

AROUND MATRIX-TREE THEOREM

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ABSTRACT. Generalizing the classical matrix-tree theorem we provide a formula counting, for a given graph, its subgraphs with a fixed 2-core. We use this generalization to obtain an analog of the matrix-tree theorem for the root system D_n (the classical theorem corresponds to the A_n -case). Several byproducts of the developed technique, such as a new formula for a specialization of the multivariate Tutte polynomial, are of independent interest.

1. Introduction

Let us first fix some definitions and notation to be used throughout the paper. The main object of our study will be an undirected graph G without multiple edges. It is understood as a subset $G \subset \{\{i, j\} \mid i, j \in \{1, 2, \dots, n\}\}$, where elements of $\{1, 2, \dots, n\}$ are vertices and elements of G itself are edges. Informally speaking, this means that we mark (i.e. distinguish) vertices but not edges of G (except for Section 6 where an edge labeling will be used). Usually we will assume that G contains no loops, i.e. edges $\{i, i\}$. Directed graphs (appearing in Sections 2 and 5 for technical purposes) are subsets of $\{1, 2, \dots, n\}^2$. Since a graph is understood as a set of edges, notation $F \subset G$ means that F is a subgraph of G .

We will denote by $n = v(G)$ the number of vertices of G , by $\#G = e(G)$ the number of its edges, and by $k(G)$ the number of connected components. For every connected component $G_i \subset G$ ($i = 1, \dots, k(G)$) it will be useful to consider its Euler characteristics $\chi(G_i) = v(G_i) - e(G_i)$. A connected graph containing no cycles will be called a *tree*; a disconnected one, a *forest*. Note that the absence of cycles is equivalent to the equality $\chi(G_i) = 1$ for all i ; if cycles are present then $\chi(G_i) \leq 0$.

We will usually supply edges of the graph G with weights. A weight $w_{ij} = w_{ji}$ of the edge $\{i, j\}$ is an element of any algebra \mathcal{A} . For a subgraph $F \subset G$ denote $w(F) \stackrel{\text{def}}{=} \prod_{\{i,j\} \in F} w_{ij}$; call it the *weight* of F . For any set U of subgraphs of G call the expression $Z(U) = \sum_{F \in U} w(F)$ the *statistical sum* of U . (By definition, we assume $w_{ij} = 0$ if G contains no edge $\{i, j\}$.)

To a graph G with weighted edges one associates its *Laplacian matrix* L_G . It is a symmetric $(n \times n)$ -matrix with the elements

$$(L_G)_{ij} = \begin{cases} -w_{ij}, & i \neq j, \\ \sum_{k \neq i} w_{ik}, & i = j. \end{cases}$$

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The Laplacian matrix is degenerate; its kernel always contains the vector $(1, 1, \dots, 1)$. However, its principal $(n - 1) \times (n - 1)$ minors are generally nonzero and enter into the classical matrix-tree theorem whose first version was proved by G. Kirchhoff in 1847:

Theorem 1 ([8]). *Let T_G be the set of all (spanning) trees of G . Then $Z(T_G)$ is equal to any principal $(n - 1) \times (n - 1)$ minor of L_G .*

This theorem has numerous generalizations (for a review, see e.g. [1] and the references therein). For our purposes the most important will be the “all-minors” theorem by S. Chaiken [2].

Call a subset $J = \{(i_1, j_1), \dots, (i_m, j_m)\} \subset \{1, 2, \dots, n\}^2$ *component-disjoint* if $i_p \neq i_q$ and $j_p \neq j_q$ for every $p \neq q$; denote $\Sigma J \stackrel{\text{def}}{=} \sum_{p=1}^m (i_p + j_p)$. Fix a numeration of the pairs $(i_p, j_p) \in J$ such that $i_1 < \dots < i_m$, and denote by τ_J the permutation of $\{1, 2, \dots, m\}$ defined by the condition $j_{\tau_J(1)} < j_{\tau_J(2)} < \dots < j_{\tau_J(m)}$.

A forest F with the vertex set $\{1, 2, \dots, n\}$ is called *J -admissible* if it has m components, and every component contains exactly one vertex from the set $\{i_1, \dots, i_m\}$, and exactly one, from $\{j_1, \dots, j_m\}$ (these two may coincide if the sets intersect). Denote by $\gamma_{F,J}$ a permutation of the set $\{1, 2, \dots, m\}$ such that i_p and $j_{\gamma_{F,J}(p)}$ lie in the same component of F , for every $p = 1, 2, \dots, m$.

For an $n \times n$ -matrix M and a component-disjoint set J denote by $M(J)$ the submatrix of M obtained by deletion of the rows i_1, \dots, i_m and the columns j_1, \dots, j_m . For any permutation σ denote by $\varepsilon(\sigma) = \pm 1$ its sign (parity).

Theorem 2 ([2]). *For any component-disjoint subset $J \subset \{1, 2, \dots, n\}^2$ one has*

$$(-1)^{\Sigma J} \det(L_G)(J) = \sum_F \varepsilon(\tau_J \circ \gamma_{F,J}) w(F)$$

where the sum is taken over the set of all J -admissible subforests F of G .

Theorem 1 is a particular case of Theorem 2 corresponding to the situation when J contains one element only.

Most of this article is devoted to various generalizations of Theorem 1. In Section 2 we consider determinant-like expressions for statistical sums of subgraphs $F \subset G$ with cycles (namely, subgraphs with a given 2-core). In Section 3 we consider the case of subgraphs with vanishing Euler characteristics. Spanning trees of a graph G can be interpreted as irreducible linearly independent subsets of roots in the root system A_n ; in Section 4 we prove an analog of Theorem 1 for the root system D_n .

Two remaining sections form a sort of appendix to the paper. In Section 5 we give an explicit formula for the number $d(G)$ of orientations of the graph G without sources and sinks (this number enters Theorem 3). In Section 6 we provide a formula for the so called *external activity polynomial* which is a specialization of the multivariate Tutte polynomial of the graph G . The latter is defined as

$$T_G(q, w) = \sum_{m=1}^n q^n Z(U_m)$$

(see [12, 16, 18] for details) where U_m is the set of subgraphs of G having m connected components and w is a collection of weights of the edges. The formula we prove

(Theorem 9) is an alternating sign summation over partitions of the set of vertices of G .

In the end of the paper we discuss several open problems related to the main topic.

2. Graphs with a given 2-core

Let G be an undirected graph (loops and multiple edges are allowed). The maximal subgraph $G' \subset G$ such that every vertex of G' is an endpoint of at least two edges or is attached to a loop (that is, there are no “hanging” vertices) is called the *2-core* of G and denoted by $\text{core}_2(G)$. A graph G is the union of $\text{core}_2(G)$ and a forests (possibly none) attached to every vertex of $\text{core}_2(G)$.

A graph G is called *negative* if it contains no loops, no multiple edges, $G = \text{core}_2(G)$, and $\chi(G_i) < 0$ where $G_i, i = 1, \dots, k(G)$ are connected components of G . A graph G is called *non-positive* if all the above is true but $\chi(G_i) \leq 0$. A non-positive graph G is the union of a negative graph G_0 and several cycles, each cycle forming a separate connected component. We will code this situation as $G = G_0 \cup 3^{k_3} \dots n^{k_n}$ where k_s stands for the total number of cycles of length s .

For any directed graph Q (with the vertex set $\{1, 2, \dots, n\}$) denote by $[Q]$ the corresponding undirected graph. Given a $(n \times n)$ -matrix M with entries $a_{ij} \in \mathcal{A}$ define

$$\langle M, Q \rangle \stackrel{\text{def}}{=} \prod_{(i,j) \in Q} a_{ij}.$$

In particular, if $[Q] \subset G$ where G is a graph without loops or multiple edges, with weights w_{ij} (like in the previous section), then $\langle L_G, Q \rangle = (-1)^{e(Q)} w(Q)$.

A directed graph Q is called *regular* if the following two conditions are satisfied:

- (1) Q contains no sources or sinks, i.e. for every vertex there is at least one incoming and one outgoing edge.
- (2) If Q contains a loop (an edge (i, i)) or a pair of antiparallel edges (edges (i, j) and (j, i)) then they form a separate connected component of Q .

If Q is a regular directed graph then $[Q]$ consists of a non-positive graph and several loops and double edges (cycles of length 2), each loop and double edge forming a separate connected component. We will denote this by $[Q] = H \cup 1^{k_1} 2^{k_2}$ where H is non-positive and k_1, k_2 are the number of loops and double edges, respectively.

In what follows it will be convenient to allow graphs to have multiple (more specifically, double) edges. If F is a graph with multiple edges we will abuse notation writing $F \subset G$ if the graph obtained from F by neglecting the multiplicities is a subgraph of G . Computing the weights, we will, however, take multiplicities into account:

$$w(F) \stackrel{\text{def}}{=} \prod_{\{i,j\} \text{ is an edge of } F} w_{ij}^{m_{ij}}$$

where $m_{ij} \in \mathbb{Z}_{\geq 0}$ is the multiplicity of the edge $\{i, j\}$.

Let $H \subset G$ be a non-positive graph plus several double edges, each double edge forming a separate component. In other words, $H = H_0 \cup 2^{k_2} 3^{k_3} \dots n^{k_n}$ where H_0 is

negative. Then denote

$$\varrho_G(H) = \sum_{\substack{\Lambda \text{ is regular} \\ [\Lambda] = H \cup 1^{n-v(H)}}} \langle L_G, \Lambda \rangle$$

(so that the total number of vertices of Λ is n). By $U(H)$ denote the set of all subgraphs $F \subset G$ such that $\text{core}_2(F) = H$.

Theorem 3. *Let $H = H_0 \cup 2^{k_2} 3^{k_3} \dots n^{k_n}$ be a non-positive graph without loops together with several double edges. Then*

(2.1)

$$\varrho_G(H) = (-1)^{e(H)} d(H_0) \sum_{l_2=k_2}^n \dots \sum_{l_n=k_n}^n \binom{l_2}{k_2} \dots \binom{l_n}{k_n} 2^{l_3+\dots+l_n} Z(U(H_0 \cup 2^{l_2} \dots n^{l_n}))$$

where $d(H_0)$ is the number of orientations of H_0 without sources and sinks.

Corollary 1. *One has*

$$(2.2) \quad \begin{aligned} Z(U(H)) &= (-1)^{e(H_0)} d(H_0) 2^{-(k_3+\dots+k_n)} \times \\ &\times \sum_{l_2=k_2}^n \dots \sum_{l_n=k_n}^n (-1)^{l_2+l_3+\dots+l_n} \binom{l_2}{k_2} \dots \binom{l_n}{k_n} \varrho_G(H_0 \cup 2^{l_2} \dots n^{l_n}). \end{aligned}$$

Remark. Corollary 1 is our closest approximation to a “matrix-subgraph” theorem, that is, the best available analog of Theorem 1 for subgraphs of arbitrary structure. Indeed, the left-hand side of (2.2) is the statistical sum over the graphs with a fixed 2-core (for trees the 2-core is empty), while the right-hand side is a polylinear function of matrix elements of the Laplacian matrix (in the case of trees it was its principal minor). Notice that, unlike Theorem 1, the right-hand side of (2.2) cannot be computed in polynomial time. This is hardly surprising: it is known that the calculation of the Tutte polynomial (and even its value at almost any point of the plane) is a sharp P -hard problem (see [17, §9]). Therefore there is no hope to obtain a formula for the statistical sum of connected subgraphs in G with any given number of edges in the form of a determinant or, in general, to get a formula of polynomial complexity in terms of the Laplacian matrix.

Proof of Theorem 3. Let $\Lambda = \Lambda_0 \cup \Lambda_1$ be a regular subgraph of G such that $[\Lambda_0] = H$ and $[\Lambda_1] = 1^{n-v(H)}$. Now, $\langle L_G, \Lambda \rangle = \langle L_G, \Lambda_0 \rangle \langle L_G, \Lambda_1 \rangle$. Since Λ_0 contains no loops, then $\langle L_G, \Lambda_0 \rangle = (-1)^{e(H)} w(H)$.

One has $(L_G)_{ii} = \sum_{k \neq i} w_{ik}$, so that the term $\langle L_G, \Lambda_1 \rangle$ can be represented as the sum of monomials $w_{i_1 k_1} \dots w_{i_s k_s}$ where $\{i_1, \dots, i_s\}$ is the vertex set of Λ_1 . In other words, $\langle L_G, \Lambda_1 \rangle = \sum_{\Theta} w(\Theta)$ where Θ is the directed graph with $[\Theta] \subset G$ satisfying the following property: if $i \in \{i_1, \dots, i_s\}$ then Θ contains exactly one edge starting from i , and if $i \notin \{i_1, \dots, i_s\}$ is a vertex of Θ then it is a sink (no edge starts from it).

One can easily see that every connected component of Θ is either a tree such that all its vertices except the root are in $\{i_1, \dots, i_s\}$, or a graph with exactly one cycle with all its vertices in $\{i_1, \dots, i_s\}$. Thus, $\text{core}_2([\Lambda_0 \cup \Theta]) = H_0 \cup 2^{l_2} \dots n^{l_n}$, where $l_2 \geq k_2, \dots, l_n \geq k_n$.

On the other hand, let $F \subset G$ be a subgraph such that $\text{core}_2(F) = H_0 \cup 2^{l_2} \dots n^{l_n}$. To identify F with $[\Lambda_0 \cup \Theta]$ one has, first, to point out which “1-cycled” connected components of F belong to Λ_0 and which to Θ — there are $\binom{l_2}{k_2} \dots \binom{l_n}{k_n}$ ways to do this. Having this choice made one must orient the 2-core of F without sources and sinks — the number of such orientations being $d(H_0 \cup 2^{l_2} \dots n^{l_n}) = 2^{l_3 + \dots + l_n} d(H_0)$. \square

Proof of Corollary 1. One has $\frac{1}{k!} \frac{d^k}{dx^k} (1-x)^l = \sum_{s=0}^l \binom{s}{k} \binom{l}{s} (-1)^s x^s = \binom{l}{k} (1-x)^{l-k}$, and therefore

$$\sum_{s=0}^l (-1)^s \binom{s}{k} \binom{l}{s} = \begin{cases} 0, & \text{if } k < l, \\ 1, & \text{if } k = l. \end{cases}$$

The corollary is now straightforward. \square

3. Graphs with vanishing Euler characteristics

Corollary 1 becomes particularly simple if H is a cycle. Namely, if $H = s^1$ (a cycle of length s) then $\lambda_G(H)$ is the statistical sum of the set of all connected subgraphs $F \subset G$ having exactly one cycle of length s . The “negative part” H_0 of the graph H is empty which implies $d(H_0) = 1$.

Denote by Σ_n the symmetric group of order n acting on $\{1, 2, \dots, n\}$, and denote by \mathcal{D}_n the set of all partitions of n . For a permutation $\sigma \in \Sigma_n$ having k_1 cycles of length 1, k_2 cycles of length 2, etc., denote $D(\sigma) \stackrel{\text{def}}{=} 1^{k_1} \dots n^{k_n} \in \mathcal{D}_n$. Finally, for any function $f : \mathcal{D} \rightarrow \mathcal{A}$ define the f -determinant of an $(n \times n)$ -matrix M with entries $a_{ij} \in \mathcal{A}$ by the formula

$$\det_f(L) = \sum_{\sigma \in \Sigma_n} f(D(\sigma)) a_{1, \sigma(1)} \dots a_{n, \sigma(n)}.$$

Now one has

$$\begin{aligned} \varrho_G(2^{l_2} \dots n^{l_n}) &= \sum_{D(\sigma) = 1^{n-2l_2-\dots-nl_n} 2^{l_2} \dots n^{l_n}} (-1)^{n+2l_2+\dots+nl_n} (L_G)_{1, \sigma(1)} \dots (L_G)_{n, \sigma(n)} \\ &= \det_{\chi_{l_2, \dots, l_n}} L_G, \end{aligned}$$

where

$$\chi_{l_2, \dots, l_n}(1^{k_1} \dots n^{k_n}) = \begin{cases} (-1)^{n+2l_2+\dots+nl_n}, & \text{if } k_2 = l_2, k_3 = l_3, \dots, k_n = l_n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, Corollary 1 for a cycle takes the following form:

Statement 1. *The statistical sum of the set of subgraphs $F \subset G$ having one cycle of length $s \geq 3$ is equal to $\frac{1}{2} \det_{\tau_s} L_G$, where $\tau_s(1^{k_1} \dots n^{k_n}) = (-1)^{n+2k_2+\dots+nk_n} k_s$. The statistical sum of the set of subgraphs $F \subset G$ having one cycle of length 2 is $\det_{\tau_2} L_G$.*

This corollary implies the following formula which is the “matrix-tree theorem” for connected subgraphs containing exactly one cycle of any length $s \geq 3$, that is, connected subgraphs $H \subset G$ with $\chi(H) = 0$:

Corollary 2. *Let U_G be the set of all connected subgraphs of $H \subset G$ such that $\chi(H) = 0$. Then*

$$Z(U_G) = \frac{1}{2} \det_{\mu}(L_G)$$

where

$$(3.1) \quad \mu(1^{k_1} 2^{k_2} \dots n^{k_n}) = (-1)^{n+k_2+2k_3+\dots+(n-1)k_n} (2k_2 + k_3 + \dots + k_n).$$

A finer result concerning graphs $H \subset G$ such that $\chi(H_i) = 0$ for any connected component H_i of H ($i = 1, \dots, k(H)$) can be obtained using Theorem 2.

For a graph G and a component-disjoint set J denote by $G - J$ the graph obtained from G by deletion of all the edges $(i_p j_p)$ where $(i_p, j_p) \in J$. Then Theorem 2 implies

Statement 2. *Let G be a graph with the vertex set $\{1, 2, \dots, n\}$, without loops and multiple edges, with weights w_{ij} defined for all the edges. Let $J = \{(i_1, j_1), \dots, (i_m, j_m)\}$ be a component-disjoint subset of $\{1, 2, \dots, n\}^2$. Then*

$$(3.2) \quad (-1)^n \varepsilon(\tau_J) w_{i_1 j_1} \dots w_{i_m j_m} \det(L_{G-J})(J) = \sum_H (-1)^{k(H)} w(H)$$

where the sum is taken over the set of all subgraphs $H \subseteq G$ such that every connected component H_i of H contains one cycle (that is, $\chi(H_i) = 0$), the edges $\{i_1, j_1\}, \dots, \{i_m, j_m\}$ enter these cycles and vertices i_p and j_q alternate along the cycle.

Proof. It follows from Theorem 2 that the product $w_{i_1 j_1} \dots w_{i_m j_m} \det(L_{G-J})(J)$ is equal to the sum of $\pm w_{i_1 j_1} \dots w_{i_m j_m} w(F)$ where F runs over the set of subforests of $G - J$ having m components and such that the p -th component contains the vertices i_p and $j_{\gamma_{F,J}(p)}$; here $\gamma_{F,J}$ is the permutation of $\{1, 2, \dots, m\}$ defined in Section 1. In other words, $w_{i_1 j_1} \dots w_{i_m j_m} \det(L_{G-J})(J)$ is equal to the sum of $\pm w(H)$ where $H = F + J$ is the result of addition to F of the edges $\{i_1, j_1\}, \dots, \{i_m, j_m\}$. Thus, H is a graph with one cycle in every connected component; all edges $\{i_p, j_p\}$ enter the cycles, and vertices i_p and j_q alternate along the cycle. The connected components of H are in one-to-one correspondence with the cycles of the permutation $\gamma_{F,J}$. The sign of the term $w(F)$ is equal to $(-1)^n \varepsilon(\tau_J) \varepsilon(\tau_J \circ \gamma_{F,J}) = (-1)^n \varepsilon(\gamma_{G,J})$. The permutation $\gamma_{G,J}$ contains $k(H)$ cycles. The sign of any permutation of $\{1, 2, \dots, n\}$ with k cycles equals $(-1)^{n+k}$, and therefore, the total sign is $(-1)^{k(H)}$. \square

Denote now

$$Q_m = \sum_{\#J=m} w_{i_1 j_1} \dots w_{i_m j_m} \det(L_{G-J})(J)$$

where the sum is taken over the set of all component-disjoint subsets $J \subset \{1, 2, \dots, n\}^2$ of cardinality m . Statement 2 allows to express the generating function for the sequence Q_m :

Theorem 4. *One has*

$$(3.3) \quad \sum_{m=1}^{\infty} Q_m t^m = (-1)^n \sum_H w(H) \prod_{i=1}^{k(H)} ((1+t)^{\ell_i(H)} - 1)$$

where the sum in the right-hand side is taken over the set of all subgraphs $H \subset G$ such that $\text{core}_2(H_i)$ is a cycle of length $l_i(H)$; here $H_1, \dots, H_{k(H)}$ are connected components of H .

Proof. By Statement 2 one has that $Q_m = \sum_H a_m(H)w(H)$ where the sum is taken over the set of all subgraphs $H \subset G$ having exactly one cycle in every connected component. The coefficient $a_m(H)$ is equal, to $(-1)^{n+k(H)}$ times the number of component-disjoint sets $J = \{(i_1, j_1), \dots, (i_m, j_m)\}$ such that

- $\{i_p, j_p\} \in \text{core}_2(H)$ for all $p = 1, \dots, m$.
- For every cycle of H there is at least one edge (i_p, j_p) entering it.
- If a cycle of H has more than one edge (i_p, j_p) in it then the vertices i_p and j_q alternate along the cycle.

This obviously implies that

$$a_m(H) = (-1)^{n+k(H)} \sum_{\substack{m_1 + \dots + m_{k(H)} = m \\ m_1, \dots, m_{k(H)} \geq 1}} \binom{\ell_1(H)}{m_1} \dots \binom{\ell_{k(H)}(H)}{m_{k(H)}}$$

and (3.3) follows. □

Corollary 3.

$$(3.4) \quad \sum_{m=1}^{\infty} (-1)^m Q_m = \sum_H (-1)^{n+k(H)} w(H).$$

4. Linearly independent subsets of the root systems A_n and D_n

The technique of Section 3 can be used to obtain results on linearly independent subsets of finite root systems, cf. [11].

The set of positive roots $R_+(A_n)$ of the reflection group A_n consists of vectors $e_{ij} = b_i - b_j$, $1 \leq i < j \leq n$ where b_1, \dots, b_n is the standard basis in \mathbb{C}^n . We will assign to every root $e_{ij} \in R_+(A_n)$ its weight $w_{ij} \in \mathcal{A}$ where \mathcal{A} is any algebra. By definition $w_{ji} = w_{ij}$. For any subset $S \subset R_+(A_n)$ of positive roots consider a graph $\Gamma(S)$ with the vertices $1, \dots, n$ such that $\{i, j\}$ is an edge of $\Gamma(S)$ wherever $e_{ij} \in S$. The edge $\{i, j\}$ bears the weight w_{ij} . The graph $\Gamma(S)$ is undirected and contains no loops or multiple edges. If $S' \subset S$ then $\Gamma(S')$ is a subgraph of $\Gamma(S)$. We will write $w(S)$ instead of $w(\Gamma(S))$ for short and denote by L_S the Laplacian matrix of the graph $\Gamma(S)$.

For a given subset $S \subset R_+(A_n)$ one can consider the group $G(S)$ generated by the reflections in the roots $e_{ij} \in S$. The group $G(S)$ is a subgroup of the Weyl group of A_n , and therefore the space $V = \{\sum_{i=1}^n x_i b_i \mid \sum_{i=1}^n x_i = 0\} \subset \mathbb{C}^n$ is $G(S)$ -invariant. S is called *irreducible* if V is an irreducible representation of $G(S)$.

The following is obvious:

Theorem 5. *A set $S \subset R_+(A_n)$ is linearly independent if and only if $\Gamma(S)$ contains no cycles. S is irreducible if and only if $\Gamma(S)$ is connected. A linearly independent set $S' \subset S$ is maximal (among linearly independent subsets of S) if and only if $\Gamma(S')$ is a forest composed of spanning trees of connected components of $\Gamma(S)$. If S is irreducible*

(that is, $\Gamma(S)$ connected) then any maximal linearly independent subset S' of S is also irreducible (that is, $\Gamma(S')$ is a spanning tree of $\Gamma(S)$).

Using matroid terminology, one can reformulate Theorem 5 as follows. (See [12, 18] for more detail about matroids.)

Corollary 4. *A submatroid of the linear matroid of \mathbb{C}^n generated by vectors $e_{ij} \in S$ is isomorphic to the graphical matroid of $\Gamma(S)$.*

One can associate a weight $w_{ij} = w_{ji} \in \mathcal{A}$ to every root $e_{ij} \in R_+(A_n)$. So, one can consider weights of the root systems and statistical sums of sets of root systems, as was done for graphs in the previous sections. Now the matrix-tree theorem (i.e. Theorem 1) and Theorem 5 imply:

Statement 3. *Let $S \subset R_+(A_n)$ be irreducible and T_S be the collection of all maximal linearly independent subsets of S . Then $Z(T_S)$ is equal to (any) principal minor of the Laplacian matrix L_S .*

Consider now a similar question for the reflection group D_n . Its set $R_+(D_n)$ of positive roots consists of the vectors $e_{ij}^+ = b_i - b_j$ (the “+”-vectors) and $e_{ij}^- = b_i + b_j$ (the “-”-vectors) for all $1 \leq i < j \leq n$. We associate to every “+”-vector e_{ij}^+ the weight $u_{ij} \in \mathcal{A}$, and to every “-”-vector e_{ij}^- the weight $v_{ij} \in \mathcal{A}$. Notions of linearly independent, maximal and irreducible subsets $S \subset R_+(D_n)$ are defined exactly as in the A_n -case.

For every set $S \subset R_+(D_n)$ consider the graph $\Gamma(S)$ with the vertices $1, \dots, n$ where the vertices i and j are joined by the edge marked “+” if $e_{ij}^+ \in S$, and by the edge marked “-” if $e_{ij}^- \in S$. Thus, the graph $\Gamma(S)$ is undirected, contains no loops, and has at most two edges joining every pair of vertices; all its edges are marked by “+” or “-”, and if two edges join the same pair of vertices then their marks are different.

A cycle in $\Gamma(S)$ is called *odd* if it contains an odd number of edges marked “-”.

Theorem 6. *A set $S \subset R_+(D_n)$ is irreducible if and only if $\Gamma(S)$ is connected. S is linearly independent if and only if every connected component of $\Gamma(S)$ is either a tree or a graph with exactly one cycle, and this cycle is odd. If S is irreducible then a linearly independent set $S' \subset S$ is maximal if and only if the following holds: if $\Gamma(S)$ contains no odd cycles then $S' = S$, otherwise every connected component of $\Gamma(S')$ is a graph containing exactly one cycle, and this cycle is odd.*

This is a D -analog of Theorem 5 and it is obvious as well. Our goal in this section is to obtain a D -analog of Statement 3.

Let $J = \{(i_1, j_1), \dots, (i_m, j_m)\} \subset \{1, 2, \dots, n\}^2$ be a component-disjoint subset. Denote $S - J^- \stackrel{\text{def}}{=} S \setminus \{e_{i_1 j_1}^-, \dots, e_{i_m j_m}^-\}$.

Theorem 7. *One has*

$$\sum_{m=1}^n t^m \sum_{\#J=m} v_{i_1 j_1} \dots v_{i_m j_m} \det(L_{S-J^-})(J) = (-1)^n \sum_H w(H) \prod_{i=1}^{k(H)} ((1+t)^{\ell_i^-(H)} - 1).$$

Here the internal sum in the left-hand side is taken over the set of all component-disjoint sets $J \subset \{1, 2, \dots, n\}^2$ of cardinality m . The sum in the right-hand side is taken over the set of all subsets $H \subset S$ such that every connected component of

the graph $\Gamma(H)$ contains exactly one cycle. Above we denote by $k(H) \stackrel{\text{def}}{=} k(\Gamma(H))$ the number of these components, and by $\ell_i^-(H)$ ($i = 1, \dots, k(H)$) the number of “-”-edges entering the cycle in the i -th component.

The proof is completely analogous to that of Theorem 4. A required analog of Statement 3 is now:

Corollary 5.

$$(4.1) \quad \sum_{m=1}^n (-2)^m \sum_{\#J=m} v_{i_1 j_1} \dots v_{i_m j_m} \det(L_{S-J})(J) = (-1)^n \sum_F (-2)^{k(F)} w(F).$$

Here the internal sum in the left-hand side is taken over the set of all component-disjoint sets $J \subset \{1, 2, \dots, n\}^2$ of cardinality m . The sum in the right-hand side is taken over the set of all maximal linearly independent subsets $F \subset S$. As usual, $k(F)$ is the number of connected components of the graph $\Gamma(F)$.

Proof. This follows directly from Theorems 7 and 6 and the equality

$$\prod_{i=1}^k ((-1)^{\ell_i} - 1) = \begin{cases} (-2)^k, & \text{if all the } \ell_i \text{ are odd,} \\ 0, & \text{if at least one } \ell_i \text{ is even.} \end{cases}$$

□

5. Orientations without sources and sinks

Let G be an undirected graph with the vertex set $\{1, 2, \dots, n\}$, without loops (multiple edges are allowed). In this section we give a combinatorial description of the number $d(G)$ of directed graphs Q such that $[Q] = G$ and Q has no sources or sinks. (Recall that the number $d(G)$ enters equation (2.1).)

For a set of vertices $P \subset \{1, 2, \dots, n\}$ of G denote by $\langle P \rangle$ the subgraph of G spanned by P (i.e. having P as its vertex set and containing all the edges of G with both endpoints in P). Denote $k(P) \stackrel{\text{def}}{=} k(\langle P \rangle)$ for short and denote by $\mu(P) = e(\langle \{1, 2, \dots, n\} \setminus P \rangle)$, that is, the number of edges in G having both endpoints outside P .

Recall that a graph F is called *bipartite* if one can split its vertices into two groups such that every edge joins two vertices from different groups. Equivalently, this means that every closed path in F contains an even number of edges.

Theorem 8. *Assume that G has no isolated vertices. Then the number $d(G)$ of orientations of G without sources and sinks (i.e. such that for every vertex there is at least one incoming and one outgoing edge) is given by the expression*

$$d(G) = \sum_{m=0}^n (-1)^m \sum_{P:\#P=m, \langle P \rangle \text{ is bipartite}} 2^{\mu(P)+k(P)}.$$

(By assumption, $k(\emptyset) = 0$ and $\mu(\emptyset)$ is equal to the total number of edges in G .)

Proof. Fix a set P of vertices, and let $N(P)$ be the number of orientations of G such that every vertex from P is either a source or a sink. Since an edge cannot join two sources or two sinks, one has $N(P) = 0$ if $\langle P \rangle$ is not bipartite.

Suppose now that $\langle P \rangle$ is bipartite. Consider the graph F obtained by adding to $\langle P \rangle$ all the edges having one vertex in P and the other outside P . Apparently, $k(F) = k(P)$. Since G has no isolated vertices, every connected component of F has 2 orientations such that every its vertex is either a source or a sink. Thus, the total number of ways to orient the edges of F is $2^{k(P)}$. The number of edges of G not belonging to F is $\mu(P)$. These edges can be oriented arbitrarily, and, therefore, $N(P) = 2^{\mu(P)+k(P)}$. The statement follows now from the inclusion-exclusion formula. \square

Corollary 6. *The number of orientations of G without sources and sinks is given by*

$$(5.1) \quad d(G) = \sum_{m=0}^n (-1)^m \sum_{P: \#P=m} 2^{\mu(P)} \text{chr}_{\langle P \rangle}(2)$$

where $\text{chr}_F(\lambda)$ is the chromatic polynomial of the graph F , that is, the number of ways to color its vertices in λ colors so that any two adjacent vertices have different colors. (One assumes $\text{chr}_{\langle \emptyset \rangle} = 1$.)

Proof. One has $\text{chr}_F(2) = 2^{k(F)}$ if the graph F is bipartite, and $\text{chr}_F(2) = 0$ otherwise. \square

Corollary 7. *The number of orientations of G without sources and sinks is given by*

$$(5.2) \quad d(G) = \sum_{F \subseteq G} 2^{\mu(F)+k(F)} (-1)^{\chi(F)}$$

where the sum is taken over the set of all subgraphs $F \subseteq G$, and $\mu(F)$ is the total number of edges in G having no common vertices with the edges from F .

Proof. A classical result (see e.g. [18] for proof) relates the multivariate Tutte polynomial to the chromatic polynomial:

$$\text{chr}_H(\lambda) = T_H(\lambda, -1) = \sum_{\substack{F \subseteq H \\ v(F)=v(H)}} \lambda^{k(F)} (-1)^{e(F)}.$$

(-1 in the argument of T_H means that one takes $w_{ij} = -1$ for every edge $\{i, j\}$ of H). Now by Corollary 6,

$$d(G) = \sum_{m=0}^n (-1)^m \sum_{\substack{F \subseteq G \\ v(F)=m}} 2^{\mu(F)} 2^{k(F)} (-1)^{e(F)} = \sum_{F \subseteq G} 2^{\mu(F)+k(F)} (-1)^{\chi(F)}.$$

where $\chi(F) = v(F) - e(F)$ is the Euler characteristics of F . \square

6. Multivariable external activity polynomial

In the previous section we made use of the fact that the chromatic polynomial of a graph is a specialization of its multivariate Tutte polynomial $T(q, w)$. Below we consider another specialization of T which we call the *external activity polynomial*.

As in Section 2 let G be a graph without loops or multiple edges and with the weights $w_{ij} \in \mathcal{A}$ assigned to its edges. Suppose that G is connected and fix an arbitrary numeration of its edges. Let T be a spanning tree of G and e be an edge not entering T . The graph $T \cup e$ has exactly one cycle, and this cycle contains the edge e . An edge e is called *externally active* for T if e is the smallest edge (with respect to the above numeration) in the cycle. The polynomial

$$C_G(w) = \sum_{T \text{ is a spanning tree of } G} w(T) \prod_{\{i,j\} \text{ is externally active for } T} (w_{ij} + 1).$$

will be called the *external activity polynomial* of G . Its specialization (re)appeared recently in the form of the Hilbert polynomial for a certain commutative algebra related to G , see [10].

Obviously, the following statement holds:

Statement 4. *If G is connected then $C_G(w) = Z(U_0)$, where U_0 is the set of all connected spanning subgraphs of G .*

Corollary 8 ([12]). *If G is connected then $C_G(w) = \lim_{q \rightarrow 0} T_G(q, w)/q$ where T_G is the multivariate Tutte polynomial.*

We present another expression for the polynomial C_G :

Theorem 9.

$$(6.1) \quad C_G(w) = \sum_{k=1}^n (-1)^{k-1} (k-1)! \sum_{\{1,2,\dots,n\}=P_1 \sqcup \dots \sqcup P_k} \prod_{\{i,j\} : \exists s \ i,j \in P_s} (w_{ij} + 1)$$

where the internal sum taken is over all partitions of the set of vertices into $k \geq 1$ pairwise disjoint subsets, and the product is taken over the set of all edges $\{i, j\}$ of the graph G such that both endpoints (i and j) belong to some P_s , $s = 1, \dots, k$.

To prove Theorem 9 we need the following technical lemma.

Lemma 1. *For any $n \geq 2$ one has*

$$(6.2) \quad \sum_{k=1}^n (-1)^{k-1} \sum_{\{1,2,\dots,n\}=P_1 \sqcup \dots \sqcup P_k} (p_1 - 1)! \dots (p_k - 1)! = 0,$$

where $p_i = \#P_i$ is the cardinality of P_i .

Proof. Denote by $a_{k,n}$ the coefficient at $(-1)^{k-1}(k-1)!$ in (6.2) and use induction on n to prove the lemma. For $n = 2$ one has $a_{1,2} = 1$ (the only possible set partition is $\{1, 2\} = \{1, 2\}$) and $a_{2,2} = 1$ (the only possible set partition is $\{1, 2\} = \{1\} \sqcup \{2\}$), so that (6.2) holds. Let now it hold for some n . Partitions of the set $\{1, 2, \dots, n+1\}$ fall into two types: either $\{1, 2, \dots, n+1\} = P_1 \sqcup \dots \sqcup P_k \sqcup \{n+1\}$, or $\{1, 2, \dots, n+1\} =$

$P_1 \sqcup \dots \sqcup (P_s \cup \{n+1\}) \sqcup \dots \sqcup P_k$, where in both cases $P_1 \sqcup \dots \sqcup P_k$ is a partition of $\{1, 2, \dots, n\}$. The sum taken over the set partitions of the first type is

$$\sum_{k=1}^n (-1)^k \sum_{\{1,2,\dots,n+1\}=P_1 \sqcup \dots \sqcup P_k \sqcup \{n+1\}} (p_1 - 1)! \dots (p_k - 1)! 0! = 0$$

by the induction hypothesis. The sum taken over the set partitions of the second type equals

$$\begin{aligned} & \sum_{k=1}^n (-1)^{k-1} \sum_{s=1}^k \sum_{\{1,2,\dots,n+1\}=P_1 \sqcup \dots \sqcup (P_s \cup \{n+1\}) \sqcup \dots \sqcup P_k} (p_1 - 1)! \dots p_s! \dots (p_k - 1)! \\ &= \sum_{s=1}^k (-1)^{k-1} \sum_{s=1}^k p_s \sum_{\{1,2,\dots,n\}=P_1 \sqcup \dots \sqcup P_s \sqcup \dots \sqcup P_k} (p_1 - 1)! \dots (p_s - 1)! \dots (p_k - 1)! \\ &= n \sum_{s=1}^k (-1)^{k-1} \sum_{\{1,2,\dots,n\}=P_1 \sqcup \dots \sqcup P_k} (p_1 - 1)! \dots (p_k - 1)! \\ &= 0. \end{aligned}$$

□

Proof of Theorem 9. Notice first that for any graph F the statistical sum of *all* subgraphs $H \subset F$ equals

$$Z(\{H \mid H \subset F\}) = \prod_{\{i,j\} \text{ is an edge of } F} (1 + w_{ij}).$$

Consider the poset \mathcal{P}_n of all set partitions of $\{1, 2, \dots, n\}$ ordered by refinement. In particular, $\min = \{1\} \sqcup \dots \sqcup \{n\}$ is the smallest element of \mathcal{P}_n , and $\text{Max} = \{1, 2, \dots, n\}$ is its largest element.

Recall (from Section 5) that for a set $P \subset \{1, 2, \dots, n\}$ one denotes by $\langle P \rangle$ the subgraph of G with the vertex set P ; edges of $\langle P \rangle$ are all the edges of G with both endpoints in P . Statement 4 implies now that

$$Z(\{H \mid H \subset F\}) = \sum_{(P_1 \sqcup \dots \sqcup P_k) \in \mathcal{P}_n} C_{\langle P_1 \rangle}(w) \dots C_{\langle P_k \rangle}(w).$$

By Lemma 1 the value $\mu(u, \text{Max}) = (-1)^{k-1} (k - 1)!$ where $\mu(u, v)$ is the Möbius function for the poset \mathcal{P}_n . Therefore theorem follows from the Möbius inversion formula, see e.g. [13]. □

Questions and final remarks

Multivariate Tutte polynomial has been studied intensively since 1970s when it was found to be related to partition functions of some important models in mathematical physics (Ising model, Potts model, and more; for more information consult [5, 3], references therein and also the review paper [12].) Of particular interest are the complex zeros of these polynomials because they are responsible for the phase transition in ferromagnetic and antiferromagnetic media. It might be very interesting to study the zeros of the external activity polynomial G_G . Many natural questions

about them (including the half-plane property, see [12]) are still open. Notice however that by Corollary 8 the polynomial C_G is related to the specialization of the Tutte polynomial at $q = 0$ while in Potts model q is interpreted as a number of states (of the spin) — we are leaving it to professional physicists to give a sensible interpretation to the polynomial C_G .

Another possible direction of study is suggested by the nature of formulas (2.1), (2.2), (3.2), (3.4), (4.1), (5.1), (5.2) and (6.1): they all contain a sign alternating summation. It is highly probable that these formulas present the Euler characteristics of suitable complexes; we pose a problem of finding these complexes. They must be related to the categorification of the Tutte polynomial obtained recently by E.F. Jasso-Hernandez and Y. Rong in [6]; see also [7, 14, 15] where the categorification of the chromatic polynomial was carried out.

Section 3 of this paper contains two different descriptions of the statistical sum of graphs G such that every connected component of G has vanishing Euler characteristics. One is given by the Statement 1 and its corollary (both are special cases of Theorem 3), and the other is contained in Theorem 4 and its corollary, based on the all-minors version of the matrix-tree theorem. The relation between these results resembles the relation between a determinant and its minors decomposition. It seems very interesting to find similar results for graphs with an arbitrary 2-core.

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