 and r_2 on $\partial_0 M$, the *distance* $\Delta(r_1, r_2)$ denotes their minimal geometric intersection number. For simple manifolds M, if both $M(r_1)$ and $M(r_2)$ fail to be simple, then the upper bounds for $\Delta(r_1, r_2)$ have been established in various cases. See [8] for more details.

For example, Oh [18] and independently Wu [23] showed that for a simple manifold M, if $M(r_1)$ is reducible and $M(r_2)$ is toroidal then $\Delta(r_1, r_2) \leq 3$. Furthermore, Wu [22] also showed that if one puts an additional condition $H_2(M, \partial M - \partial_0 M) \neq 0$, then $\Delta(r_1, r_2) \leq 1$. In particular, this homological condition holds if M has a boundary component with genus greater than one or if M has more than two boundary tori. Note that M has no boundary sphere, for M is simple. It is natural then to consider the following question raised by Gordon [8, Question 5.1]; if ∂M consists of two tori, is it possible that $\Delta(r_1, r_2) = 3$? In this paper we give a negative answer to the question.

Theorem 1.1. *Let M be a simple 3-manifold with boundary a union of two tori. If r_1 and r_2 are slopes on one boundary component $\partial_0 M$ such that $M(r_1)$ is reducible and $M(r_2)$ is toroidal, then $\Delta(r_1, r_2) \leq 2$.*

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Our theorem is sharp because Eudave-Muñoz and Wu [5, Theorem 2.6] have given infinitely many simple manifolds which are bounded by two tori and admit reducing and toroidal Dehn fillings at distance 2.

Oh [19] showed that if one Dehn filling yields a reducible manifold and another yields a manifold containing a Klein bottle, then the distance between their filling slopes is not greater than 3. On the other hand, Boyer and Zhang [1, p.286] gave an example of a simple manifold showing Oh's result is sharp. This simple manifold, which we shall denote by $W(6)$, is obtained from the exterior W of the Whitehead link by performing Dehn filling on its one boundary component with slope 6 under the standard meridian-longitude coordinates. In this paper, we shall show that $W(6)$ is the only simple manifold having two such Dehn fillings at distance 3.

Theorem 1.2. *Let M be a simple manifold. If $M(r_1)$ is reducible and $M(r_2)$ contains a Klein bottle with $\Delta(r_1, r_2) = 3$, then M is homeomorphic to $W(6)$.*

Corollary 1.3. *Let M be a simple manifold. If $M(r_1)$ is reducible and $M(r_2)$ is a Seifert fibered manifold over the 2-sphere with three exceptional fibers of orders $2, 2, n$, then $\Delta(r_1, r_2) \leq 2$.*

It is still unknown whether or not the upper bound 2 is the best possible.

2. The intersection graphs

From now on we assume that M is a simple 3-manifold with a torus boundary component $\partial_0 M$ and that r_1 and r_2 are slopes on $\partial_0 M$ of distance 3 apart such that $M(r_1)$ is reducible and $M(r_2)$ contains an essential torus or a Klein bottle.

Over all reducing spheres in $M(r_1)$ which intersect the attached solid torus J_1 in a family of meridian disks, we choose a 2-sphere $\widehat{F}_1$ so that $F_1 = \widehat{F}_1 \cap M$ has the minimal number, say n_1, of boundary components. Similarly let $\widehat{F}_2$ be either an essential torus or a Klein bottle in $M(r_2)$ which intersects the attached solid torus J_2 in a family of meridian disks, the number of which, say n_2, is minimal over all such surfaces and let $F_2 = \widehat{F}_2 \cap M$. Let $u_1, u_2, \ldots, u_{n_1}$ be the disks of $\widehat{F}_1 \cap J_1$, labelled as they appear along J_1. Similarly let $v_1, v_2, \ldots, v_{n_2}$ be the disks of $\widehat{F}_2 \cap J_2$. Then F_1 is an essential planar surface, and F_2 is an essential punctured torus or a punctured Klein bottle in M. We may assume that F_1 and F_2 intersect transversely and the number of components in $F_1 \cap F_2$ is minimal over all such surfaces. Then no circle component of $F_1 \cap F_2$ bounds a disk in either F_1 or F_2 and no arc component is boundary-parallel in either F_1 or F_2. The components of ∂F_i are numbered $1, 2, \ldots, n_i$ according to the labels of the corresponding disks of $\widehat{F}_i \cap J_i$. We obtain a graph G_i in $\widehat{F}_i$ by taking as the (fat) vertices of G_i the disks in $\widehat{F}_i \cap J_i$ and as the edges of G_i the arc components of $F_1 \cap F_2$ in F_i. Each endpoint of an edge of G_i has a label, that is, the number of the corresponding component of ∂F_j, $i \neq j$. Since each component of ∂F_i intersects each component of ∂F_j in $\Delta(= \Delta(r_1, r_2) = 3)$ points, the labels $1, 2, \ldots, n_j$ appear in order around each vertex of G_i repeatedly Δ times.

For a graph G, the *reduced graph* $\overline{G}$ of G is defined to be the graph obtained from G by amalgamating each family of parallel edges into a single edge. For an edge α of $\overline{G}$, the *weight* of α, denoted by $w(\alpha)$, is the number of edges of G represented by α.

Although F_2 may be non-orientable, we can establish a parity rule. In fact, this is a natural generalization of the usual one. First, orient all components of ∂F_i so that they are mutually homologous on $\partial_0 M$, $i = 1, 2$. Let e be an edge in G_i. Since e is a properly embedded arc in F_i, it has a disk neighborhood D in F_i with $\partial D = a \cup b \cup c \cup d$, where a and c are arcs in ∂F_i with induced orientation from ∂F_i. On D, if a and c have opposite directions, then e is called *positive*, otherwise *negative*. See Figure 1. Then we have the following.

Parity rule. *An edge is positive on one graph if and only if it is negative on the other graph.*

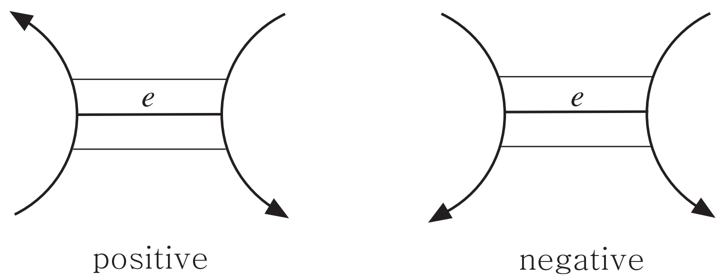


FIGURE 1

Orient the core of J_i. If $\widehat{F}_i$ is orientable, we can give a sign to each vertex of G_i according to the sign of its intersection with the core of J_i. Two vertices (possibly equal) of G_i are called *parallel* if they have the same sign, otherwise *antiparallel*. A positive edge connects parallel vertices, while a negative one connects antiparallel vertices. Let G_i^+ denote the subgraph of G_i consisting of all the vertices and all the positive edges of G_i.

Let G be G_1 or G_2 and let x be a label of G. An x*-edge* is an edge of G with label x at one endpoint. An x*-cycle* is a cycle of positive x-edges which can be oriented so that the tail of each edge has label x. A cycle in G is a *Scharlemann cycle* if it bounds a disk face, and the edges in the cycle are all positive and have the same label pair. If the label pair is $\{x, y\}$, then we refer to such a Scharlemann cycle as an (x, y)*-Scharlemann cycle*. In particular, a Scharlemann cycle of length 2 is called an S*-cycle*. An edge in G is called *level* if its endpoints have the same label. A set of four parallel edges $\{e_1, e_2, e_3, e_4\}$ of G_2 is called an *extended* S*-cycle* if $\{e_2, e_3\}$ is an S-cycle and e_k is adjacent to e_{k+1} $(k = 1, 2, 3)$.

Lemma 2.1. (1) *G_2 has no positive level edge.*
(2) *G_2 has no extended S-cycle.*
(3) *Suppose $\widehat{F}_j$ is not a Klein bottle. If G_i has a Scharlemann cycle, $\widehat{F}_j$ is separating, $i \neq j$.*

(4) *Any two Scharlemann cycles of* G_2 *have the same label set.*
(5) *Any positive edge* α *of* $\overline{G}_2$ *has* $w(\alpha) \leq n_1/2 + 1$.
(6) *Any edge* α *of* $\overline{G}_2$ *has* $w(\alpha) < n_1$.
(7) *Let* $\{e_1, \ldots, e_k\}$ *be a set of parallel positive edges of* G_2 *with* e_l *adjacent to* e_{l+1} $(l = 1, \ldots, k-1)$. *If the sets of labels at two ends of* $\{e_1, \ldots, e_k\}$ *have a label in common, then either* $\{e_1, e_2\}$ *or* $\{e_{k-1}, e_k\}$ *forms an S-cycle. Moreover, the common label belongs to the label set of the S-cycle.*

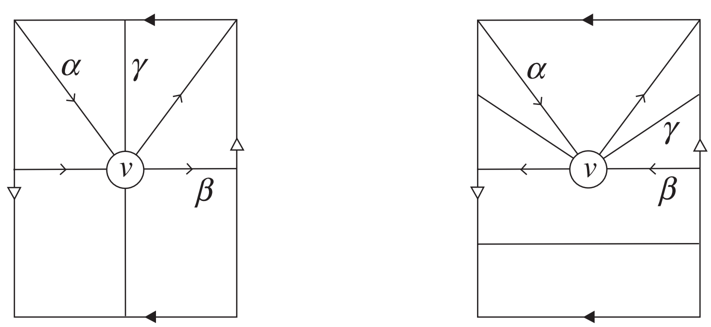
Proof. (1) By the parity rule a positive level edge in G_2 is a negative loop in G_1, which has a Möbius band neighborhood in $\widehat{F}_1$, contradicting that $\widehat{F}_1$ is a sphere. (2)–(4) follow from [23, Lemma 1.2], (5) and (6) follow from [23, Lemma 1.5], and (7) follows from (2),(4) and [4, Lemma 2.6.6]. □

Lemma 2.2. $n_2 = 2$ *when* $\widehat{F}_2$ *is a torus, and* $n_2 = 1$ *when* $\widehat{F}_2$ *is a Klein bottle.*

Proof. This is a part of the main result in [17]. □

3. Klein bottle

Throughout this section we assume that $\widehat{F}_2$ is a Klein bottle. Then G_2 has a single vertex v by Lemma 2.2. The reduced graph $\overline{G}_2$ is a subgraph of the graphs shown in Figure 2. Whether $\overline{G}_2$ is a subgraph of the graph in Figure 2(a) or (b), there are three *edge classes*, α, β and γ. An edge in G_1 or G_2 is called an α*-edge,* β*-edge* or γ*-edge* according as, being regarded as an edge in G_2, it lies in class α, β or γ. In G_2, all γ-edges are positive, while the others are negative.



(a) (b)

FIGURE 2

Lemma 3.1. *The weights of the reduced edges* α *and* β *in* $\overline{G}_2$ *are positive.*

Proof. Assume $w(\alpha) = 0$. By Lemma 2.1(5) and (6), $w(\beta) = n_1 - 1$ and $w(\gamma) = n_1/2 + 1$. If $\overline{G}_2$ is a subgraph of the graph in Figure 2(a), then G_2 has two positive level edges, contradicting Lemma 2.1(1). If $\overline{G}_2$ is a subgraph of the graph in Figure 2(b), then G_2 contains an S-cycle, so $\widehat{F}_1$ is separating and n_1 is even. However, for any β-edge in G_2, which is negative, the two labels at its endpoints have distinct parities. This contradicts the parity rule. Similarly, $w(\beta) > 0$. □

Thus the edges in $\overline{G}_2$ cut F_2 into one or two disks, so there is no circle component of $F_1 \cap F_2$.

Orienting the negative edges in G_2 as shown in Figure 2, we can think of G_1^+ as a directed graph. If a disk face of G_1^+ is bounded by a circuit of consistently oriented edges, we call it a *cycle face.* Throughout this section, let $I_1(\alpha)$ (resp. $I_2(\alpha)$) denote the shortest interval on ∂v containing α-edge endpoints at the head of α (resp. at the tail of α). And similarly for $I_1(\beta)$ and $I_2(\beta)$.

Lemma 3.2. *G_1^+ has a cycle face.*

Proof. First, assume that $\overline{G}_2$ is a subgraph of the graph in Figure 2(a). Then $w(\alpha) + w(\beta) \geq n_1$. Otherwise, $w(\alpha) + w(\beta) = n_1 - 1$ or $\leq n_1 - 2$. In the first case, the outermost edges in the family of γ-edges would be positive level edges, and in the latter case $w(\gamma) \geq n_1/2 + 2$. Both are absurd by Lemma 2.1(1) and (5). Hence each label x appears at least once on each of $I_1(\alpha) \cup I_1(\beta)$ and $I_2(\alpha) \cup I_2(\beta)$. This means that in G_1^+, each vertex u_x of G_1^+ has an edge pointing in and an edge pointing out. Starting at any vertex of G_1^+, one can construct a path through the oriented edges always consistent with orientations. Ultimately the path hits the same vertex to create a cycle. Among such cycles, an innermost one bounds a disk face of G_1^+ and we are done.

Next, assume that $\overline{G}_2$ is a subgraph of the graph in Figure 2(b).

Claim. If G_1^+ has a sink or source at a vertex u_x, then x is a label of an S-cycle of G_2.

Proof. Suppose for example that G_1^+ has a sink at u_x. We first show that u_x is univalent in G_1^+. If u_x were trivalent in G_1^+, then u_x would have two (say) α-edges pointing in. This means that label x would appear twice on $I_1(\alpha)$, so $w(\alpha) > n_1$, contradicting Lemma 2.1(6). Thus u_x has valency at most 2 in G_1^+. Suppose u_x is bivalent in G_1^+. Then an α-edge and a β-edge are incident to u_x (otherwise, two (say) α-edges would be incident, contradicting Lemma 2.1(6) as above). Since u_x is trivalent in G_1, a γ-edge is incident to u_x in G_1. Orient γ so that its head lies between the tail of α and the head of β. Then in G_2, label x appears at the heads of α, β and either at the head of γ or at the tail of γ, say, at the head. Then x appears twice at the heads of β and γ, implying $w(\beta) + w(\gamma) > n_1$, and x does not appear at the tails of β and γ, implying $w(\beta) + w(\gamma) < n_1$. Two inequalities give a contradiction. Thus u_x is univalent in G_1^+.

In G_1, two γ-edges are incident to u_x. In G_2, by Lemma 2.1(6), label x appears at both ends of γ. By Lemma 2.1(7) the γ-edge family contains an S-cycle and x is a label of this S-cycle. □

Suppose G_1^+ has no cycle face. Then G_1^+ has a sink or source by [10, Lemma 2.3.1]. The above claim and [19, Lemma 2.3(1)] imply that there are exactly two sinks and sources in total. Let u_x and u_y be vertices of G_1 at which these sinks and sources occur (then labels x, y form a label pair of an S-cycle in G_2). Then two γ-edges running from u_x to u_y divide $\widehat{F}_1$ into two disks and each of them contains neither sink nor source in its interior. The two disks have the same number of vertices of G_1 in their interiors by [21, Lemma 2.1]. One can choose a disk whose interior contains no positive edge incident to a sink. Then there would be a cycle face in the disk. □

Orient all components of $\partial F_1 = \{\partial u_1, \ldots, \partial u_{n_1}\}$ homologously on $\partial_0 M$ and orient $\partial F_2 = \partial v$. Let $u \in \{u_1, \ldots, u_{n_1}\}$. If P and Q are two points in $\partial u \cap \partial v$, denote by $\mu_1(P, Q)$ (resp. $\mu_2(P, Q)$) the arc in ∂u (resp. ∂v) going from P to Q with respect to the chosen orientation. As in [6, p.1720] we define $\tau_i(P, Q) = |\mu_i(P, Q) \cap \partial F_j| - 1$ ($\{i, j\} = \{1, 2\}$).

Lemma 3.3. *Let $u, u' \in \{u_1, \ldots, u_{n_1}\}$. Suppose $P, Q \in \partial u \cap \partial v$ and $R, S \in \partial u' \cap \partial v$. If $\tau_1(P, Q) = \tau_1(R, S)$, then $\tau_2(P, Q) = \tau_2(R, S)$.*

Proof. This follows from [6, Lemma 2.4]. □

Lemma 3.4. *Let f be a cycle face with vertices $u_{x_1}, \ldots, u_{x_n}$ and with corners λ_i at u_{x_i}, i.e. the intervals $f \cap u_{x_i}$ on ∂u_{x_i}. Let $\partial^1 \lambda_i$ be one endpoint of λ_i at the head of an oriented edge of f and $\partial^2 \lambda_i$ the other endpoint (automatically at the tail of another edge of f). Then we have $\tau_2(\partial^1 \lambda_1, \partial^2 \lambda_1) = \cdots = \tau_2(\partial^1 \lambda_n, \partial^2 \lambda_n)$.*

Proof. Since u_{x_i}'s are all parallel, an orientation of F_1 induces orientations of ∂u_{x_i}'s which are mutually homologous on $\partial_0 M$, so $\tau_1(\partial^1 \lambda_1, \partial^2 \lambda_1) = \cdots = \tau_1(\partial^1 \lambda_n, \partial^2 \lambda_n)$. By Lemma 3.3 we have $\tau_2(\partial^1 \lambda_1, \partial^2 \lambda_1) = \cdots = \tau_2(\partial^1 \lambda_n, \partial^2 \lambda_n)$. □

Proposition 3.5. *$\overline{G}_2$ is a subgraph of the graph in Figure 2(b).*

Proof. Assume for contradiction that $\overline{G}_2$ is a subgraph of the graph in Figure 2(a). Let f be a cycle face of G_1^+ guaranteed by Lemma 3.2 and $u_{x_1}, \ldots, u_{x_n}$ the vertices of f. Let λ_i be the corner of f at u_{x_i} with one endpoint, $\partial^1 \lambda_i$, at the head of an oriented edge of f and the other, $\partial^2 \lambda_i$, at the tail of another edge of f. On ∂v, choose the shortest interval I_j such that $\{\partial^j \lambda_1, \ldots, \partial^j \lambda_n\} \subset I_j$ for $j = 1, 2$. Since $\{\partial^j \lambda_1, \ldots, \partial^j \lambda_n\} \subset I_j(\alpha) \cup I_j(\beta)$, $I_1 \cap I_2 = \emptyset$. Label $x_1, \ldots, x_n$ so that $\partial I_1 = \{\partial^1 \lambda_1, \partial^1 \lambda_n\}$. Using Lemma 3.4, one can verify that $\partial I_2 = \{\partial^2 \lambda_1, \partial^2 \lambda_n\}$. Hence $I_1 \cup I_2 \cup \lambda_1 \cup \lambda_n$ bounds a disk D on $\partial_0 M$. As below the proof of [11, Claim 7.5], one can use D and f to construct a new Klein bottle in $M(r_2)$, on which the core of J_2 can be isotoped to lie. This implies that M contains a properly embedded Möbius band and hence fails to be simple. □

By Lemma 3.2, G_1^+ has a disk face f bounded by a cycle of consistently oriented edges $e_1, \ldots, e_n$, labelled so that the head of e_i is adjacent to the tail of e_{i+1} modulo n. The edges $e_1, \ldots, e_n$ do not totally belong to one edge class, α or β, since otherwise, the argument in [9, Section 5] would show that M contains a cable space.

Lemma 3.6. *n is even and $\{e_1, \dots, e_n\}$ is an alternating sequence of α-edges and β-edges.*

Proof. If $n = 2$, it is obvious, so we assume $n > 2$. Assume for contradiction that e_1, e_2 are α-edges and e_3 is a β-edge. Let u_{x_1} be the vertex to which e_1 and e_2 are incident and let u_{x_2} be the vertex to which e_2 and e_3 are incident. Let λ_i be the corner of f at u_{x_i} with endpoints $\partial^j \lambda_i = e_{i+j-1} \cap u_{x_i}$ $(i, j = 1, 2)$. Then in G_2, the points $\partial^1\lambda_1, \partial^2\lambda_1, \partial^1\lambda_2, \partial^2\lambda_2$ are on $I_1(\alpha), I_2(\alpha), I_1(\alpha), I_2(\beta)$, respectively.

Orient ∂v clockwise. By Lemma 3.4 $\tau_2(\partial^1\lambda_1, \partial^2\lambda_1) = \tau_2(\partial^1\lambda_2, \partial^2\lambda_2)$. Recall that $\overline{G}_2$ is a subgraph of the graph in Figure 2(b). From the fact that the points $\partial^1\lambda_1, \partial^2\lambda_1, \partial^1\lambda_2, \partial^2\lambda_2$ are respectively on $I_1(\alpha), I_2(\alpha), I_1(\alpha), I_2(\beta)$, one can obtain two inequalities $\tau_2(\partial^1\lambda_1, \partial^2\lambda_1) \le 2w(\alpha) < 2n_1$ and $\tau_2(\partial^1\lambda_2, \partial^2\lambda_2) > w(\alpha) + w(\beta) + w(\gamma) = 3n_1/2$. Since $\partial^1\lambda_i, \partial^2\lambda_i$ are labelled x_i, $\tau_2(\partial^1\lambda_1, \partial^2\lambda_1) = n_1$ and $\tau_2(\partial^1\lambda_2, \partial^2\lambda_2) = 2n_1$, giving a contradiction. □

Lemma 3.7. *f is a bigon.*

Proof. Assume that $n = 2m > 2$. Label the vertices of f $u_{x_1}, u_{y_1}, \dots, u_{x_m}, u_{y_m}$ along ∂f so that an α-edge (resp. β-edge) is incident to u_{x_i} (resp. u_{y_i}) at its head. See Figure 3. Then each label x_i (resp. y_i) appears once on each of $I_1(\alpha)$ and $I_2(\beta)$ (resp. $I_1(\beta)$ and $I_2(\alpha)$). Since $w(\alpha), w(\beta) < n_1$, any label cannot occur twice on each interval.

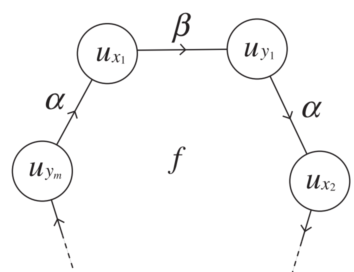


FIGURE 3

Orient ∂v clockwise. Relabelling x_i's (and correspondingly y_i's), if necessary, we may assume that among x_i's, x_1 appears first on $I_1(\alpha)$ with respect to the orientation of ∂v. Then x_1 precedes other x_i's on $I_2(\beta)$ by Lemma 3.4. The vertices u_{x_i} and u_{y_i} are connected by a β-edge for each i. Since β-edges are negative in G_2, y_1 also precedes other y_i's on $I_1(\beta)$. Again by Lemma 3.4, y_1 precedes other y_i's on $I_2(\alpha)$. In particular, y_m follows y_1 on $I_2(\alpha)$. Since u_{y_m}, u_{x_1} and u_{y_1}, u_{x_2} are connected by α-edges, respectively and since α-edges are negative in G_2, x_1 follows x_2 on $I_1(\alpha)$. This contradicts the choice of x_1. □

Without loss of generality we may assume that the labels around v are ordered in the clockwise direction. Then one can see that if an α-edge (resp. a β-edge) in G_2 has label x at its head, then its tail is labelled $x + w(\alpha)$ (resp. $x + w(\beta)$) modulo n_1.

By Lemmas 3.6 and 3.7, f has exactly two vertices u_x, u_y along with two edges e_α, e_β (e_i is an i-edge, $i = \alpha, \beta$) such that e_α and e_β are incident to u_x and u_y at their heads, respectively. Then $y \equiv x + w(\alpha), x \equiv y + w(\beta) \pmod{n_1}$, so $w(\alpha) + w(\beta) \equiv 0$. Since $0 < w(\alpha) + w(\beta) < 2n_1$, we get $w(\alpha) + w(\beta) = n_1$. Thus $w(\gamma) = n_1/2$. By Lemma 2.1(7), either the γ-edge family in G_2 contains an S-cycle or the sets of labels at its two ends are disjoint.

Suppose that the γ-edge family contains an S-cycle σ with label pair $\{1, 2\}$, say. Then σ consists of either the first two edges or the last two edges of the family. By examining the labels around the vertex v, one can see that either $w(\alpha) = n_1/2 - 1$ or $w(\beta) = n_1/2 - 1$ holds. We conclude that the three numbers $w(\alpha), w(\beta), w(\gamma)$ cannot be all even.

We shall rechoose $\widehat{F}_1$ to rule out this case. Let f be the disk face bounded by σ. The edges of σ cut $\widehat{F}_1 - \mathrm{Int}(u_1 \cup u_2)$ into two disks, D' and D'', say. Put $D = D' \cup u_1$. Then D contains $n_1/2$ fat vertices by [21, Lemma 2.1], and $f \cup D$ is a Möbius band whose boundary bounds a disk B on ∂H, where H is the part of J_1 between the consecutive vertices u_1 and u_2. After being slightly pushed off H, $\widehat{P} = f \cup D \cup B$ becomes a projective plane which intersects J_1 in $n_1/2$ meridian disks. For a thin I-bundle neighborhood N of $\widehat{P}$ in $M(r_1)$, its boundary is a reducing sphere intersecting J_1 in n_1 meridian disks. Using ∂N instead of $\widehat{F}_1$, we obtain a new graph pair G_1, G_2, where each edge of $\overline{G}_2$ has an even weight by the I-bundle structure of N. In particular, the above observation shows that G_2 cannot have an S-cycle.

Therefore we may assume that the label sets at two ends of the γ-edge family are disjoint and hence $w(\alpha) = w(\beta) = w(\gamma) = n_1/2$.

Lemma 3.8. *$n_1 = 4$ and $\widehat{F}_1$ is separating.*

Proof. Since $\Delta n_1 = 3n_1 = 2(w(\alpha) + w(\beta) + w(\gamma))$, n_1 is even. Let $n_1 = 2m$. We may assume that the labels around v are as shown in Figure 4. Notice that $n_1/2 = m$ is also even, for otherwise the central edge in the γ-edge family would have the same label pair as the central edge in the α-edge family and they would form an orientation-reversing loop in $\widehat{F}_1$.

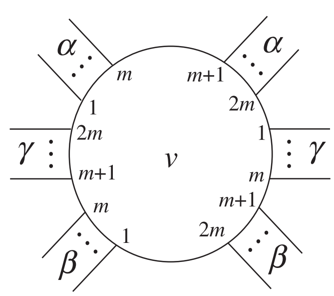


FIGURE 4

Let e_i (resp. e'_i) be a γ-edge (resp. an α-edge) in G_2 with label i at one endpoint, $i = 1, \ldots, m$. Let D_i be the disk in F_2, realizing the parallelism of e_i

and e_{m-i+1} with $\partial D_i = e_i \cup a_i \cup e_{m-i+1} \cup b_i$, where a_i is an arc on ∂v from label i to $m-i+1$. And similarly for D_i', a_i', b_i'. The four edges $e_i, e_{m-i+1}, e_i', e_{m-i+1}'$ together with vertices $u_i, u_{m-i+1}, u_{m+i}, u_{2m-i+1}$ form a loop σ_i on $\widehat{F}_1$. These four vertices divide J_1 into four parts. Let U_i (resp. V_i) be the part of J_1 between u_i and u_{m-i+1} (resp. u_{m+i} and u_{2m-i+1}). A regular neighborhood of $\widehat{F}_1 \cup U_{m/2} \cup V_{m/2} \cup D_{m/2} \cup D_{m/2}'$ in $M(r_1)$ is a punctured projective space with two boundary spheres one of which is parallel to $\widehat{F}_1$ and the other intersects J_1 in fewer components than $\widehat{F}_1$, so bounds a 3-ball. Hence $\widehat{F}_1$ is separating.

Now assume that $n_1 > 4$. Among the loops σ_i's, choose an innermost one, say, σ_k. Then a_k and a_k' are properly imbedded essential arcs in the annulus $\partial U_k - \mathrm{Int}(u_k \cup u_{m-k+1})$, and $I_2(\beta)$ intersects the annulus in a spanning arc. The arcs a_k and a_k' cut the annulus into two disks and one can choose a component B disjoint from $I_2(\beta)$. Similarly after cutting the annulus $\partial V_k - \mathrm{Int}(u_{m+k} \cup u_{2m-k+1})$ along the arcs b_k and b_k', one can choose a component B' disjoint from $I_1(\beta)$. Then $D_k \cup D_k' \cup B \cup B'$ is a Möbius band whose boundary bounds a disk in $\widehat{F}_1$ containing exactly two vertices, either $\{u_k, u_{m+k}\}$ or $\{u_{m-k+1}, u_{2m-k+1}\}$. Hence we can find a projective plane in $M(r_1)$ intersecting the core of J_1 in two points. The boundary of a thin regular neighborhood of this projective plane is a reducing sphere of $M(r_1)$, intersecting J_1 in fewer components than $\widehat{F}_1$, which contradicts our choice of $\widehat{F}_1$ at the beginning of Section 2. □

Proof of Theorem 1.2. By Lemma 3.8 and the argument just above it, G_2 is uniquely determined as illustrated in Figure 5(a). Let A, B, C, D, E, F be the edges of G_2 as shown in Figure 5(a). The correspondence between the edges of G_1 and G_2 uniquely determines G_1 up to a homeomorphism of the underlying sphere $\widehat{F}_1$, as shown in Figure 5(b).

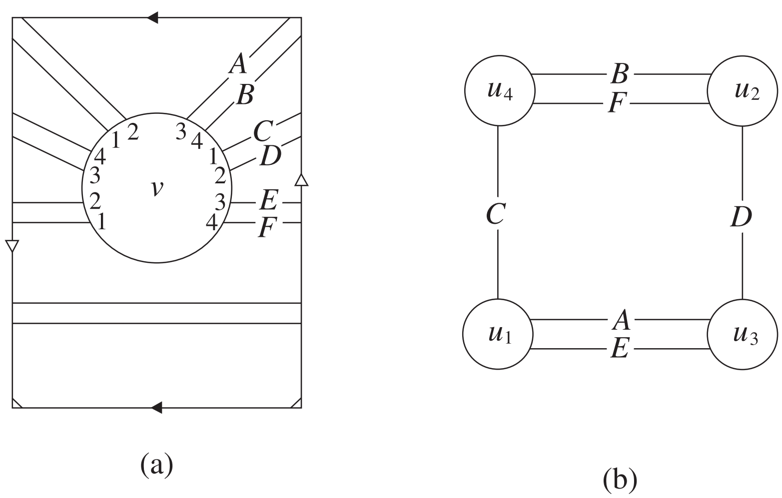


FIGURE 5

The graph G_2 has two trigons f_1, f_2 bounded by A, B, C and D, E, F, respectively. Let g_1, g_2 be bigons of G_2 bounded by A, B and C, D, respectively.

These trigons and bigons lie on the opposite sides of $\widehat{F}_1$. Let X be a regular neighborhood of $\partial_0 M \cup F_1 \cup f_1 \cup f_2 \cup g_1 \cup g_2$ in M. Then one can verify that ∂X consists of two spheres. Capping off these spheres with 3-balls gives M. This shows that M is uniquely determined by the pair of the graphs in Figure 5. Hence $M = W(6)$. □

Proof of Corollary 1.3. Since $M(r_2)$ contains a Klein bottle, $\Delta(r_1, r_2) \le 3$ by [19, Theorem 1.1]. Assume for contradiction that $\Delta(r_1, r_2) = 3$. Then $M = W(6)$ by Theorem 1.2.

Note that $M(\infty) = L(6,1)$ and $M(2)$ is a small Seifert fibered space with a finite fundamental group. See [3, Example 7.8]. We have $\Delta(r_1, \infty) \le 1$ and $\Delta(r_1, 1) \le 1$ by [3, Theorem 1.2(1)] and [12, Theorem 1.2]. Hence $r_1 = 0, 1$ or 2. However, $M(2)$ is irreducible. The slope 0 is a boundary slope of an essential once-punctured torus in M, which extends to an essential torus in $M(0)$, and $\dim H_1(M; \mathbb{Q}) = 1$. The conclusions (ii), (iii) and (iv) in [4, Theorem 2.0.3] do not hold for M and the slope 0, so $M(0)$ is irreducible. Therefore $r_1 = 1$. By [2, Theorem 1.5(1)] we have $\Delta(r_2, \infty) = 1$ and hence the assumption $\Delta(r_1, r_2) = 3$ yields $r_2 = -2$ or 4. However, $M(-2)$ is hyperbolic by [7, Example (5)]. Therefore $r_2 = 4$. □

4. Torus

Throughout this section we assume that $\widehat{F}_2$ is a torus. Then G_2 has exactly two vertices, v_1 and v_2, by Lemma 2.2. We may assume that $M(r_2)$ is irreducible and boundary-irreducible by [12, 20]. By Lemma 2.1(4), we also assume that G_2 has only $(1,2)$-Scharlemann cycles if it has Scharlemann cycles.

Lemma 4.1. *The vertices of G_2 are antiparallel.*

Proof. Assume that v_1 and v_2 are parallel. Then all the edges of G_2 are positive. For a label $x \ne 1, 2$, consider the subgraph Γ of G_2 consisting of all vertices and all x-edges of G_2. Let V, E and F be the number of vertices, edges and disk faces of Γ, respectively. Since $V < E$, we have $0 = \chi(\widehat{F}_2) \le V - E + F < F$, so Γ contains a disk face, which is an x-face in G_2 in terms of [17]. This contradicts [17, Theorem 4.4]. □

The graph G_2 has at most 6 *edge classes*, which we call $\alpha, \beta, \gamma, \delta, \varepsilon, \varepsilon'$ as illustrated in Figure 6. See [6, Lemma 5.2]. An edge in G_1 or G_2 is called an *η-edge* if, being regarded as an edge in G_2, it lies in class η, $\eta \in \{\alpha, \beta, \gamma, \delta, \varepsilon, \varepsilon'\}$. An edge in G_1 or G_2 is called *of type* 1 if it is an α-edge or a β-edge, and *of type* 2 if it is a γ-edge or a δ-edge. The ε-edges and ε'-edges are positive in G_2, while the others are negative by Lemma 4.1. Since v_1 and v_2 have the same valency, we have $w(\varepsilon) = w(\varepsilon')$. Without loss of generality we assume that the ordering of the labels around v_1 is anticlockwise, while the ordering around v_2 is clockwise.

Lemma 4.2. *Let $x (\ne 1, 2)$ be a label of G_2. Then there exist an edge of type* 1 *and an edge of type* 2 *incident to v_i with label x, $i = 1, 2$.*

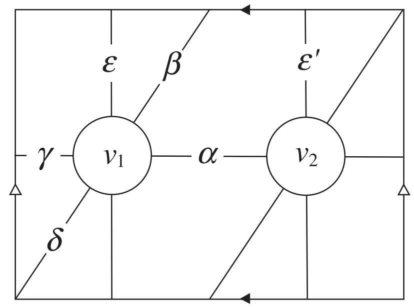


FIGURE 6

Proof. Let $\varepsilon = \{e_1, \ldots, e_{w(\varepsilon)}\}$, $\varepsilon' = \{e'_1, \ldots, e'_{w(\varepsilon')}\}$ as shown in Figure 7. There are two cases.

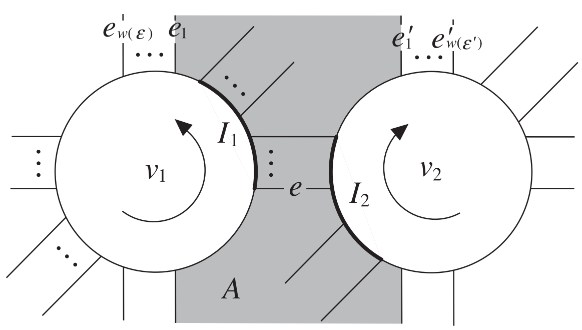


FIGURE 7

Case I. *The sets of labels at two ends of* ε *have a label in common.* By Lemma 2.1(7) we may assume that $\{e_1, e_2\}$ is an S-cycle with label pair $\{1, 2\}$. Let I_i be the shortest interval on ∂v_i containing the endpoints of the edges of type 1 at v_i, $i = 1, 2$. Since I_1 and I_2 have the same number of edge endpoints, $\{e'_1, e'_2\}$ is also an S-cycle with label pair $\{1, 2\}$. Then $w(\alpha) + w(\beta) = n_1 - 2$ or $2n_1 - 2$. In the first case for any label $x \neq 1, 2$, exactly one edge of type 1 is incident to v_i with label x, $i = 1, 2$. On the other hand, since $w(\varepsilon) \leq n_1/2 + 1$ by Lemma 2.1(5), $w(\gamma) + w(\delta) = 3n_1 - w(\alpha) - w(\beta) - 2w(\varepsilon) \geq n_1$. This means that for any label x, an edge of type 2 is incident to v_i with label x, $i = 1, 2$, so we have the desired result. In the latter case $w(\alpha) = w(\beta) = n_1 - 1$ by Lemma 2.1(6). Let e be the lowest α-edge as in Figure 7. Since $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ are $(1, 2)$-S-cycles, the point $e \cap \partial v_1$ is labelled 3, while the point $e \cap \partial v_2$ is labelled 2. Hence e connects vertices u_2 and u_3 in G_1. Since G_2 contains S-cycles, $\widehat{F}_1$ is separating by Lemma 2.1(3), so u_2 and u_3 are antiparallel. Therefore the edge e is negative in both graphs G_1 and G_2, which is impossible by the parity rule.

Case II. *The sets of labels at two ends of* ε *are disjoint.* Suppose, for example, that no edge of type 1 is incident to v_1 with label $x \neq 1, 2$, that is, the label x does not appear on I_1. Since $w(\gamma), w(\delta) < n_1$ by Lemma 2.1(6), $w(\alpha)+w(\beta)+2w(\varepsilon) = 3n_1 - w(\gamma) - w(\delta) > n_1$, so x appears at one end of ε, say, at the bottom of v_1 (then x does not appear at the top of v_1). Let y be the label of the point in $\partial v_1 \cap e_{w(\varepsilon)}$ at the bottom of v_1 (y is possibly equal to x). Then y does not appear on I_1, otherwise x must appear at the end of ε at the top of v_1. Since $w(\alpha) + w(\beta) + 2w(\varepsilon) > n_1$, y must appear at the end of ε at the top of v_1. This contradicts our assumption. □

In particular, we have $w(\alpha) + w(\beta) > 0$ and $w(\gamma) + w(\delta) > 0$. And if $w(\varepsilon) = w(\varepsilon') = 0$, then we have $w(\alpha), w(\beta), w(\gamma), w(\delta) > 0$ by Lemma 2.1(6). Thus the reduced edges in $\overline{G}_2$ cut F_2 into disks, so $F_1 \cap F_2$ has no circle component.

Proposition 4.3. *G_1^+ contains a connected subgraph Λ satisfying that*

(1) *for all vertices u_x of Λ but at most one vertex, there are an edge of type 1 and an edge of type 2 in Λ which are incident to u_x with label i for each $i = 1, 2$;*
(2) *for the other vertex u_{x_0}, if it exists, there are two edges in Λ incident to u_{x_0}; and*
(3) *there is a disk $D_\Lambda \subset \widehat{F}_1$ such that $D_\Lambda \cap G_1^+ = \Lambda$.*

The vertex u_{x_0} is called an *exceptional vertex* of Λ.

Proof. Let Γ be an *extremal component* of G_1^+. That is, Γ is a component of G_1^+ having a disk support D such that $D \cap G_1^+ = \Gamma$.

Recall that all the Scharlemann cycles in G_2 are $(1, 2)$-Scharlemann cycles. If G_2 contains a Scharlemann cycle, then the vertices u_1 and u_2 are antiparallel and hence they belong to distinct components of G_1^+. So, we may assume that Γ does not contain u_2 in this case.

A vertex of Γ is a *cut vertex* if it splits Γ into more components. If Γ has no cut vertex, we take Γ as Λ. Then all vertices of Λ, possibly except one (when u_1 is contained in Λ), satisfy the property (1) by Lemma 4.2.

Suppose that Γ has a cut vertex. If u_1 is a single cut vertex of Γ, cut Γ off at u_1 and take any component as Λ, and then all vertices of Λ but u_1 satisfy the property (1) again by Lemma 4.2. Otherwise, after cutting Γ off at all cut vertices we can find a component, which does not contain u_1, with a single cut vertex. We take this component as Λ and then the cut vertex may be the exceptional vertex of Λ, while the other vertices of Λ satisfy the property (1) by Lemma 4.2. □

Let D_Λ be a disk support of Λ as given in Proposition 4.3. A vertex of Λ is a *boundary vertex* if there is an arc connecting it to ∂D_Λ whose interior is disjoint from Λ, and an *interior vertex* otherwise.

Each face of Λ is a disk, for Λ is connected. Since G_1 has only two labels, $\{1, 2\}$, each edge of Λ has label 1 at one endpoint and label 2 at the other. Hence every face of Λ determines a Scharlemann cycle of G_1. By Lemma 2.1(3)

$\widehat{F}_2$ divides $M(r_2)$ into a *black* side X and a *white* side Y. A face of G_1 is black or white according as it lies in X or Y.

From now on we assume that $\partial_1 M = \partial M - \partial_0 M$ is a torus and eventually we will get a contradiction. Assume that Y contains $\partial_1 M$. By Theorem 1.2 we may also assume that $M(r_2)$ contains no Klein bottle.

Lemma 4.4. *The edges of a face in G_1^+ cannot lie in a disk in $\widehat{F}_2$.*

Proof. At the beginning of this section we assumed that $M(r_2)$ is irreducible. Hence the statement follows from the proof of [11, Lemma 3.1]. □

Let $H_X = J_2 \cap X$ and $H_Y = J_2 \cap Y$.

Lemma 4.5. *For any white face of G_1^+, its edges cannot lie in an annulus in $\widehat{F}_2$.*

Proof. Suppose to the contrary that G_1^+ contains a white face f whose edges lie in an annulus A in $\widehat{F}_2$. By Lemma 4.4, A is essential in $\widehat{F}_2$. Let N be a regular neighborhood of $A \cup H_Y \cup f$ in Y. Then ∂N is a torus, and $T = (\partial N - \text{Int}A) \cup (\widehat{F}_2 - A)$ is a torus disjoint from J_2, so it must be inessential in $M(r_2)$. If T were compressible, a compression would result in a sphere separating $\widehat{F}_2$ and $\partial_1 M$, and hence a reducing sphere of $M(r_2)$. This contradicts our assumption that $M(r_2)$ is irreducible. Suppose T is parallel to $\partial_1 M$. Then there is an annulus A' in the region between the two tori T and $\partial_1 M$ such that one component of $\partial A'$ lies in $\widehat{F}_2 - A$ and the other component lies in $\partial_1 M$. The circle $A' \cap \partial_1 M$ is an essential curve on $\partial_1 M$, otherwise $\widehat{F}_2$ would be compressible. Surgering $\widehat{F}_2$ along A' gives a properly embedded annulus A'' in $M(r_2)$. Since $\widehat{F}_2$ is essential in $M(r_2)$, so is A''. Our assumption $\Delta(r_1, r_2) = 3$ contradicts [22, Theorem 5.1]. □

Lemma 4.6. Λ *contains a face bounded only by edges of type* 1 *or only by edges of type* 2.

Proof. Let u_x be the exceptional vertex of Λ, if it exists, and any vertex of Λ otherwise. Without loss of generality we may assume that an edge e of type 1 is incident to u_x with label 1. Orient every edge of type 1 from the endpoint with label 1 to the other. Then by Proposition 4.3 any non-exceptional vertex of Λ has an edge pointing in and an edge pointing out. Starting with e, one can construct a path through the oriented edges of type 1 always consistent with orientations. Ultimately the path hits the same vertex to create a cycle. This shows that Λ contains cycles of oriented edges of type 1. Choose an innermost cycle σ. Then σ bounds a disk D in $\widehat{F}_1$ with no vertex in its interior. If D has no edge of Λ in its interior, then we are done. Otherwise, all the edge in Int$D$ are of type 2. Since σ is a 1-cycle, some of these edges bound a desired face of Λ. □

Lemma 4.7. *Let f be an n-sided face of G_1^+, $n = 2$ or 3. Then the edges of f lie in an essential annulus, A, in $\widehat{F}_2$, f is black, and X is a Seifert fibered space* *over the disk with two exceptional fibers one of which has order n. The core of A is a Seifert fiber.*

Proof. This follows from [11, Lemma 3.7 and Theorem 3.8] and Lemma 4.5. □

Note that each of non-exceptional boundary vertices of Λ has valency at least 4 in Λ by the property (1) of Proposition 4.3.

Lemma 4.8. *Λ contains a black bigon.*

Proof. Let V, E and F be the number of vertices, edges and faces of Λ, respectively. Let V_b, V_i and V_e denote the number of boundary vertices, interior vertices and exceptional vertex of Λ, respectively. Then $V = V_b + V_i$ and $V_e = 0$ or 1. Now suppose that Λ contains no bigon. Then each face has at least 3 sides and hence $3F + V_b \leq 2E$. Combining with $1 = \chi(\text{disk}) = V - E + F$, we get $E \leq 3V - V_b - 3 = 3V_i + 2V_b - 3$.

Since all boundary vertices but the exceptional vertex have valency at least 4 and all interior vertices have valency 6, we have $4(V_b - V_e) + 2V_e + 6V_i \leq 2E$. These two inequalities give $3 \leq V_e$, a contradiction. Hence Λ contains a black bigon by Lemma 4.7. □

As in [13], we may label an edge e of G_1 by class of the corresponding edge of G_2. We refer to this label as the *edge class label* of e. Then an edge of type 1 has edge class label α or β, and an edge of type 2 has edge class label γ or δ.

Let $M_X = M \cap X$ and $M_Y = M \cap Y$.

Lemma 4.9. *Any two bigons in G_1^+ have the same pair of edge class labels. Furthermore the pair is either $\{\alpha, \beta\}$ or $\{\gamma, \delta\}$.*

Proof. If there are two bigons of G_1^+ with distinct pairs of edge class labels, the argument in the proof of [13, Lemma 5.2] shows that $M(r_2)$ contains a Klein bottle. This contradicts our assumption just above Lemma 4.4.

Now we shall show that the pair is either $\{\alpha, \beta\}$ or $\{\gamma, \delta\}$. By Lemma 4.6 there is a face f of Λ bounded only by edges of type 1, say. Thus the edges of f lie in an essential annulus on $\widehat{F}_2$, so f is black by Lemma 4.5. Let $u_{x_1}, \ldots, u_{x_n}$ be the vertices of f and let λ_i be the corner of f at u_{x_i}. As illustrated in in Figure 7, let I_j be the shortest interval on ∂v_j containing the endpoints of edges of type 1 at v_j for $j = 1, 2$.

As in the proof of Proposition 3.5, one can find a disk D on the annulus $\partial H_X - \text{Int}(v_1 \cup v_2)$ such that $D \supset \lambda_1 \cup \cdots \cup \lambda_n$ and $\partial D = a \cup b \cup c \cup d$, where a and c are respectively subarcs of I_1 and I_2, and b and d are essential arcs in the annulus $\partial H_X - \text{Int}(v_1 \cup v_2)$, parallel to each of λ_i. Let A be the annulus in F_2 bounded by the edges e_1 and e_1' in Figure 7 along with subarcs on ∂v_1 and ∂v_2 containing I_1 and I_2. Then $A \cup D$ is a once punctured torus. Enlarging D slightly in $\partial H_X - \text{Int}(v_1 \cup v_2)$ we may assume that ∂f lies in $\text{Int}(A \cup D)$. Notice that ∂f is a non-separating curve on $A \cup D$, since the vertices of f are all parallel. Surgering $A \cup D$ along f gives a disk B. Pushing Int$B$ into the interior of M_X rel. boundary gives a properly embedded disk B' in M_X. Here

$\partial B' \cap \partial H_X = b \cup d$ and $\partial B' - \text{Int}(b \cup d)$ consists of two arcs in ∂A. Notice that an orientation of $\partial B'$ induces orientations of b and d which are opposite in the annulus $\partial H_X - \text{Int}(v_1 \cup v_2)$. Now by shrinking H_X to its core, $H_X \cup B'$ becomes a properly embedded annulus A' in X.

The annulus A' divides X into two regions Z_1 and Z_2. We claim that both Z_1 and Z_2 are solid tori. Since the core of H_X lies on A', we can isotope the core of J_2 slightly so that it is disjoint from the torus ∂Z_1. Then the minimality of $|\widehat{F}_2 \cap J_2|$ guarantees that Z_1 is a solid torus. Similarly so is Z_2. Thus X is a Seifert fibered space over the disk with the core of A' (and hence that of A) a Seifert fiber. Since $M(r_2)$ (and hence X) contains no Klein bottle, the Seifert fibration of X is unique by [15, Theorem VI.18]. Therefore the pair of edge class labels of any bigon in G_1^+ is either $\{\alpha, \beta\}$ or $\{\gamma, \delta\}$ by Lemma 4.7. □

Lemma 4.10. (1) *All interior vertices of Λ have valency at least 4 in $\overline{\Lambda}$.*
(2) *All boundary vertices of Λ but the exceptional vertex have valency at least 3 in $\overline{\Lambda}$.*

Proof. (1) If Λ had a set of three parallel edges, the set would contain a white bigon, contradicting Lemma 4.7. Since any interior vertex has valency 6 in Λ, it has valency at least 3 in $\overline{\Lambda}$. Suppose that an interior vertex has valency exactly 3 in $\overline{\Lambda}$. Then the vertex is incident to three pairs of parallel edges in Λ, which have the same pair of edge class labels, say, $\{\alpha, \beta\}$ by Lemma 4.9. This contradicts the property (1) of Proposition 4.3.

(2) Suppose that a non-exceptional boundary vertex of Λ has valency 2 in $\overline{\Lambda}$. Then the vertex is incident to two pairs of parallel positive edges and two negative edges in G_1. Then we get a contradiction as above. □

Lemma 4.11. Λ *contains a black trigon.*

Proof. Let V, E and F be the number of vertices, edges and faces of $\overline{\Lambda}$, respectively. Let V_b, V_i and V_e denote the number of boundary vertices, interior vertices and the exceptional vertex of Λ, respectively. Now suppose that $\overline{\Lambda}$ contains no 3-sided face. Then each face of $\overline{\Lambda}$ has at least 4 sides and hence $4F + V_b \leq 2E$. Combining $1 = V - E + F$, we get $2E \leq 4V_i + 3V_b - 4$.

By Lemma 4.10 we have $3(V_b - V_e) + 2V_e + 4V_i \leq 2E$. These two inequalities give $4 \leq V_e$, a contradiction. Thus $\overline{\Lambda}$ (and hence Λ) has a 3-sided face, which must be black by Lemma 4.7. □

Lemma 4.12. *M_X is a handlebody of genus 2 and M_Y is a compression body with the boundary a union of a genus 2 surface and $\partial_1 M$.*

Proof. Since F_2 is a twice punctured torus, both ∂M_X and $\partial M_Y - \partial_1 M$ are surfaces of genus 2. By Lemma 4.10(2) one easily sees that Λ contains black and white faces simultaneously. A black face compresses ∂M_X to result in a torus in M_X, which bounds a solid torus since M_X contains no incompressible torus. Hence M_X is a handlebody of genus 2. Similarly a white face compresses $\partial M_Y - \partial_1 M$ to result in a torus parallel to $\partial_1 M$. □

Travelling around the boundary of a disk face of G_1^+ gives rise to a cyclic sequence of edge class labels. We shall say that two disk faces of G_1^+ of the same color are *isomorphic* if the cyclic sequences obtained by travelling in some directions are equal.

Let $A_X = H_X \cap M_X$ and $A_Y = H_Y \cap M_Y$. Then $\partial M_X = A_X \cup F_2$ and $\partial M_Y - \partial_1 M = A_Y \cup F_2$. Since all the vertices of Λ are parallel, each face of Λ is a non-separating disk in M_X or M_Y. Note that any two faces of Λ of the same color are disjoint.

Lemma 4.13. *If two disk faces of G_1^+ are parallel in M_X or M_Y, then they are isomorphic.*

Proof. Suppose, for example, that two disk faces f and g of G_1^+ are parallel in M_X. The curves ∂f and ∂g cobound an annulus A in ∂M_X. Note that each component of $\partial A - \mathrm{Int} A_X$ is an edge of G_2 in ∂f or ∂g, while each component of $\partial A \cap A_X$ is a corner of f or g. The boundary circles of A_X must intersect A in spanning arcs, otherwise some edge of G_2 in ∂f or ∂g would be a trivial loop in G_2. Thus $A - \mathrm{Int} A_X$ is a union of disjoint rectangles $R_1, \ldots, R_n$. Each R_i realizes a parallelism between two edges $\partial f \cap R_i$ and $\partial g \cap R_i$, so these edges have the same edge class label. □

By Lemmas 4.8 and 4.11, G_1^+ contains black bigons and trigons. Without loss of generality we may assume that all bigons of G_1^+ have the edge class label pair, $\{\alpha, \beta\}$, by Lemma 4.9. Then we have the following.

Lemma 4.14. *All trigons of G_1^+ have the same pair of edge class labels $\{\gamma, \delta\}$, i.e. they are bounded by edges of type* 2.

Proof. Let f and g be a bigon and a trigon of G_1^+, respectively. Let A be the annulus in F_2 bounded by e_1 and e_1' along with two subarcs in ∂F_2 as shown in Figure 7. Then f is bounded by an α-edge and a β-edge, and X is a Seifert fibered space over the disk with two exceptional fibers, whose Seifert fibration is unique because X does not contain a Klein bottle. Here, the core of A is a Seifert fiber.

By Lemma 4.7, g is bounded either by edges of type 1 or by edges of type 2. In the first case, surgering a twice punctured torus $A \cup A_X$ using f and g gives two disks in X, since ∂f and ∂g are non-separating and not mutually parallel in the surface. The boundary circles of the disks lie in $\widehat{F}_2$ and are isotopic to the core of A. This implies that $\widehat{F}_2$ is compressible in $M(r_2)$, a contradiction. Thus g is bounded by edges of type 2. □

We will assume that each trigon has two γ-edges and a δ-edge.

Lemma 4.15. *If two edges e_1, e_2 of G_1^+ are incident to a vertex with the same label, then they have distinct edge class labels.*

Proof. If e_1 and e_2 have the same edge class labels, then the corresponding edge class in G_2 contains more than n_1 edges, contradicting Lemma 2.1(6). □

Lemma 4.16. *There is no triple of mutually non-isomorphic black disk faces of* G_1^+.

Proof. Suppose that G_1^+ has such a triple (f_1, f_2, f_3). Then these faces cut M_X into two 3-balls by Lemma 4.13.

Claim. $G_1 = G_1^+$ and it is connected.

Proof. Note that G_1 is connected if and only if every face of G_1 is a disk. Let f be a black face of G_1 other than f_1, f_2, f_3. Then f lies in the complement of $f_1 \cup f_2 \cup f_3$ in M_X, so f must be a disk, otherwise it would be compressible in M_X, so F_1 would be compressible in M and one could find a new essential sphere in $M(r_1)$ which meets J_1 in fewer components than $\widehat{F}_1$. Each component of $\partial f \cap F_2$ is an edge of G_2. The circle ∂f must be an essential curve in $\partial M_X = F_2 \cup A_X$, otherwise some component of $\partial f \cap F_2$ would be a trivial loop in G_2. Hence f is an essential disk in M_X, which must be parallel to one of the faces f_1, f_2 and f_3. Therefore any black face of G_1 is a disk face isomorphic to one of f_1, f_2 and f_3 by Lemma 4.13. It follows that all the edges of G_1 are positive, i.e. $G_1 = G_1^+$.

It remains to show that any white face of G_1 is a disk. Suppose to the contrary that a white face g of G_1 is not a disk. Then g is incompressible in M_Y as above. Recall that Λ contains a white disk face and that M_Y is a compression body with the outer boundary a surface of genus 2 and the inner boundary a torus, $\partial_1 M$. Let g_1 be a white disk face of Λ. Cutting M_Y along g_1 gives rise to a manifold V, homeomorphic to $S^1 \times S^1 \times I$, with $\partial_1 M$ one component of ∂V. Here, g lies in V and $\partial g \subset \partial V - \partial_1 M$. Since g is incompressible in M_Y, g is also incompressible in V. Hence g must be an annulus parallel to $\partial V - \partial_1 M$. Thus V contains an annulus A such that one component of ∂A is the core curve of g and the other lies in $\partial_1 M$, where $\partial A \cap \partial_1 M$ is an essential curve, otherwise g would be compressible in M_Y. Surgering $\widehat{F}_1$ along A gives two compressing disks for $\partial_1 M$ in $M(r_1)$. This shows $M(r_1)$ is boundary-reducible. The assumption $\Delta = 3$ contradicts [13, Theorem 1.1]. □

Hence, $\widehat{F}_1$ is a non-separating sphere in $M(r_1)$. This contradicts [16, Theorem 1.1]. □

Proof of Theorem 1.1. Orient the edges of G_1 as shown in Figure 8 so that G_1 becomes a directed graph in the 2-sphere $\widehat{F}_1$. (For example, all α-edges are oriented so that their left hand sides are black.)

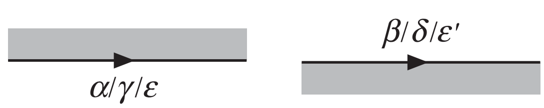


FIGURE 8

Note that any disk face of G_1 has the same number of ε-edges and ε'-edges. Hence if there were a cycle face in G_1, then it would be a face of G_1^+, since the

ε-edges and ε'-edges are oppositely oriented. Moreover, it would be bounded either by α-edges and γ-edges or by β-edges and δ-edges and hence it would be white by Lemma 4.16. This contradicts Lemma 4.5.

Therefore it is enough to show that G_1 has neither a sink nor a source. Assume for contradiction that G_1 has a source at a vertex u_x. The local view at u_x must be like one of the pictures in Figure 9 by Lemma 4.15. We shall show that any of them is impossible.

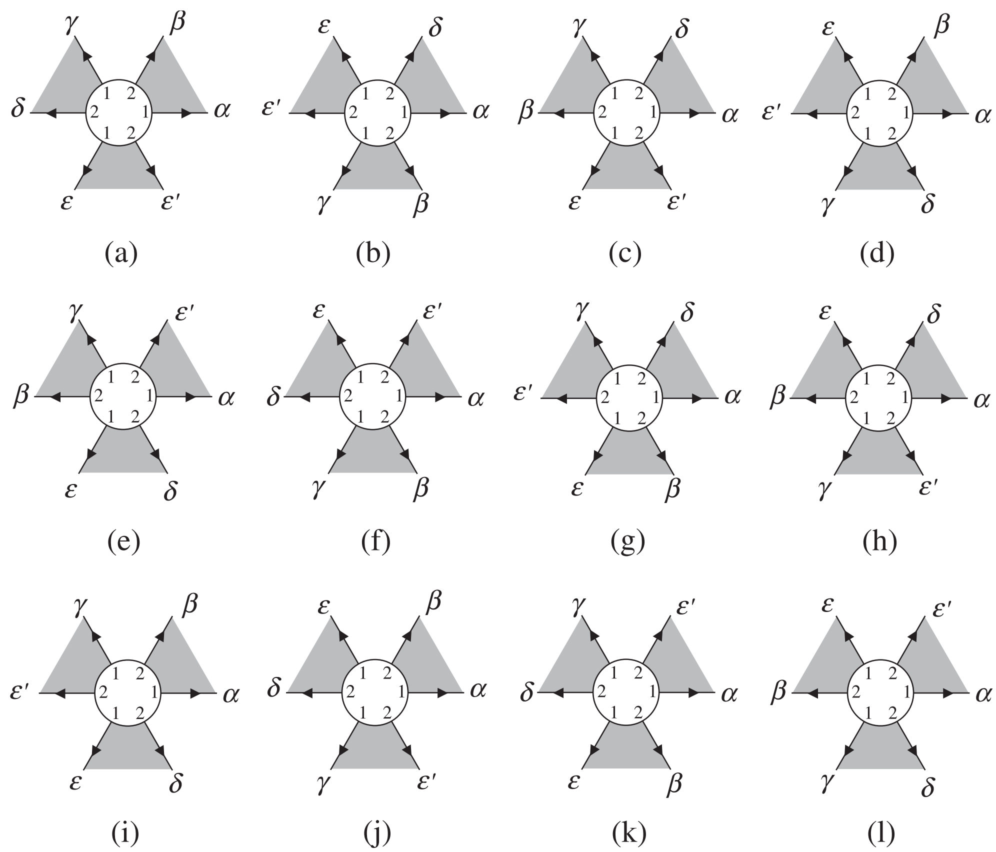


FIGURE 9

In G_2, the label x appears three times around the vertex v_1 at ends of α-, γ-, and ε-edge families and three times around the vertex v_2 at ends of β-, δ-, and ε'-edge families.

Claim. In G_2, the label x appears at the northern (resp. southern) end of ε-edge family if and only if it appears at the southern (resp. northern) end of ε'-edge family.

Proof. Assume, for example, that x appears at the northern ends of ε- and ε'-edge families. See Figure 10. Around the vertex v_1, x does not appear at an end of δ-edge family and at the southern end of ε-edge family, implying $w(\delta) + w(\varepsilon) < n_1$. Around the vertex v_2, x appears once at an end of δ-edge family and once at the northern end of ε'-edge family, implying $w(\delta)+w(\varepsilon') > n_1$. Since $w(\varepsilon) = w(\varepsilon')$, these two inequalities conflict. □

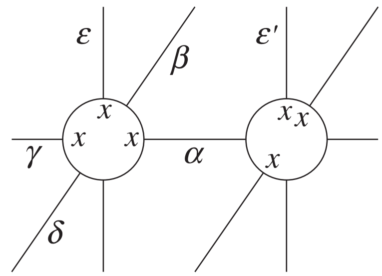


FIGURE 10

Let f be a black bigon of Λ with vertices u_y, u_z and g a black trigon with vertices u_p, u_q, u_r, as shown in Figure 11. Let C_1 be the corner of f at the vertex u_z and C_2 the corner of g at the vertex u_q.

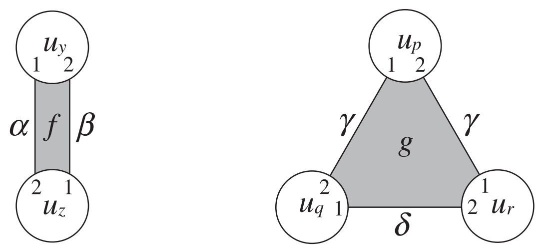


FIGURE 11

Assume the source at u_x looks like Figure 9(a), (b), (c) or (d). Let C be the black corner in ∂u_x running from an ε-edge endpoint to an ε'-edge endpoint. Then C, C_1, C_2 contradict [14, Lemma 3.2] by the above claim.

Assume the source at u_x looks like Figure 9(e) or (f). Let C be the black corner in ∂u_x running from a γ-edge endpoint to a β-edge endpoint. Then C, C_1, C_2 contradict [14, Lemma 3.2].

Assume the source at u_x looks like Figure 9(g) or (h). Let C be the black corner in ∂u_x running from an α-edge endpoint to a δ-edge endpoint. Then C, C_1, C_2 contradict [14, Lemma 3.2].

Assume the source at u_x looks like Figure 9(i) or (j). Let C be the black corner in ∂u_x running from a γ-edge endpoint to an ε'-edge endpoint and C' the black corner running from an ε-edge endpoint to a δ-edge endpoint. Then C, C', C_2 contradict [14, Lemma 3.2] by the above claim.

Assume the source at u_x looks like Figure 9(k) or (l). Let C be the black corner in ∂u_x running from an ε-edge endpoint to a β-edge endpoint and C' the black corner running from an α-edge endpoint to an ε'-edge endpoint. Then C, C', C_1 contradict [14, Lemma 3.2] by the above claim.

Using the same argument as above, we can see that G_1 has no sink. □

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