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EXAMPLES CONCERNING WHITNEY'S $\mathcal{C}^m$ EXTENSION PROBLEM

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Abstract. We present several examples connected with our articles on Whitney's extension problem. The first shows that Glaeser iterations cannot be avoided in the $\mathcal{C}^m$ extension criterion of [F2], and the remaining three are counterexamples to $\mathcal{C}^m$ extension assuming a weaker criterion that was used to prove a $\mathcal{C}^\infty$ extension theorem for closed subanalytic sets, in [BMP1].

1. Introduction

We present several examples connected with our articles [BMP1, BMP2], [F1, F2] on Whitney's extension problem (cf. [W]). The $\mathcal{C}^m$ extension criterion of [F2] is a variant of that in [BMP1]. Both involve iterated "Glaeser operations" on generalized finite difference operators. Example 2.1 below shows that Glaeser iterations cannot be avoided in the criterion of [F2]. Examples 3.1–3.3 are counterexamples to $\mathcal{C}^m$ extension assuming the weaker criterion of [BMP1], used in the latter to prove a $\mathcal{C}^\infty$ extension theorem (or a $\mathcal{C}^m$ theorem with loss of differentiability) for closed subanalytic sets.

Let $\mathcal{P} = \mathcal{P}^m(\mathbb{R}^n)$ denote the vector space of real polynomial functions of degree $\leq m$ on $\mathbb{R}^n$. Let S denote a finite subset of $\mathbb{R}^n$. The space $W^m(S)$ of *Whitney* $\mathcal{C}^m$ *functions* on S is the space of sections of $S \times \mathcal{P}$, with the *Whitney* $\mathcal{C}^m$ *norm*

$$\|P\|_{W^m(S)} = \max\left\{ \max_{\substack{a\in S\\|\alpha|\leq m}} |\partial^\alpha P_a(a)|,\ \max_{\substack{a\neq b \text{ in } S\\|\alpha|\leq m}} \frac{|\partial^\alpha (P_a - P_b)(a)|}{|a-b|^{m-|\alpha|}} \right\},$$

where $P = (P_a)_{a\in S} \in W^m(S)$. (Each $P_a \in \mathcal{P}$).

Given a Banach space B, with norm $\|\cdot\|_B$, we write $\|\cdot\|_{B^*}$ for the dual norm on B^*.

Elements $\xi \in W^m(S)^*$ can be identified with sections $\xi = (\xi_a)_{a\in S}$ of $S \times \mathcal{P}^*$. Let $P = (P_a)_{a\in S} \in W^m(S)$ and $\xi = (\xi_a)_{a\in S} \in W^m(S)^*$. Fix a point $a_0 \in S$ (a *reference point*). Then

$$\xi(P) = \sum_{a\in S} \xi_a (P_a - P_{a_0}) + \left(\sum_{a\in S} \xi_a\right)(P_{a_0}).$$

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Writing

$$\xi_a\left(P_a - P_{a_0}\right) = \sum_{|\alpha|\le m} \xi_a\left(\frac{(x-a)^\alpha}{\alpha!}\right)|a-a_0|^{m-|\alpha|}\frac{\partial^\alpha\left(P_a - P_{a_0}\right)(a)}{|a-a_0|^{m-|\alpha|}},$$

for each $a \in S\backslash\{a_0\}$, we get

$$(1.1)\quad |\xi(P)| \le \left(\sum_{\substack{a\in S\backslash\{a_0\}\\ |\alpha|\le m}}\left|\xi_a\left(\frac{(x-a)^\alpha}{\alpha!}\right)\right||a-a_0|^{m-|\alpha|} + \left\|\sum_{a\in S}\xi_a\right\|_{\mathcal{P}^*}\right)\cdot\|P\|_{W^m(S)}.$$

The expression in big brackets is a norm on $W^m(S)^*$ which, by (1.1), majorizes the dual Whitney norm $\|\xi\|_{W^m(S)^*}$.

Suppose that $S \subset L$, where L is a closed cube in $\mathbb{R}^n$. Then there is a natural surjection $\mathcal{C}^m(L) \ni F \mapsto P \in W^m(S)$, where $P = (P_a)_{a\in S}$ and P_a is the Taylor polynomial $T^m_a F$. It follows from Whitney's classical extension theorem (cf. [M, Complement 3.5]) that $\|\cdot\|_{W^m(S)}$ is equivalent to the quotient norm from $\|\cdot\|_{\mathcal{C}^m(L)}$, uniformly with respect to finite subsets S of L. Examples 3.1–3.3 show that the norm given by the expression in brackets in (1.1) is not equivalent to $\|\cdot\|_{W^m(S)^*}$, uniformly with respect to $S \subset L$, $\#S \le k$. The criterion for $\mathcal{C}^m$ extension of [F2] (as formulated in [BMP2]) involves bounds on $\|\cdot\|_{W^m(S)^*}$, whereas the criterion of [BMP1] is formulated in the same way using bounds on the norm given by (1.1). (See §1.1, §1.2 below.)

One can obtain a norm uniformly equivalent to $\|\cdot\|_{W^m(S)^*}$ by using a version of (1.1) that depends on a "clustering" of S, with an arbitrary choice of reference point in each cluster [BM].

In the examples following, we use several notions from [BMP2] that we briefly recall. Let E denote a closed subset of $\mathbb{R}^n$. We define the $\mathcal{C}^m$ *Zariski paratangent bundle* $\mathcal{T}^m(E)$ as

$$\mathcal{T}^m(E) = \{(a,\xi) \in E\times\mathcal{P}^* : \ \xi(T^m_a F) = 0,\ F \in \mathcal{I}^m(E)\}$$

where $\mathcal{I}^m(E) \subset \mathcal{C}^m(\mathbb{R}^n)$ denotes the ideal of $\mathcal{C}^m$ functions vanishing on E. $\mathcal{T}^m(E)$ is a linear subbundle of $E \times \mathcal{P}^*$.

(If V is a finite-dimensional vector space, then a *linear subbundle* of $E\times V$ means a subset Γ of $E \times V$ such that, for all $a \in E$, the *fibre* $\Gamma(a) := \{v \in V : \ (a, v) \in E\}$ is a linear subspace of V.)

1.1. Glaeser operation. Fix a positive integer k. Given a linear subbundle T of $E \times \mathcal{P}^*$, we define a new linear subbundle $g(T)$ of $E \times \mathcal{P}^*$, as follows: The fibre $g(T)(a_0)$, where $a_0 \in E$, is defined as the linear span of all elements $\xi \in \mathcal{P}^*$ that are obtained in the following way: There are subsets $S_i \subset E$, $\#S_i \le k$, and elements $\xi_i \in W^m(S_i)^*$, $i = 1, 2, \dots$, such that

(1) All $a \in S_i$ converge to a_0 as $i \to \infty$.
(2) For each i, $\xi_i = (\xi_{ia})_{a\in S_i}$, where each $\xi_{ia} \in T(a) \subset \mathcal{P}^*$.
(3) $\|\xi_i\|_{W^m(S_i)^*} \le c$, for all i, where c is a constant;
(4) $\xi = \lim_{i\to\infty}\sum_{a\in S_i}\xi_{ia}$ in $\mathcal{P}^*$.

Then $T \mapsto g(T)$ is a *Glaeser operation* in the sense of [BMP1, Def. 3.2].

Let $\varphi : T \to \mathbb{R}$ denote a function which is linear on the fibres of T. Let $a_0 \in E$. Suppose there exists a linear function $g(\varphi)(a_0, \cdot) : g(T)(a_0) \to \mathbb{R}$ such that

$$g(\varphi)(a_0, \xi) \;=\; \lim_{i\to\infty} \sum_{a \in S_i} \varphi(a, \xi_{ia})$$

whenever $\xi \in g(T)(a_0)$ is obtained as above. Clearly, $g(\varphi)(a_0, \cdot)$ is unique if it exists. If $g(\varphi)(a, \cdot)$ exists for all $a \in E$, then we call the resulting mapping $g(\varphi) : g(T) \to \mathbb{R}$ the *Glaeser extension* of φ.

1.2. Higher-order tangent bundle. Fix k. We define a *higher-order tangent bundle* (or *paratangent bundle*) $T_k^m(E) \subset E \times \mathcal{P}^*$ as follows: We begin with the line bundle $T_0 \subset E \times \mathcal{P}^*$ defined by

$$T_0 \;=\; \{(a, \lambda\delta_a) :\; a \in E,\; \lambda \in \mathbb{R}\}\,,$$

where δ_a is the *delta function* $\delta_a(P) := P(a)$, $P \in \mathcal{P}$. We then define a sequence of linear subbundles $T_0 \subset T_1 \subset \cdots$ of $E \times \mathcal{P}^*$, by iterated Glaeser operations: $T_l = g(T_{l-1})$, $l = 1, 2, \ldots$. Let $r = \dim \mathcal{P}$. By Glaeser's lemma [G], [BMP1, Lemma 3.3], T_{2r} is a closed linear subbundle $T_k^m(E)$ of $E \times \mathcal{P}^*$, and $T_l = T_{2r}$, for all $l \geq 2r$.

Now consider $f : E \to \mathbb{R}$. We define $\varphi_0 : T_0 \to \mathbb{R}$ by $\varphi_0(a, \lambda\delta_a) = \lambda\varphi(a)$. Clearly, φ_0 is linear on the fibres of T_0. We inductively define $\varphi_l : T_l \to \mathbb{R}$ by $\varphi_l = g(\varphi_{l-1})$, $l = 1, 2, \ldots$, provided that the Glaeser extension $g(\varphi_{l-1})$ exists. If φ_l exists for all l, then we denote φ_{2r} by $\nabla_k^m f$ and we say that $\nabla_k^m f : T_k^m(E) \to \mathbb{R}$ is the *Glaeser extension* of f.

We can define a second Glaeser operation $T \mapsto \rho(T)$ by replacing (3) in the definition of $g(T)$ above by the condition:

(3′) $|a - a_i|^{m-|\alpha|}\, |\xi_{ia}\,((x-a)^\alpha/\alpha!)| \leq c$, for all $i = 1, 2, \ldots$, $a \in S_i \backslash \{a_i\}$, $|\alpha| \leq m$, where c is a constant and $a_i \in S_i$, for all i.

Furthermore, for every $\varphi : T \to \mathbb{R}$ linear on the fibres, we can define a Glaeser extension $\rho(\varphi) : \rho(T) \to \mathbb{R}$ as above, using the Glaeser operation ρ instead of g. Then $\rho(T) \subset g(T)$, by (1.1), and, if $g(f)$ exists, then $\rho(f) = g(f)|\rho(T)$.

Let $\tau_k^m(E)$ denote the paratangent bundle defined as above, using the Glaeser operation ρ in place of g. Then

$$\tau_k^m(E) \subset T_k^m(E) \subset \mathcal{T}^m(E)\,.$$

The main results of [F2] (in the dual formulation of [BMP2]) are the following:

Theorem 1.1. *There is a positive integer $k = k^\#(m, n)$ such that, if $f : E \to \mathbb{R}$, then:*

(a) *f is the restriction of a $\mathcal{C}^m$ function if and only if f extends to $\nabla_k^m f : T_k^m(E) \to \mathbb{R}$. Moreover, if $F \in \mathcal{C}^m(\mathbb{R}^n)$ and $F|E = f$, then, for all $a \in E$ and $\xi \in T_k^m(E)(a)$, $\nabla_k^m f(a)(\xi) = \xi(T_a^m F)$.*

(b) *Suppose that f extends to $\nabla_k^m f :\; T_k^m(E) \to \mathbb{R}$. If $a_0 \in E$ and $(\nabla_k^m f)(a_0) = 0$, then there exists $F \in \mathcal{C}^m(\mathbb{R}^n)$ such that $F|E = f$ and $T_{a_0}^m F = 0$.*

Corollary 1.2. *If $k = k^\#(m, n)$, then $T_k^m(E) = \mathcal{T}^m(E)$.*

In these results, one can take $k^{\#}(m,n) = 2^{\dim \mathcal{P}}$ [BM]. (See also [S].)

In [BMP1] (p. 330), it was conjectured that the preceding results hold using $\tau_k^m(E)$ (for suitable k) in place of $T_k^m(E)$; Examples 3.1–3.3 below are counterexamples.

2. On Glaeser iterations

We give an example of a closed subset E of $\mathbb{R}^2$ for which the $\mathcal{C}^1$ paratangent bundle $T_k^1(E)$ cannot be defined without iterated Glaeser operations, no matter how large we take k; in other words, for any k, the bundle $g(T_0)$ defined in §1.2 is a proper subbundle of the Zariski paratangent bundle $\mathcal{T}^1(E)$, so that $g(T_0) \subsetneqq T_k^1(E)$, by Corollary 1.2. Klartag and Zobin have recently showed that, for any $k \geq 2$, there is a closed subset E of $\mathbb{R}^n$ for which n Glaeser iterations are necessary, and that $n+1$ iterations are enough to obtain $\mathcal{T}^1(E)$ for any $E \subset \mathbb{R}^n$ [KZ]. ([BMP1, Example 1.8] shows that Glaeser iterations cannot be avoided in the criterion of [BMP1].)

Example 2.1. Let $\{\gamma_l\}_{l\geq 1}$ and $\{\theta_l\}_{l\geq 1}$ denote decreasing sequences of positive numbers, both with limit 0. (Assume that $\theta_l < \pi/4$, $l = 1, 2, \ldots$.) For each $l \geq 1$, let

$$E_l := \{(\gamma_l, 0)\} \cup \{b_{lm} : m \geq 1\} \subset \mathbb{R}^2,$$

where $b_{l1} = (\delta_l, 0)$, $\gamma_l < \delta_l < \gamma_{l-1}$, and, for $m \geq 1$:

(1) If m is odd, then $b_{l,m+1}$ denotes the intersection point of the line $L_l :\ y = (\tan\theta_l)(x - \gamma_l)$ with the line through b_{lm} with slope $-\tan\theta_l$.
(2) If m is even, then $b_{l,m+1}$ denotes the intersection of the x-axis with the line through b_{lm} with slope $\tan 2\theta_l$.

Let

$$E := \bigcup_l E_l \cup \{(0,0)\}.$$

Clearly, if $f \in \mathcal{C}^1(\mathbb{R}^2)$ and $f = 0$ on E, then $(\operatorname{grad} f)(\gamma_l, 0) = 0$, for all l, so the $\mathcal{C}^1$ Zariski paratangent space $\mathcal{T}^1(E)(0)$ of E at 0 equals $\mathcal{P}^1(\mathbb{R}^2)^*$ (i.e., $\mathcal{T}^1(E)(0)$ is spanned by δ_0, $(\partial/\partial x)|_0$, and $(\partial/\partial y)|_0$).

Claim. Given any positive integer k, there does not exist a sequence

$$\xi_i = \sum_{j=1}^{k} \lambda_{ij}\delta_{a_{ij}} \quad i = 1, 2, \ldots,$$

such that all $a_{ij} \in E \setminus \{0\}$, $a_{ij} \to 0$ as $i \to \infty$ (for each j), and $\xi_i \to (\partial/\partial y)|_0$ on $\mathcal{C}^1(\mathbb{R}^2)$ (as $i \to \infty$).

It follows that $(\partial/\partial y)|_0 \notin g(T_0)(0)$, no matter how large we take k.

Proof of claim. Only finitely many points a_{ij} lie in each cluster E_l. It is therefore easy to find a $\mathcal{C}^1$ function $y = \varphi(x)$, $x \in (0, \infty)$, such that the graph of φ contains all points a_{ij}, $\varphi'(\gamma_l, 0) = 0$ for all l, and $\max\{|\varphi'(x)| :\ x \in [\gamma_{l+1}, \gamma_l]\} \to 0$ as $l \to \infty$.

Clearly, φ extends to a $\mathcal{C}^1$ function φ on $\mathbb{R}$ such that $\varphi(0) = 0 = \varphi'(0)$. So $f(x,y) := y - \varphi(x)$ vanishes on all a_{ij}, but $(\partial f/\partial y)(0) = 1$. The claim follows.

3. On the criterion of [BMP1]

The following three examples are all counterexamples to the analogue of Theorem 1.1 above using the Glaeser operation ρ in place of g. In Example 3.1, E is given by a convergent sequence of points in $\mathbb{R}^2$, and $\tau_k^2(E) \subsetneqq \mathcal{T}^2(E)$, no matter how large we take k. In Example 3.2, we use a similar idea to define a closed subset E of the line $\mathbb{R}$ with the property that, for any $k \geq 2$, $\tau_k^3(E) = \mathcal{T}^3(E)$, but there is a function $f : E \to \mathbb{R}$ which admits an extension $\nabla_k^3 f : \tau_k^3(E) \to \mathbb{R}$, although f is not the restriction of a $\mathcal{C}^3$ function. In the final example 3.3, E is a union of 4 analytic arcs in $\mathbb{R}^3$, but $\tau_k^2(E) \subsetneqq \mathcal{T}^2(E)$, for every k.

We use Whitney's theorem [W] in Example 3.2. Whitney's theorem asserts that, for a closed set $E \subset \mathbb{R}$, a function $f : E \to \mathbb{R}$ is $\mathcal{C}^m$ if and only if the limiting values of all m'th finite differences $\Delta^m(x_0, x_1, \ldots, x_m)(f)$ (where the x_j are distinct points of E) define a continuous function on the diagonal $\{x_0 = \cdots = x_m\}$. (See (3.4) below.)

Example 3.1. Consider four decreasing sequences of positive numbers,

$$\{d_1^i\}\,,\ \{d_2^i\}\,,\ \{d_3^i\}\,,\ \{d_4^i\}\,,\quad i = 1, 2, \ldots\,,$$

where

$$d_1^i \ll d_2^i \ll d_3^i < d_4^i \leq \mathrm{const}\cdot d_3^i\,,\quad d_3^i = d_4^i - d_4^{i+1}\,.$$

("$\ll$" means "much less than"; here it is enough to define $a_i \ll b_i$ by $\lim_{i\to\infty} a_i/b_i^2 = 0$.) For example, we can take

$$d_1^i = \frac{1}{2^{7i}}\,,\ d_2^i = \frac{1}{2^{3i}}\,,\ d_3^i = \frac{1}{2^i}\,,\ d_4^i = \frac{1}{2^{i-1}}\,.$$

For each i, let

$$R^i = \{p_{11}^i, p_{21}^i, p_{12}^i, p_{22}^i\} \subset \mathbb{R}^2\,,$$

where

$$p_{11}^i = (d_4^i, d_4^i)\,,\qquad p_{21}^i = (d_4^i + d_2^i, d_4^i)\,,$$
$$p_{12}^i = (d_4^i, d_4^i + d_1^i)\,,\qquad p_{22}^i = (d_4^i + d_2^i, d_4^i + d_1^i)\,.$$

Let

$$E = \{0\} \cup \bigcup_{i=1}^{\infty} R^i\,,$$

where 0 denotes the origin $(0,0) \in \mathbb{R}^2$. Clearly, the fibre $\mathcal{T}^2(E)(0)$ of the $\mathcal{C}^2$ Zariski paratangent bundle of E at the origin is $\mathcal{P}^* = \mathcal{P}^2(\mathbb{R}^2)^*$.

Fix a positive integer $k \geq 3$. Consider the subbundle T of $E \times \mathcal{P}^*$ whose fibre over every nonzero $a \in E$ is the 1-dimensional subspace of $\mathcal{P}^*$ spanned by the delta-function δ_a, and whose fibre over 0 is the codimension 1 subspace spanned by the elements

$$(3.1)\qquad \delta_0\,,\ \frac{\partial}{\partial x}\bigg|_0\,,\ \frac{\partial}{\partial y}\bigg|_0\,,\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y}\right)\bigg|_0\,,\ \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\right)\bigg|_0\,.$$

(It is easy to see that all these elements belong to $\tau_k^2(E)(0)$; i.e., they can be realized from the bundle T_0 spanned by delta-functions, using iterations of the Glaeser operation ρ.)

We will show that the bundle T is stable under Glaeser operations (i.e., $\rho(T) = T$), so it coincides with $\tau_k^2(E)$. (Therefore, $\tau_k^2(E)(0) \neq \mathcal{T}^2(E)(0)$.) It is enough to show that every $\xi \in \rho(T)(0)$ vanishes on $(x-y)^2 \in \mathcal{P}$. Of course, by (3.1), all elements of $T(0)$ vanish on $(x-y)^2$.

Consider $\xi \in \mathcal{P}^*$, $\xi = \lim_{l\to\infty} \xi_l$, as in §1.1, where the condition (3) is replaced by (3′) of §1.2, and where ξ_l here means $\sum_{a\in S_l} \xi_{la}$ (in the notation of condition (4)). Write $\delta^i_{qr} := \delta_{p^i_{qr}}$, for each $i = 1, 2, \dots$ and $q, r = 1, 2$. We can write each ξ_l as a sum

$$\xi_l \;=\; \eta_l + \sum_i \left(\mu^i_{11}\delta^i_{11} + \mu^i_{21}\delta^i_{21} + \mu^i_{12}\delta^i_{12} + \mu^i_{22}\delta^i_{22}\right)\,,$$

where η_l is a linear combination of terms (3.1) and at most k coefficients are nonzero. (Each coefficient depends on l, but we simplify our notation by not writing this dependence.)

We can rewrite any linear combination

$$\mu^i_{11}\delta^i_{11} + \mu^i_{21}\delta^i_{21} + \mu^i_{12}\delta^i_{12} + \mu^i_{22}\delta^i_{22}$$

as a linear combination

$$\lambda^i_0\delta^i_{11} + \lambda^i_1\frac{\delta^i_{21}-\delta^i_{11}}{d^i_2} + \lambda^i_2\frac{\delta^i_{12}-\delta^i_{11}}{d^i_1} + \lambda^i_3\frac{\delta^i_{22}-\delta^i_{21}}{d^i_1}\,.$$

Of course,

$$\delta^i_{11}\,,\ \frac{\delta^i_{21}-\delta^i_{11}}{d^i_2}\,,\ \frac{\delta^i_{12}-\delta^i_{11}}{d^i_1}\,,\ \frac{\delta^i_{22}-\delta^i_{21}}{d^i_1}$$

are all bounded as elements of $\mathcal{P}^*$ (or $\mathcal{C}^2(L)^*$, where L is a closed rectangle containing E).

For each l, there is a reference point $a_l \in S_l$ (as in the condition (3′) of §1.2); either $a_l = 0$ or $a_l = p^{i_0}_{qr}$ for some $i_0 = i_0(l)$, q, r. Passing to a subsequence of the ξ_l if necessary, we can assume that either $a_l = 0$ for all l, or $a_l = p^{i_0(l)}_{qr}$, for all l, where q, r are independent of l. In the second case, we will assume that $a_l = p^{i_0(l)}_{11}$, for all l. (The other possibilities are similar.)

Let us first consider the second case. We will simplify our notation by dropping the superscript i whenever $i = i_0$ (the superscript for the reference point); i.e., we write $\mu_{qr} = \mu^{i_0(l)}_{qr}$, $d_j = d^{i_0(l)}_j$, etc. Now,

$$(d_2)^2\mu_{22} \;=\; (d_2)^2\frac{\lambda_3}{d_1}$$

is bounded (i.e., bounded in absolute value, uniformly with respect to l), by the condition (3′); therefore,

$$|\lambda_3| \;\lesssim\; d_1 d_2^{-2} \;\to\; 0 \quad (\text{as } l\to\infty).$$

($\lesssim$ means bounded by (the following term), up to a multiplicative constant (independent of l).) For any other i occurring (as the superscript of a nonzero coefficient) in this element ξ_l of the sequence,

$$(d^i_3)^2\mu^i_{22} = (d^i_3)^2\frac{\lambda^i_3}{d^i_1},\ (d^i_3)^2\mu^i_{12} = (d^i_3)^2\frac{\lambda^i_2}{d^i_1},\ (d^i_3)^2\mu^i_{21} = (d^i_3)^2\left(\frac{\lambda^i_1}{d^i_2} - \frac{\lambda^i_3}{d^i_1}\right)$$

are all likewise bounded; therefore,

$$|\lambda_3^i| \lesssim d_1^i (d_3^i)^{-2} \to 0\,, \qquad |\lambda_2^i| \lesssim d_1^i (d_3^i)^{-2} \to 0\,,$$
$$|\lambda_1^i| \lesssim \frac{d_2^i}{d_1^i}|\lambda_3^i| + d_2^i (d_3^i)^{-2} \lesssim d_2^i (d_3^i)^{-2} \to 0\,.$$

It follows that we can reduce to the case that each term ξ_l in our sequence is of the form

$$\sum_{i\ge 0} c_i \delta_{11}^i + a\frac{\delta_{21}-\delta_{11}}{d_2} + b\frac{\delta_{12}-\delta_{11}}{d_1} + a_0 \frac{\partial}{\partial x}\bigg|_0 + b_0 \frac{\partial}{\partial y}\bigg|_0 + d_0 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y}\right)\bigg|_0 + e_0 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\right)\bigg|_0 . \tag{3.2}$$

(The reference superscript and all coefficients depend on l; δ_{11}^0 means δ_0.)

(In the first case above ($a_l = 0$, for all l), by the same argument, we can reduce to the case that each term ξ_l in our sequence is of the form (3.2) where, in addition, $a = b = 0$, for all l. Then ξ_l vanishes on $(x-y)^2$, for all l, so the desired result is verified in this case.)

Now, d_0 and e_0 are bounded (by condition (3′)), so (passing to a subsequence if necessary), we can assume that the sequence $\{\zeta_l\}$, where ζ_l is given by the sum of the last two terms in (3.2), converges. We can therefore assume that $d_0 = 0 = e_0$; i.e., it is enough to consider the case that ξ_l is of the form

$$\sum_{i\ge 0} c_i \delta_{11}^i + a\frac{\delta_{21}-\delta_{11}}{d_2} + b\frac{\delta_{12}-\delta_{11}}{d_1} + a_0 \frac{\partial}{\partial x}\bigg|_0 + b_0 \frac{\partial}{\partial y}\bigg|_0 . \tag{3.3}$$

We want to show that the limit of this sequence evaluated on $(x-y)^2$ is zero. When we evaluate (3.3) on $(x-y)^2$, we get a nonzero contribution only from

$$\left(\frac{a}{d_2}\delta_{21} + \frac{b}{d_1}\delta_{12}\right)\left((x-y)^2\right) = \frac{a}{d_2}(d_2)^2 + \frac{b}{d_1}(d_1)^2 = ad_2 + bd_1\,,$$

so that this sequence converges. But $(d_2)^2 a/d_2$ and $(d_1)^2 b/d_1$ are both bounded (from (3′)); i.e., ad_2 and bd_1 are bounded, so that, passing to a subsequence if necessary, we can assume that ad_2 and bd_1 separately converge.

Let us consider the limit of (3.3) also when evaluated on $x(x-y)$ and on xy. On $x(x-y)$, the only nonzero contribution of (3.3) is

$$\left(\frac{a}{d_2}\delta_{21} + \frac{b}{d_1}\delta_{12}\right)(x(x-y)) = \frac{a}{d_2}(d_4+d_2)d_2 + \frac{b}{d_1}d_4(-d_1) = (a-b)d_4 + ad_2\,.$$

Therefore, $(a-b)d_4$ is bounded.

On xy, we get

$$\sum_{i\ge 0} c_i (d_4^i)^2 + ad_4 + bd_4\,.$$

Now, $\sum_{i\ge 0} c_i (d_4^i)^2$ is bounded, as follows: For any "rectangle" R^i (including $R^\infty := \{0\}$) lying "below" the reference rectangle $R = R^{i_0}$ (i.e., $i > i_0$), $c_i (d_4)^2$ is bounded (using (3′)); therefore $c_i(d_4^i)^2$ is bounded. For a rectangle R^i lying above the reference rectangle R, $c_i(d_4^i)^2$ is bounded (and it follows that $c_i(d_4)^2$ is bounded). For the reference rectangle R itself, it then follows that $c(d_4)^2$ is bounded, because $\sum_i c_i$ is bounded (as we see by evaluating on the constant polynomial 1). Therefore, $(a+b)d_4$ is also bounded.

It follows that ad_4 and bd_4 are bounded. Therefore, $ad_2 = ad_4(d_2/d_4) \to 0$ and $bd_1 = bd_4(d_1/d_4) \to 0$, as required.

Example 3.2. Consider

$$d_1^i \ll d_2^i \ll d_3^i \ll \frac{1}{i}.$$

(It will be enough to take, for example, $d_3^i = 1/2^i$, $d_2^i = 1/2^{4i}$, $d_1^i = 1/2^{5i}$.) For each i, let

$$a_0^i = d_3^i - d_2^i, \qquad a_1^i = d_3^i - d_2^i + d_1^i,$$
$$a_2^i = d_3^i + d_2^i - d_1^i, \qquad a_3^i = d_3^i + d_2^i.$$

Take

$$E = \{0\} \bigcup \{a_j^i : i = 1, 2, \dots, j = 0, 1, 2, 3\}$$

For each fixed $j = 0, 1, 2, 3$, write $E_j^* = E \setminus \{a_j^i\}_{i=1,2,\dots}$.

Given $m+1$ distint points $x_0, x_1, \dots, x_m \in \mathbb{R}$, let $\Delta^m(x_0, x_1, \dots, x_m)$ denote the m'th finite difference operator

$$\Delta^m(x_0, x_1, \dots, x_m) = \sum_{l=0}^{m} \frac{\delta_{x_l}}{\prod_{h \neq l}(x_l - x_h)}. \tag{3.4}$$

Fix $k \geq 2$. Then it is easy to see that

$$\tau_k^3(E) = \mathcal{T}^3(E) = T,$$

where T is the linear subbundle of $E \times \mathcal{P}^*$ ($\mathcal{P} = \mathcal{P}^3(\mathbb{R})$) such that the fibre of T over a nonzero point $a \in E$ is the one-dimensional subspace of $\mathcal{P}^*$ spanned by the delta-function δ_a, and the fibre over 0 is $\mathcal{P}^*$. In particular, if $a \in E\setminus\{0\}$ and $\xi \in T(a)$, then $\xi = \lambda\delta_a$, $\lambda \in \mathbb{R}$, so that $\xi(f)$ is well-defined as $\lambda f(a)$. We will show:

(A) There exists $f: E \to \mathbb{R}$ such that:
  (1) $\Delta^3(a_0^i, a_1^i, a_2^i, a_3^i)(f) \to \infty$ (as $i \to \infty$), so that f is not the restriction to E of a $\mathcal{C}^3$ function.
  (2) $f \in \mathcal{C}^1(E)$ (i.e., f is the restriction of a $\mathcal{C}^1$ function).
  (3) For each $j = 0, 1, 2, 3$, $f|_{E_j^*}$ is the restriction to E_j^* of a $\mathcal{C}^3$ function that is flat at 0.
  (4) $\dfrac{d_1^i}{d_2^i}\Delta^3(a_0^i, a_1^i, a_2^i, a_3^i)(f) \to 0$.

(B) If $\xi \in T(0)$, set $\xi(f) := 0$. Consider $\xi = \lim_{l\to\infty} \xi_l$ in $\mathcal{P}^*$ as in §1.1, with the condition (3) replaced by (3′) (cf. Example 3.1 above). Then $\lim_{l\to\infty} \xi_l(f) = 0$.

Therefore, the criterion of [BMP1] is not sufficient to guarantee that a given function on E is the restriction of a $\mathcal{C}^3$ function.

We will first prove (B), assuming (A). We can write each ξ_l as

$$\xi_l \;=\; \xi_l^0 + \sum_{i,j} \mu_j^i \delta_{a_j^i}\,,$$

where ξ_l^0 is a linear combination of derivatives of orders 0 through 3 at the origin (and there are at most k nonzero coefficients). As in Example 3.1 each coefficient depends on l, but we do not indicate this dependence. We will write $\delta_j^i := \delta_{a_j^i}$.

For each l, we have a reference point which is either $a_{j_0}^{i_0}$, for some $i_0 = i_0(l)$ and $j_0 = j_0(l)$, or 0. Replace $\{\xi_l\}$ by any subsequence for which every reference point is either $a_{j_0}^{i_0}$, where j_0 is independent of l, or 0. In the first case, take $j_1 = 3$ if $j_0 = 0$ or 1, and take $j_1 = 0$ if $j_0 = 2$ or 3; in the second case, take $j_1 = 3$. Let $E^* = E_{j_1}^*$. Let us say, for example, that $j_1 = 3$ (so that $j_0 = 0$ or 1 in the first case).

(For each l and) for each i, write

$$\sum_{j=0}^{3} \mu_j^i \delta_j^i \;=\; \lambda_0^i \delta_0^i + \lambda_1^i \frac{\delta_1^i - \delta_0^i}{d_1^i} + \lambda_2^i \frac{\delta_2^i - \delta_0^i}{d_2^i} + \lambda_3^i \frac{\delta_3^i - \delta_0^i}{d_2^i}$$

(so that $\lambda_1^i / d_1^i = \mu_1^i$ and $\lambda_j^i / d_2^i = \mu_j^i$, $j = 2, 3$).

Consider $i \neq i_0$ (or any i, if the reference point is 0). If $j = 2$ or 3, then

$$\frac{|\lambda_j^i|}{d_2^i} \;=\; |\mu_j^i| \;\lesssim\; (d_3^i)^{-3}\,,$$

by (3′), so that $\lambda_j^i \to 0$ (as $l \to \infty$). Also,

$$\frac{|\lambda_1^i|}{d_1^i} \;=\; |\mu_1^i| \;\lesssim\; (d_3^i)^{-3}\,,$$

so that $\lambda_1^i \to 0$.

Consider the sequence $\{\eta_l\}$, where each η_l is obtained from ξ_l by setting $\lambda_j^i = 0$ for all $i \neq i_0$ and $j = 1, 2, 3$ (or, for all i and for $j = 1, 2, 3$, if the reference point is 0). Then $\{\eta_l\}$ satisfies our conditions (using (3′)). Since f is $\mathcal{C}^1$ (by (A)(2)), if $\lim \eta_l(f) = 0$, then $\lim \xi_l(f) = 0$. Therefore, we can assume that (for each l) $\lambda_j^i = 0$ for all $i \neq i_0$ and $j = 1, 2, 3$ (or, for all i and $j = 1, 2, 3$, if the reference point is 0). In the second case, if follows that $\lim_{l \to \infty} \xi_l(f) = 0$, by (A)(3).

In the first case, let us now consider the reference terms in ξ_l (which we denote again by dropping the superscript $i_0 = i_0(l)$). Write

$$\sum_{j=0}^{3} \mu_j \delta_j \;=\; \lambda_0 \delta_0 + \lambda_1 \frac{\delta_1 - \delta_0}{d_1} + \lambda_2 \Delta^2(a_0, a_1, a_2) + \lambda_3 \Delta^3(a_0, a_1, a_2, a_3)\,.$$

Every denominator in the expression (3.4) for $\Delta^3(a_0, a_1, a_2, a_3)$ is comparable to $d_1(d_2)^2$. Therefore,

$$\frac{|\lambda_3|}{d_1(d_2)^2} \;\simeq\; |\mu_3| \;\lesssim\; d_2^{-3}\,,$$

so that $|\lambda_3| \lesssim d_1 d_2^{-1} \to 0$. Moreover, $\lambda_3 \Delta^3(a_0, a_1, a_2, a_3)(f) \to 0$ (as $l \to \infty$), by (A)(4).

Now consider the sequence $\{\eta_l\}$, where each η_l is obtained from ξ_l by setting $\lambda_3 = \lambda_3^{i_0(l)} = 0$ (so that η_l has its coefficient μ_3 equal to 0). Then $\{\eta_l\}$ converges and satisfies our conditions. Moreover, $\{\eta_l\}$ is supported in E^*, so that $\lim \eta_l(f) = 0$. It follows that $\lim \xi_l(f) = 0$, as required.

It remains to verify (A). For each i, set

$$f_i(x) = i\left(x - d_3^i\right)^3 .$$

Define f by $f(a_j^i) = f_i(a_j^i)$, for all i, j, and $f(0) = 0$. Then:

(1) $\Delta^3(a_0^i, a_1^i, a_2^i, a_3^i)(f) = \Delta^3(a_0^i, a_1^i, a_2^i, a_3^i)(f_i) = i \to \infty$, as $i \to \infty$.
(2) It is easy to check that, given $b \neq c$ in E, $\Delta^1(b,c)(f) \to 0$ as $b, c \to 0$. Therefore, $f \in \mathcal{C}^1(E)$.
(3) It is again easy to check that, for each $j = 0,1,2,3$, all $\Delta^3(b_0, b_1, b_2, b_3)(f)$, where the $b_j \in E_j^*$, tend to 0 as the $b_j \to 0$. (Thus (A)(3) follows from Whitney's theorem [W].)
(4) $(d_1^i/d_2^i)\Delta^3(a_0^i, a_1^i, a_2^i, a_3^i)(f) = i d_1^i (d_2^i)^{-1} \to 0$.

Example 3.3. Let $E \subset \mathbb{R}^3$ denote the union of the (images of) the four arcs

$$\gamma_{jk}(t) = \left((-1)^j t^3,\, (-1)^k t^7,\, t\right), \quad t \geq 0, \quad \text{where } j,k = 1,2;$$

i.e., E is the zero set in $z \geq 0$ of the polynomials

$$f_1(x,y,z) = x^2 - z^6, \quad f_2(x,y,z) = y^2 - z^{14}.$$

We will compare E with the union E^* of three of the four arcs, the arcs $\gamma_{11}, \gamma_{12}, \gamma_{21}$, say. Then E^* is the zero set in $z \geq 0$ of the three polynomials f_1, f_2 and

$$f_3(x,y,z) = (x + z^3)(y + z^7).$$

Clearly, the fibre $\mathcal{T}^2(E)(0)$ of the $\mathcal{C}^2$ Zariski paratangent bundle of E at the origin is the orthogonal complement in $\mathcal{P}^*$ of the linear span of $x^2, y^2 \in \mathcal{P} := \mathcal{P}^2(\mathbb{R}^3)$; i.e., $\mathcal{T}^2(E)(0) \subset \mathcal{P}^*$ is spanned by all partial derivatives of order ≤ 2 except for

$$\left.\frac{\partial^2}{\partial x^2}\right|_0, \quad \left.\frac{\partial^2}{\partial y^2}\right|_0 .$$

On the other hand, $\mathcal{T}^2(E^*)(0)$ is the orthogonal complement of the linear span of x^2, y^2, xy; i.e., $\mathcal{T}^2(E^*)(0)$ is spanned by all partials of order ≤ 2 except

$$\left.\frac{\partial^2}{\partial x^2}\right|_0, \quad \left.\frac{\partial^2}{\partial y^2}\right|_0, \quad \left.\frac{\partial^2}{\partial x \partial y}\right|_0 . \tag{3.5}$$

Fix any positive integer $k \geq 3$. It is easy to see that $\mathcal{T}^2(E^*)(0) \subset \tau_k^2(E)(0)$; i.e., $\tau_k^2(E)(0)$ includes the span of all partials of order ≤ 2 except (3.5). We will show that $\tau_k^2(E)(0) = \mathcal{T}^2(E^*)(0)$; i.e., $\tau_k^2(E)(0) \subsetneqq \mathcal{T}^2(E)(0)$.

Consider any of the four arcs γ_{jk}. Then $\gamma_{jk}'(t) = \left((-1)^j 3t^2,\, (-1)^k 7t^6,\, 1\right)$, so that

$$\frac{\partial}{\partial \tau} := (-1)^j 3z^2 \frac{\partial}{\partial x} + (-1)^k 7z^6 \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

is tangent to γ_{jk} at every point $(x,y,z) \in \gamma_{jk}$. Write $\theta_1 := 3z^2$, $\theta_2 := 7z^6$. At a nonzero point $a \in E$, $\tau_k^2(E)_a = \mathcal{T}^2(E)_a$ is spanned by

$$\delta_a, \quad \left.\frac{\partial}{\partial \tau}\right|_a, \quad \left.\frac{\partial^2}{\partial \tau^2}\right|_a .$$

For each fixed $z \geq 0$, there are four points of E, forming the vertices of a rectangle. Let $d_1 = d_1(z)$ and $d_2 = d_2(z)$ denote the height and width (respectively) of this rectangle, so that $d_2 \approx d_1^{3/7}$, $\theta_1 \approx d_1^{2/7}$, $\theta_2 \approx d_1^{6/7}$. In particular, $d_1 d_2^{-2} \to 0$ as $z \to 0$.

As in Example 3.1, consider the limit of a sequence $\{\xi_l\}$ in $\mathcal{P}^*$ satisfying our condition $(3')$. There is a sequence of positive numbers $z^i \to 0$ such that, if $a^i_{jk} := \gamma_{jk}(z^i)$, $j,k = 1,2$, and $\delta^i_{jk} := \delta_{a^i_{jk}}$, then each ξ_l is of the form $\sum_{i,j,k} \mu^i_{jk}\delta^i_{jk}$ plus a linear combination of $\partial/\partial\tau$, $\partial^2/\partial\tau^2$ at the points a^i_{jk}, plus a linear combination of terms (3.5) (at the origin).

For each l, there is a reference point which is either 0 or a^i_{jk}, for some i,j,k. We can assume that the reference point is either 0 or $a^{i_0}_{11}$, for some $i_0 = i_0(l)$. We will write $a_{11} = a^{i_0}_{11}$, for each l, and we will consider both cases at once by allowing a_{11} to mean 0.

For each i, we write

$$\sum_{j,k} \mu^i_{jk}\delta^i_{jk} = \sum_{(j,k)\neq(2,2)} \lambda^i_{jk}\delta^i_{jk} + \lambda^i_3 \frac{\delta^i_{22} - \delta^i_{21}}{d^i_1} ;$$

i.e., $\lambda^i_3/d^i_1 = \mu^i_{22}$, $\lambda^i_{21} = \mu^i_{21} + \lambda^i_3/d^i_1$, and $\lambda^i_{jk} = \mu^i_{jk}$ when $(j,k) = (1,1),(1,2)$.

According to condition $(3')$,

$$\left|a_{11} - a^i_{22}\right|^2 \left(\mu^i_{22}\delta^i_{22} + * \left.\frac{\partial}{\partial\tau}\right|_{a^i_{22}} + * \left.\frac{\partial^2}{\partial\tau^2}\right|_{a^i_{22}} \right)(1) \leq \text{const}\,,$$

where the asterisks indicate the coefficients of the corresponding terms in the expression for ξ_l above. But (by the Pythagorean theorem)

$$\left|a_{11} - a^i_{22}\right|^2 = \left(\frac{d_1 + d^i_1}{2}\right)^2 + \left(\frac{d_2 + d^i_2}{2}\right)^2 + \left(d^i_3\right)^2 \geq \frac{\left(d^i_2\right)^2}{4}\,,$$

where $d^i_3 = |z^i - z^{i_0}|$, so that

$$\left|\frac{\lambda^i_3}{d^i_1}\right| = \left|\mu^i_{22}\right| \lesssim \left(d^i_2\right)^{-2} ;$$

i.e., $\left|\lambda^i_3\right| \lesssim d^i_1 \left(d^i_2\right)^{-2} \to 0$. We can therefore assume that each term ξ_l in our sequence is of the form

$$\sum_{\substack{i,j,k \\ (j,k)\neq(2,2)}} \mu^i_{jk}\delta^i_{jk} + \sum_{\substack{i,j,k \\ (j,k)\neq(2,2)}} \nu^i_{jk} \left.\frac{\partial}{\partial\tau}\right|_{a^i_{jk}} + \sum_i \nu^i_{22} \left.\frac{\partial}{\partial\tau}\right|_{a^i_{22}} + \sum_{i,j,k} \rho^i_{jk} \left.\frac{\partial^2}{\partial\tau^2}\right|_{a^i_{jk}} \tag{3.6}$$

plus terms supported at the origin. (Here and in a further reduction at the end of our argument, condition $(3')$ is preserved because we subtract off sequences satisfying $(3')$.)

Note first that the $\left|\rho^i_{jk}\right|$ are bounded (by our condition), so (passing to a subsequence if necessary) we can assume that $\sum \rho^i_{jk} \left(\partial^2/\partial\tau^2\right)\big|_{a^i_{jk}}$ converges to a multiple of $\left(\partial^2/\partial z^2\right)$ at 0. It follows that we can assume that all $\rho^i_{jk} = 0$.

Now, for each i. rewrite the sum of the middle two terms in (3.6) as

$$\sum_{(j,k)\neq(2,2)} \tilde{\nu}^i_{jk} \left.\frac{\partial}{\partial\tau}\right|_{a^i_{jk}} + \nu^i_3 \frac{\left.\frac{\partial}{\partial\tau}\right|_{a^i_{22}} - \left.\frac{\partial}{\partial\tau}\right|_{a^i_{21}}}{d^i_1}\,.$$

(In particular, $\tilde{\nu}^i_{21} = \nu^i_{21} + \nu^i_3/d^i_1$.) It is easy to check that

$$\left|\frac{\nu^i_3}{d^i_1}\right| = \left|\nu^i_{22}\right| \lesssim \left(d^i_2\right)^{-1},$$

using condition ($3'$) on $\eta_{a^i_{22},\alpha} := \eta\left(\left(x - a^i_{22}\right)^\alpha/\alpha\right)$, where $|\alpha| = 1$ and

$$\eta = \left.\frac{\partial}{\partial\tau}\right|_{a^i_{22}} = \left.\left(\frac{\partial}{\partial z} + \theta_1\frac{\partial}{\partial x} + \theta_2\frac{\partial}{\partial y}\right)\right|_{a^i_{22}};$$

therefore,

$$\left|\nu^i_3\right| \lesssim d^i_1\left(d^i_2\right)^{-1} \to 0\,.$$

Now,

$$\frac{\left.\frac{\partial}{\partial\tau}\right|_{a^i_{22}} - \left.\frac{\partial}{\partial\tau}\right|_{a^i_{21}}}{d^i_1} = \frac{\left.\frac{\partial}{\partial z}\right|_{a^i_{22}} - \left.\frac{\partial}{\partial z}\right|_{a^i_{21}}}{d^i_1} + \theta_1\frac{\left.\frac{\partial}{\partial x}\right|_{a^i_{22}} - \left.\frac{\partial}{\partial x}\right|_{a^i_{21}}}{d^i_1} + \theta_2\frac{\left.\frac{\partial}{\partial y}\right|_{a^i_{22}} + \left.\frac{\partial}{\partial y}\right|_{a^i_{21}}}{d^i_1}\,.$$

On the right-hand side, the first term tends to $\left.\left(\partial^2/\partial y\partial z\right)\right|_0$, the second term tends to 0, and ν^i_3 times the third term tends to 0 (because $|\nu^i_3|/d^i_1 \lesssim (d^i_2)^{-1}$ and $\theta_2 \approx (d^i_2)^2$). Therefore,

$$\nu^i_3 \cdot \frac{\left.\frac{\partial}{\partial\tau}\right|_{a^i_{22}} - \left.\frac{\partial}{\partial\tau}\right|_{a^i_{21}}}{d^i_1} \to 0$$

so we can assume that each term ξ_l in our sequence is of the form

$$\sum_{\substack{i,j,k\\(j,k)\neq(2,2)}} \mu^i_{jk}\delta^i_{jk} + \sum_{\substack{i,j,k\\(j,k)\neq(2,2)}} \nu^i_{jk}\left.\frac{\partial}{\partial\tau}\right|_{a^i_{jk}}$$

plus terms with support $\{0\}$. But now the support of each term lies in E^* (union of three arcs), so the limit is in the linear span of all partials of order ≤ 2 at the origin, except (3.5).

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