Math. Res. Lett. **13** (2005), *no.* 4, 531–537

A GENERAL BOUND FOR OSCILLATORY INTEGRALS WITH A POLYNOMIAL PHASE OF DEGREE k

O. Robert and P. Sargos

Abstract. Let $f \in \mathbb{R}[X_1, ..., X_n]$ be a polynomial of degree $k \geqslant 2$. We consider the oscillatory integral $I(\lambda) = \int \varphi(\mathbf{x})e^{i\lambda f(\mathbf{x})}d\mathbf{x}$, where φ is a C^1 function with compact support. A classical result due to E.M. Stein implies that $I(\lambda) = O(\lambda^{-1/k})$, as $\lambda \to +\infty$. The exponent $1/k$ is best possible, as shown by the example $f(\mathbf{x}) = f(\mathbf{x}_0) \pm L(\mathbf{x}-\mathbf{x}_0)^k$, where $\mathbf{x}_0$ is any point in $\mathbb{R}^n$ and L is any nonzero linear form on $\mathbb{R}^n$. In this paper, we show that, if f is precisely not of the above form, then the stronger bound $I(\lambda) = O(\lambda^{-1/(k-1)})$ holds, and the exponent $-1/(k-1)$ is best possible.

1. Statement of the result

We consider a polynomial $f \in \mathbb{R}[X_1, ..., X_n]$ of total degree k, i.e. $f(\mathbf{x}) = \sum_\alpha a_\alpha \mathbf{x}^\alpha$, with $\alpha = (\alpha_1, ..., \alpha_n)$, such that $|\alpha| = \alpha_1 + ... + \alpha_n \leqslant k$, for each α in the summation, and such that there exists at least one α with $|\alpha| = k$ and $a_\alpha \neq 0$; here, as usual, we have set $\mathbf{x}^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. Let us denote by $C_c^1(\Omega)$ the set of all functions φ which are C^1 with compact support contained in the open set Ω. Let Δ denote the largest size of those $|a_\alpha|$ for which $|\alpha| = k$. Then for any $\varphi \in C_c^1(\mathbb{R}^n)$, the general bound of Stein's Lemma (cf Lemma 1 below) applies here in the following form :

$$(1.1) \qquad \left|\int \varphi(\mathbf{x})e^{i\lambda f(\mathbf{x})}d\mathbf{x}\right| \leqslant C_1(k)(\Delta\lambda)^{-1/k}(\|\varphi\|_{L^\infty} + \|\varphi'\|_{L^1}), \text{ for all } \lambda > 0$$

where $C_1(k)$ is a positive constant that depends only on the degree k. Such a bound is quite uniform. Our aim is to improve the exponent $-1/k$ in $-1/(k-1)$, providing that f cannot be written as

$$(1.2) \qquad f(\mathbf{x}) = f(\mathbf{x}_0) \pm L(\mathbf{x} - \mathbf{x}_0)^k,$$

for some $\mathbf{x}_0 \in \mathbb{R}^n$ and some linear form L on $\mathbb{R}^n$. If so it is, we shall say that f satisfies the hypothesis $H_{k,n}$. Our analog of (1.1) is as follows.

Theorem 1. *We suppose that f satisfies the hypothesis $H_{k,n}$ and that Ω is a bounded open set in $\mathbb{R}^n$. We then have, for any $\varphi \in C_c^1(\Omega)$,*

$$(1.3) \qquad \left|\int \varphi(\mathbf{x})e^{i\lambda f(\mathbf{x})}d\mathbf{x}\right| \leqslant C_2(f, \Omega)\, \|\varphi\|_1\, \lambda^{-1/(k-1)}, \text{ for all } \lambda > 0$$

where $C_2(f, \Omega)$ is a positive constant which depends only on f and Ω, and where we have set

$$\|\varphi\|_1 = \|\varphi\|_{L^\infty} + \|\varphi'\|_{L^\infty}.$$

Received by the editors January 12, 2005.

This bound is far from being as uniform as (1.1), the constant $C_2(f,\Omega)$ depending on an abstract partition of unity. However, the exponent $-1/(k-1)$ is best possible, as shown by the example $f(x,y)=x^{k-1}y$, where in that case the true order of the oscillatory integral (restricted to a neighbourhood of the origin) is given by the stationnary phase method, see e.g. Theorem 3 in §8.3, CH.II of [1].

Now, we present a slight generalisation of Theorem 1, which may be of some interest, and which does not increase our proof. We consider a Lebesgue-measurable subset Γ of $\mathbb{R}^n$ which has the following property :

$$\text{"For any line } D, \text{ the set } D\cap\Gamma \text{ is the union of at most } N \text{ segments".} \tag{1.4}$$

The example we have in mind is the following : Γ is the intersection of a compact convex subset of $\mathbb{R}^n$ with the set

$$\{\mathbf{x}\in\mathbb{R}^n; Q(\mathbf{x})\geqslant 0\}\,, \text{ where } Q\in\mathbb{R}[X_1,...,X_n] \text{ has degree at most } N.$$

Theorem 2. *We suppose that all hypotheses of Theorem 1 are satisfied and that* Γ *satisfies* (1.4). *We then have, for any* $\varphi\in C_c^1(\Omega)$:

$$\left|\int_\Gamma \varphi(\mathbf{x})\mathrm{e}^{i\lambda f(\mathbf{x})}\mathrm{d}\mathbf{x}\right|\leqslant NC_3(f,\Omega)\left\|\varphi\right\|_1\lambda^{-1/(k-1)}, \text{ for all } \lambda>0 \tag{1.5}$$

where $C_3(f,\Omega)$ *is a positive constant which depends at most on* f *and* Ω.

Of course, Theorem 2 contains Theorem 1 and the rest of this paper is devoted to its proof.

2. Basic lemmas

We first recall Stein's fundamental lemma (cf [2], Proposition 5, page 342), with a slight modification concerning the domain Γ of integration.

Lemma 1. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$ *and let* $g:\Omega\to\mathbb{R}$ *be a regular function such that the derivative* $\partial^\alpha g=\frac{\partial^k g}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$*,with* $k=|\alpha|$*, and* $k\geqslant 1$*, satisfies*

$$|\partial^\alpha g(\mathbf{x})|\geqslant 1, \;\; \textit{for all } \mathbf{x}\in\Omega \tag{2.1}$$

and let Γ *satisfy* (1.4). *Then, for any* $\varphi\in C_c^1(\Omega)$*, one has*

$$\left|\int_\Gamma \varphi(\mathbf{x})\mathrm{e}^{i\lambda g(\mathbf{x})}\mathrm{d}\mathbf{x}\right|\leqslant NC_4(\left\|g\right\|_{k+1},k)(\left\|\varphi\right\|_{L^\infty}+\left\|\varphi'\right\|_{L^1})\lambda^{-1/k} \;\textit{for } \lambda>0 \tag{2.2}$$

where $C_4(\left\|\varphi\right\|_{k+1},k)$ *is a positive constant which depends at most on* k *and on the maximal size of the derivatives of order* $k+1$ *of* g.

Proof. In the case $N=1$, this lemma is exactly Stein's Lemma. For proving it in the case $N>1$, note that Stein's proof uses a linear change of variables which does not alter the property (1.4) and which reduces the problem to a one dimensional oscillatory integral, but, in our case, with several (at most N) intervals. It is thus obvious that the case $N>1$ reduces to the case $N=1$ by multiplying the final bound by N. □

The following elementary one dimensional lemma is also needed.

Lemma 2. *Let $k \geqslant 2$ be an integer, $s \geqslant 1$ be real, $\chi : [a,b] \to \mathbb{C}$ be a C^1 function, with $0 \leqslant a < b \leqslant 1$. We then have the bound*

$$\int_a^b \chi(t) \mathrm{e}^{i\lambda t^k} t^s \mathrm{d}t = O\left(\|\chi\|_1 \lambda^{-1/(k-1)}\right) \tag{2.3}$$

where the implied constant depends only on k.

Proof. We make a change of variable by setting $t = \tau^{(k-1)/k}$, and we get

$$\int_a^b \chi(t) \mathrm{e}^{i\lambda t^k} t^s \mathrm{d}t = \int_{a_1}^{b_1} \chi_1(\tau) \mathrm{e}^{i\lambda \tau^{k-1}} \mathrm{d}\tau,$$

where we have introduced obvious notations. As we have assumed $s \geqslant 1$ and $k \geqslant 2$, we have $\|\chi_1\|_{L^1} = O(\|\chi\|_1)$ and $\|\chi_1\|_{L^\infty} = O(\|\chi\|_1)$, so that we may apply the Corollary of Proposition 2 in page 332 of [2]. □

We need also an immediate algebraic lemma.

Lemma 3. *Let $P \in \mathbb{R}[X_1, ..., X_n]$, with $n \geqslant 2$, be a homogeneous polynomial which satisfies*

$$P(x_1, ..., x_{n-1}, 1) = 0, \text{ for all real numbers } x_1, ..., x_{n-1}.$$

Then $P(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. Of course, applying Taylor's formula over the x_n variable, we see that P is divisible by $x_n - 1$. □

3. The local form of the theorem

We establish now the main intermediate result in the proof of Theorem 2 ; we show how to divide the domain of integration according to the local properties of f, considering here the worst case.

Theorem 3. *Let $P \in \mathbb{R}[X_1, ..., X_n]$be a homogeneous polynomial of degree $k \geqslant 2$, satisfying the property*

(3.1) " *There does not exist a linear form L on $\mathbb{R}^n$ such that $P(x) = \pm L(x)^k$.*"

Then there exists an open neighbourhood V of 0 in $\mathbb{R}^n$ such that, for any set Γ satisfying (1.4) and any $\psi \in C_c^1(V)$, one has

$$\left| \int_\Gamma \psi(\mathbf{x}) \mathrm{e}^{i\lambda P(\mathbf{x})} \mathrm{d}\mathbf{x} \right| \leqslant N C_5(P) \|\psi\|_1 \lambda^{-1/(k-1)}, \tag{3.2}$$

where $C_5(P)$ is a positive constant which depends at most on P.

Proof. We divide the proof in several steps.

1) *Splitting the domain of integration*

We fix a C^1 test function ψ whose support is contained in $[-1,1]^n$. From now on, we shall use the symbol $u \ll v$ to mean that there exists a constant C (which depends at most on P and on other parameters that will be recalled when necessary), such that one has $|u| \leqslant Cv$. We then set

$$I(\lambda) = \int_\Gamma \psi(\mathbf{x}) \mathrm{e}^{i\lambda P(\mathbf{x})} \mathrm{d}\mathbf{x}$$

and we have to prove that

$$I(\lambda) \ll N \left\|\psi\right\|_1 \lambda^{-1/(k-1)}, \text{for all } \lambda > 0, \tag{3.3}$$

providing that ψ has its support contained in a sufficiently small neighbourhood of 0. We split the domain of integration into 2^n parts, writing, for each $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in \{-1, 1\}^n$,

$$\Gamma_\varepsilon = \{\mathbf{x} \in \Gamma; \varepsilon_j x_j \geqslant 0, j = 1, ..., n\}$$

so that we have $\Gamma = \cup_\varepsilon \Gamma_\varepsilon$. Moreover, for each ε, we split Γ_ε into n parts : for each $r = 1, 2, ..., n$, we define

$$\Gamma_{\varepsilon,r} = \{\mathbf{x} \in \Gamma_\varepsilon; |x_j| \leqslant |x_r| \text{ for } j = 1, ..., n\}$$

so that we have

$$|I(\lambda)| \leqslant n2^n \max_{\varepsilon,r} \left| \int_{\Gamma_{\varepsilon,r}} \psi(\mathbf{x}) \mathrm{e}^{i\lambda P(\mathbf{x})} \mathrm{d}\mathbf{x} \right|. \tag{3.4}$$

We are going to bound, for instance,

$$I_0(\lambda) = \int_{\Gamma_0} \psi(\mathbf{x}) \mathrm{e}^{i\lambda P(\mathbf{x})} \mathrm{d}\mathbf{x}, \text{ with } \Gamma_0 = \{\mathbf{x} \in \Gamma; 0 \leqslant x_j \leqslant x_n,\ j = 1, 2, ..., n-1\}$$

and we have to prove that there exists a neighbourhood V of 0 in $\mathbb{R}^n$ such that the bound

$$I_0(\lambda) \ll N \left\|\psi\right\|_1 \lambda^{-1/(k-1)}, \text{ for all } \lambda > 0 \text{ and for all } \psi \in C_c^1(V) \tag{3.5}$$

holds, the implied constant depending at most on P (and thus on n and k).

2) *A change of variables*

We want to prove Theorem 3 by induction on n. We note that, for $n = 1$, there is nothing to prove, because a homogeneous polynomial of degree k in one variable cannot satisfy (3.1). Thus, we suppose $n \geqslant 2$. If $n \geqslant 3$, we assume that the theorem have been proved up to the dimension $n - 1$, and if $n = 2$, we have nothing to assume, Lemma 1 being a sufficient reference. In order to make a change of variables which will reduce the dimension (in some way, at least), we define the sets $S = \{\mathbf{x} \in [0, 1]^n; x_j \leqslant x_n \text{ for } j = 1, ..., n-1\}$ and $T = [0, 1]^{n-1}$ We set $x_1 = tu_1, ..., x_{n-1} = tu_{n-1}, x_n = t$, so that we have the general formula

$$\int_S \chi(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_0^1 \int_T \chi(t\mathbf{u}, t) t^{n-1} \mathrm{d}t \mathrm{d}\mathbf{u}. \tag{3.6}$$

for any integrable function χ. Now, we define $\gamma(\mathbf{x})$ as being the characteristic function of Γ, so that we may write

$$I_0(\lambda) = \int_0^1 \int_T \gamma(t\mathbf{u}, t) \psi(t\mathbf{u}, t) \mathrm{e}^{it^k \lambda Q(\mathbf{u})} t^{n-1} \mathrm{d}t \mathrm{d}\mathbf{u}, \tag{3.7}$$

where we have set $Q(\mathbf{u}) = P(\mathbf{u}, 1)$.

3) *An intermediate property*

Let $\mathbf{a} \in T$ be fixed. We want to ensure the existence of a neighbourhood $V(\mathbf{a})$ of $\mathbf{a}$ in $\mathbb{R}^{n-1}$ such that

$$\int_0^1 \int_{T \cap V(\mathbf{a})} \gamma(t\mathbf{u}, t) \psi(t\mathbf{u}, t) \mathrm{e}^{it^k \lambda Q(\mathbf{u})} t^{n-1} \mathrm{d}t \mathrm{d}\mathbf{u} \ll N \left\|\psi\right\|_1 \lambda^{-1/(k-1)}. \tag{3.8}$$

Suppose first that $Q(\mathbf{a}) \neq 0$, say $|Q(\mathbf{a})| = 2\delta$, with $\delta > 0$. Then we choose $V(\mathbf{a})$ so small that $|Q(\mathbf{u})| \geqslant \delta$ throughout $V(\mathbf{a})$. For each fixed $\mathbf{u} \in V(\mathbf{a})$, we bound the integral $\int_0^1 \gamma(t\mathbf{u},t)\psi(t\mathbf{u},t)\exp(i\lambda Q(\mathbf{u})t^k)\mathrm{d}t$ by means of Lemma 2 ; for this, we have to recall that we have fixed $n \geqslant 2$, and to note that the function $t \to \gamma(t\mathbf{u},t)$ is the characteristic function of a union of at most N intervals. Integrating then over $\mathbf{u}$, we obtain (3.8) in the case $Q(\mathbf{a}) \neq 0$.

4) We consider now the more difficult case where $Q(\mathbf{a}) = 0$. We set $\mathbf{u} = \mathbf{a} + \mathbf{v}$ and $R(\mathbf{v}) = Q(\mathbf{a}+\mathbf{v})$; R is thus a polynomial in $n-1$ variables, of degree $\leqslant k$, with $R(\mathbf{0}) = 0$.

We write $R(\mathbf{v}) = \sum_\alpha b_\alpha \mathbf{v}^\alpha$, with $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in \mathbb{N}^{n-1}$, $|\alpha| \leqslant k$. We dismiss the case $R(\mathbf{v}) \equiv 0$; indeed, this would mean that $P(x_1 - a_1, ..., x_{n-1} - a_{n-1}, 1) \equiv 0$, which is impossible by Lemma 3.

Thus, we know that there is at least one index $\alpha \neq \mathbf{0}$ such that $b_\alpha \neq 0$.We first consider the case where this α satisfies $|\alpha| = l$, with $1 \leqslant l \leqslant k-1$.

For each fixed $t \in [0,1]$, we note that the function $\mathbf{u} \to \gamma(t\mathbf{u},t)$ is the characteristic function of a set in $\mathbb{R}^{n-1}$ which satisfies (1.4).

Now, the derivative $\partial^\alpha R(\mathbf{v})$ is equal to the constant term $(\alpha_1!)...(\alpha_{n-1}!)b_\alpha$ plus non constant monomials that will be small if we restrict $\mathbf{v}$ to a sufficiently small neighbourhood of $\mathbf{0}$ in $\mathbb{R}^{n-1}$. We have shown that there exists a neighbourhood W of $\mathbf{0}$ in $\mathbb{R}^{n-1}$ and a real $\delta > 0$, both depending only on $\mathbf{a}$ and P (and, in particular, not on t) such that $|\partial^\alpha R(\mathbf{v})| \geqslant \delta$ throughout W.

We set $V(\mathbf{a}) = \mathbf{a} + W$, so that we have $|\partial^\alpha Q(\mathbf{u})| \geqslant \delta$ throughout $V(\mathbf{a})$, and we apply Lemma 1 :

$$\int_{T\cap V(\mathbf{a})} \gamma(t\mathbf{u},t)\psi(t\mathbf{u},t)\exp(i\lambda t^k Q(\mathbf{u}))\mathrm{d}\mathbf{u} \ll N \|\psi\|_1 (\delta t^k \lambda)^{-1/l}, \tag{3.9}$$

for all $\lambda > 0$ and each $t \in [0,1]$.

Integrating this inequality over t, we set $A(t) = t^{n-1}\min\{1, (t^k\lambda)^{-1/l}\}$, and we write

$$\begin{aligned}\int_0^1 A(t)\mathrm{d}t &= \int_0^\tau A(t)\mathrm{d}t + \int_\tau^1 A(t)\mathrm{d}t \\ &\leqslant \int_0^\tau t^{n-1}dt + \lambda^{-1/l}\int_\tau^1 t^{n-1-k/l}dt \\ &\ll \tau^n(1+\lambda^{-1/l}\tau^{-k/l}) + \lambda^{-1/l}\end{aligned}$$

In this last bound, we take $\tau = \lambda^{-1/k}$ (assuming $\lambda \geqslant 1$, otherwise there is nothing to prove), and we get

$$\int_0^1 A(t)\mathrm{d}t \ll \lambda^{-n/k} + \lambda^{-1/l}.$$

From this, we recover (3.8).

5) For proving (3.8), it remains to consider the case where $Q(\mathbf{a}) = 0$ and where $R(\mathbf{v})$ is a homogeneous polynomial of degree k.

But such a situation cannot occur in the case $n = 2$. Indeed, $R(v)$ is a homogeneous polynomial of degree k and can be written as $R(v) = bv^k$, and thus, $P(x,1) =$

$b(x-a)^k$. By Lemma 3, the only polynomial $P(x,y)$, homogeneous of degree k, which satisfies $P(x,1)=b(x-a)^k$ is $P(x,y)=b(x-ay)^k$, so that (3.1) is not satisfied.

Now, we suppose $n \geqslant 3$. In the same way as above, we show that $R(\mathbf{v})$ cannot be written as $R(\mathbf{v}) = \pm L(\mathbf{v})^k$: otherwise we should have

$$P(x_1,...,x_{n-1},1) = \pm L(x_1-a_1,...,x_{n-1}-a_{n-1})^k$$

and this would imply that

$$P(x_1,...,x_{n-1},x_n) = \pm L(x_1-a_1x_n,...,x_{n-1}-a_{n-1}x_n)^k + P_0(\mathbf{x})$$

where $P_0(\mathbf{x})$ is a homogeneous polynomial which satisfies $P_0(x_1,...,x_{n-1},1)=0$, for all $x_1,...,x_{n-1}$.By Lemma 3, this is possible only if $P_0(\mathbf{x}) \equiv 0$. Thus R satisfies (3.1) in the lower dimension $n-1$.

From our recurrence hypothesis (that the theorem is true in the $n-1$ dimensional case), there exists a neighbourhood W of $\mathbf{0}$ in $\mathbb{R}^{n-1}$ such that, setting $V(\mathbf{a}) = \mathbf{a}+W$, we have

$$\int_{T\cap V(\mathbf{a})} \gamma(t\mathbf{u},t)\psi(t\mathbf{u},t)\mathrm{e}^{i\lambda t^k Q(\mathbf{u})}\mathrm{d}\mathbf{u} \ll N\,\|\psi\|_1\,(t^k\lambda)^{-1/(k-1)}. \tag{3.10}$$

We integrate this inequality over t as previously (see the corresponding proof in step 4), and we recover (3.8).

We have finally proved (3.8) unconditionally when $n=2$, and also for $n>2$, providing that the theorem is true in dimension $n-1$.

6) *Conclusion*

We treat together the cases $n=2$ and $n>2$ because they are identical, but one should have to prove firstly the case $n=2$, and then, the case $n>2$ by induction on n.

We have to prove (3.5). For each $\mathbf{a}\in T$, we choose a neighbourhood $V(\mathbf{a})$ as in (3.8). Let $\chi_1,...,\chi_s$ be C^1 functions on $\mathbb{R}^{n-1}$, each one having his support included in one of the $V(\mathbf{a})$, and such that $\sum_{r=1}^{s}\chi(\mathbf{u})=1$ for all $\mathbf{u}\in T$. We have

$$I_0(\lambda) = \sum_{r=1}^{s}\int_0^1\int_T \gamma(t\mathbf{u},t)\chi_r(\mathbf{u})\psi(t\mathbf{u},t)\mathrm{e}^{it^k\lambda Q(\mathbf{u})}t^{n-1}\mathrm{d}t\mathrm{d}\mathbf{u}.$$

We bound each integral in the sum using (3.8) and we obtain (3.5). The proof is complete. □

4. Proof of Theorem 2

Let f, φ, Ω and Γ be as in Theorem 2. For each $\mathbf{x}_0$ in the compact $\overline{\Omega}$, we are going to construct a neighbourhood $V(\mathbf{x}_0)$ of $\mathbf{x}_0$ in $\mathbb{R}^n$, so that, for each $\chi \in C_c^1(V(\mathbf{x}_0))$ and each $\lambda>0$, we have

$$\int_\Gamma \chi(\mathbf{x})\mathrm{e}^{i\lambda f(\mathbf{x})}\mathrm{d}\mathbf{x} \ll N\,\|\chi\|_1\,\lambda^{-1/(k-1)}, \tag{4.1}$$

where the implied constant depends at most on f. Assuming (4.1), it is easy to deduce (1.5) with a partition of unity, in the same way as above. Now, our aim is to prove (4.1).

We fix $\mathbf{x}_0 \in \overline{\Omega}$. We set $P(\mathbf{y}) = f(\mathbf{y}+\mathbf{x}_0) - f(\mathbf{x}_0)$; P is a polynomial of degree k, which is not of the form $\pm L(\mathbf{y})^k$. We have to find a neighbourhood W of $\mathbf{0}$ in $\mathbb{R}^n$ such that we have

$$\int_{\widetilde{\Gamma}} \chi(\mathbf{y}) \mathrm{e}^{i\lambda P(\mathbf{y})} \mathrm{d}\mathbf{y} \ll N \left\|\chi\right\|_1 \lambda^{-1/(k-1)}, \text{ for } \chi \in C_c^1(W) \text{ and } \lambda > 0, \tag{4.2}$$

where we have set $\widetilde{\Gamma} = -\mathbf{x}_0 + \Gamma$, and where the implied constant depends at most on f.

As P vanishes at $\mathbf{0}$, either P is a homogeneous polynomial of degree k, or we have

$$P(\mathbf{y}) = \sum_{\alpha} a_\alpha \mathbf{y}^\alpha, \text{ where } a_\alpha \neq 0 \text{ for some } \alpha \text{ with } 1 \leqslant |\alpha| \leqslant k-1. \tag{4.3}$$

In the first case, which is the more difficult, (4.2) is precisely the conclusion of Theorem 3, so that we may suppose that (4.3) holds. Let $\alpha \in \mathbb{N}^n$ such that $1 \leqslant |\alpha| \leqslant k-1$, and $|a_\alpha| = 2\delta$ for some $\delta > 0$. Then there exists a sufficiently small neighbourhood W of $\mathbf{0}$ in $\mathbb{R}^n$ such that the derivative $\partial^\alpha P$ satisfies$|\partial^\alpha P(\mathbf{y})| \geqslant \delta$ throughout W.

Thus we may apply Lemma 1 and this implies precisely (4.2). The proof of Theorem 2 is now complete. □

References

[1] V. Arnold, A. Varchenko, S. Goussein-Zade, *Singularités des applications différentiables*, Vol. II. (Russian) "Nauka", Moscow, 1984. Translation: Birkhuser Boston, Inc., Boston, MA (1988).

[2] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, **43**, Princeton University Press, Princeton, NJ, 1993.

Institut Elie Cartan, Université Henri Poincaré, Nancy 1 B.P. 239, 54506 Vandoeuvre-les-Nancy, France

E-mail address: `robert@iecn.u-nancy.fr, sargos@iecn.u-nancy.fr`