

ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS VIA THE KOOSIS THEOREM

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ABSTRACT. The main aim of this short paper is to advertise the Koosis theorem in the mathematical community, especially among those who study orthogonal polynomials. We (try to) do this by proving a new theorem about asymptotics of orthogonal polynomials for which the Koosis theorem seems to be the most natural tool. Namely, we consider the case when a Szegő measure on the unit circumference is perturbed by an arbitrary measure inside the unit disk and an arbitrary Blaschke sequence of point masses outside the unit disk.

1. Introduction and statement of results

Consider a measure μ on the complex plane \mathbb{C} of the form $\mu = \nu + w \, dm + \sum_k \mu_k \delta_{z_k}$ where ν is an arbitrary finite measure in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, m is the Haar measure on the unit circumference \mathbb{T} , $w \in L^1(m)$ is a strictly positive function satisfying the Szegő condition $\int_{\mathbb{T}} \log w \, dm > -\infty$, $\mu_k > 0$ satisfy $\sum_k \mu_k < +\infty$, and, at last, the points z_k are taken in the exterior of the unit disk, i.e., $|z_k| > 1$ for each k , and satisfy the Blaschke condition $\sum_k (|z_k| - 1) < +\infty$.

Let

$$p_n(z) = \tau_n z^n + \dots$$

be the n -th orthogonal polynomial with respect to the measure μ normalized by the conditions $\|p_n\|_{L^2(\mu)} = 1$, $\tau_n > 0$.

Theorem.

$$\lim_{n \rightarrow \infty} \tau_n = \exp \left\{ -\frac{1}{2} \int_{\mathbb{T}} \log w \, dm \right\} \prod_k \frac{1}{|z_k|}.$$

Let us introduce two auxiliary functions: the outer function ψ in the exterior of the unit disk such that $|\psi|^2 = \frac{1}{w}$ on \mathbb{T} and $\psi(\infty) > 0$, and the Blaschke product

$$B(z) = \prod_k \frac{\bar{z}_k}{|z_k|} \frac{z - z_k}{z \bar{z}_k - 1}.$$

Corollary. For every $z \in \mathbb{C}$ with $|z| > 1$, we have

$$\lim_{n \rightarrow \infty} \frac{p_n(z)}{z^n} = (B\psi)(z).$$

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A few words about the history of the problem may be in order. For finitely many point masses lying on the real line, the theorem was proved by Nikishin [4]. Nikishin's result has been generalized in various ways by Benzine and Kaliaguine [5] and by Li and Pan in [6]. Peherstorfer and Yuditskii [2] seem to be the first to consider the case of infinitely many point masses. They proved the theorem for the case when all masses lie on the real line.

An attempt to deal with the general case was made by Peherstorfer, Volberg, and Yuditskii in [3]. It was proved there that an analog of our theorem holds for orthonormal *rational functions*. It is unclear to us at this moment whether the approach in [3] can yield the asymptotics of orthogonal *polynomials* too.

In the recent years several important developments had happened in the spectral theory of finite difference and differential operators. One of them is related to the extension of the perturbation theory to the case when the perturbed operator may have *infinitely* many spectral points in the resolvent domain of the unperturbed operator. The second one deals with including in the family of "physical interest" so called CMV matrices, actually, this can be reformulated in the following way: "orthogonal polynomials on the unit circle came from the "outskirts" to the center of interest of people working in the spectral theory".

Concerning the first direction we have to mention at least the following three results that we formulate for a Jacobi matrix J with the spectrum on an interval E plus a system of points X outside of E :

1. It was shown that Blaschke condition on the set X and the Szegő condition on the absolutely continuous component of spectrum supported on E implies that $J = J_0 + J_1$, where J_0 is the free Hamiltonian (matrix with a constant coefficients) and J_1 is a compact operator. Moreover asymptotic behavior of the generalized eigenvectors were obtained. On the language of orthogonal polynomials this is exactly extension of the Szegő asymptotic that we consider here, meaning with point masses, but for orthogonal polynomials on the real axis [2].

2. One can go out of the scope of Szegő theorem not only by adding point masses, but also by replacing logarithmic summability condition by certain weaker assumptions! Quasi-Szegő and quasi-Blaschke conditions are *equivalent!* to $J = J_0 + J_1$, where J_1 is a Hilbert-Schmidt perturbation, or some more general perturbations described by so-called "sum rules", see [8], [9], [11]. Notice that the asymptotics of orthogonal polynomials obtained in [2] were used already for obtaining these asymptotics of orthogonal polynomials with weights which do not have Szegő logarithmic summability condition, see e.g. [11].

3. If absolutely continuous spectrum of J covers the whole interval E and spectral points of X accumulate only to the ends of the given interval then $J = J_0 + J_1$, and J_1 is a compact operator [7]. In the language of orthogonal polynomials this is an extension of the Rakhmanov's theorem (no information on asymptotic behavior of the generalized eigenvectors is available in the moment, and it might be that actually no information can be obtained).

Many details concerning the second phenomena can be found in the 1000 pages monograph dedicated to orthogonal polynomials on the unite circle [8]. We only want to add reference [10] where the subject is discussed from the other point of view. It

represents the scattering theory for the same operators, while orthogonal polynomials are related to the spectral theory.

There exists a constant interplay between different branches of the spectral theory (Jacobi matrices, Sturm – Liouville and Dirac operators, and now, CMV matrices). Sometimes it is just straightforward constructions, sometimes it is not. The current generalization of Peherstorfer-Yuditskii's theorem is another example of this kind, when the proof had to be completely rebuilt (the proof in [2] would work only for Blaschke–Stolz sets).

Coming back to point 2, we can mention the most probable future application of our result. As it is mentioned in 2, the asymptotics of orthogonal polynomials with weights which do not have Szegő logarithmic summability condition were obtained in e.g. [11] in the case of polynomials on the interval (plus point masses on the same line). And this paper used [2]. The analog for orthogonal polynomials on the circle (again with no summability of logarithm) were obtained in [12]. But this work does not allow to add point masses. And because the weight is on the circle it is much more natural to allow the point masses to accumulate to the circle in an arbitrary manner, not just along one line. The extension of [12] should use the present paper exactly as [11] used [2]. In other words, in the language of fractions, [2] divided by [11] should be equal to the present paper divided by [12] and its future extension.

2. Proof of Theorem

Let us show first that $\limsup_{n \rightarrow \infty} \tau_n$ does not exceed the right hand side. To this end, let us observe that the right hand side can be rewritten as $\psi(\infty)B(\infty)$ where, as before, ψ is the outer function in the exterior of the unit disk such that $|\psi|^2 = \frac{1}{w}$ on \mathbb{T} and $\psi(\infty) > 0$, and $B(z) = \prod_k \frac{\bar{z}_k}{|z_k|} \frac{z - z_k}{z\bar{z}_k - 1}$ is the Blaschke product with zeroes z_k . Now fix $\ell > 0$ and put $B_\ell(z) = \prod_{k: k \leq \ell} \frac{\bar{z}_k}{|z_k|} \frac{z - z_k}{z\bar{z}_k - 1}$. Consider the

integral $\int_{\mathbb{T}} \frac{p_n}{z^n \psi B_\ell} dm$. On one hand, its absolute value does not exceed

$$\int_{\mathbb{T}} \frac{|p_n|}{|\psi|} dm \leq \left(\int_{\mathbb{T}} \frac{|p_n|^2}{|\psi|^2} dm \right)^{\frac{1}{2}} = \|p_n\|_{L^2(w dm)} \leq \|p_n\|_{L^2(\mu)} = 1.$$

On the other hand, this integral can be easily computed using the residue theorem. It equals

$$\frac{\tau_n}{\psi(\infty)B_\ell(\infty)} - \sum_{k: k \leq \ell} \frac{p_n(z_k)}{z_k^{n+1} \psi(z_k) B'_\ell(z_k)}.$$

Note now that $|p_n(z_k)| \leq \mu_k^{-\frac{1}{2}} \|p_n\|_{L^2(\mu)} = \mu_k^{-\frac{1}{2}}$ for any n and $z_k^{n+1} \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, $\sum_{k: k \leq \ell} \frac{p_n(z_k)}{z_k^{n+1} \psi(z_k) B'_\ell(z_k)} \rightarrow 0$ as $n \rightarrow \infty$ and we conclude that

$\limsup_{n \rightarrow \infty} \tau_n \leq \psi(\infty)B_\ell(\infty)$. Since this conclusion is true for any ℓ , we can pass to the limit as $\ell \rightarrow \infty$ and get $\limsup_{n \rightarrow \infty} \tau_n \leq \psi(\infty)B(\infty)$.

Now let us prove that $\liminf_{n \rightarrow \infty} \tau_n \geq \psi(\infty)B(\infty)$. To this end, observe that

$$\tau_n = \sup\{\tau : \text{there exists } p(z) = \tau z^n + \dots \text{ with } \|p\|_{L^2(\mu)} \leq 1\}.$$

This means that it would suffice to construct a sequence of polynomials q_n such that the leading coefficients of q_n are arbitrarily close to $\psi(\infty)B(\infty)$ and

$$\limsup_{n \rightarrow \infty} \|q_n\|_{L^2(\mu)} \leq 1.$$

The construction is extremely easy and well known when $\psi, B \in C^\infty(\mathbb{T})$, which corresponds to the case when $w \in C^\infty(\mathbb{T})$ and B is a finite Blaschke product. In this case all one needs to do is to expand the analytic (in the exterior of the unit disk) function $F(z) = \psi(z)B(z)$ into its Taylor series at infinity: $F(z) = t_0 + t_1 z^{-1} + t_2 z^{-2} + \dots$ and put $q_n(z) = z^n S_n(z)$ where $S_n(z) = \sum_{j=0}^n t_j z^{-j}$ is the n -th partial sum of this series. Clearly, the leading coefficient of q_n is exactly $t_0 = F(\infty) = \psi(\infty)B(\infty)$ for all n . On the other hand, since $F \in C^\infty(\mathbb{T})$, the partial sums S_n converge to F uniformly on \mathbb{T} , which allows to estimate the norms $\|q_n\|_{L^2(\mu)}$ as follows.

First,

$$\int_{\mathbb{T}} |q_n|^2 w \, dm = \int_{\mathbb{T}} |S_n|^2 w \, dm \rightarrow \int_{\mathbb{T}} |F|^2 w \, dm = 1.$$

Second, for each k ,

$$|q_n(z_k)| = |z_k^n S_n(z_k)| = |z_k^n (S_n(z_k) - F(z_k))| \leq \max_{\mathbb{T}} |S_n - F| \rightarrow 0 \text{ as } n \rightarrow \infty$$

(the last inequality is just the maximum principle for the function $z^n(S_n(z) - F(z)) = -t_{n+1}z^{-1} - t_{n+2}z^{-2} - \dots$), which is bounded and analytic in the exterior of the unit disk. Thus, $\sum_k \mu_k |q_n(z_k)|^2 \rightarrow 0$ as $n \rightarrow \infty$ (let us remind the reader that the sum is assumed to be finite here).

At last, for every $z \in \mathbb{D}$, we have $|q_n(z)| \leq \max_{\mathbb{T}} |q_n| = \max_{\mathbb{T}} |S_n| \rightarrow \max_{\mathbb{T}} |F|$ as $n \rightarrow \infty$, so the functions q_n are uniformly bounded in \mathbb{D} . On the other hand, it is fairly easy to see that, for any ℓ^2 sequence $\{t_j\}_{j \geq 0}$ and any $z \in \mathbb{D}$, the sequence $\sum_{j=0}^n t_j z^{n-j}$ ($n = 1, 2, \dots$) tends to 0 as $n \rightarrow \infty$ and, moreover, this convergence is uniform on any compact subset of \mathbb{D} . Indeed, we have

$$\begin{aligned} \left| \sum_{j=0}^n t_j z^{n-j} \right| &\leq \sum_{0 \leq j \leq \frac{n}{2}} |t_j| \cdot |z|^{n-j} + \sum_{\frac{n}{2} < j \leq n} |t_j| \cdot |z|^{n-j} \\ &\leq \left(\sum_{j \geq 0} |t_j|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{0 \leq j \leq \frac{n}{2}} |z|^{2n-2j} \right)^{\frac{1}{2}} + \left(\sum_{j > \frac{n}{2}} |t_j|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{0 \leq j \leq n} |z|^{2n-2j} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j \geq 0} |t_j|^2 \right)^{\frac{1}{2}} \cdot \left(\frac{|z|^n}{1 - |z|^2} \right)^{\frac{1}{2}} + \left(\sum_{j > \frac{n}{2}} |t_j|^2 \right)^{\frac{1}{2}} \cdot \left(\frac{1}{1 - |z|^2} \right)^{\frac{1}{2}}. \end{aligned}$$

It remains to note that $|z|^n \rightarrow 0$ and $\sum_{j > \frac{n}{2}} |t_j|^2 \rightarrow 0$ as $n \rightarrow \infty$. Thus, $q_n(z) \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$ and, by the dominated convergence theorem, $\int_{\mathbb{D}} |q_n|^2 d\nu \rightarrow 0$ as $n \rightarrow \infty$.

Combining these 3 estimates, we conclude that, as $n \rightarrow \infty$,

$$\|q_n\|_{L^2(\mu)}^2 = \int_{\mathbb{T}} |q_n|^2 dm + \int_{\mathbb{D}} |q_n|^2 d\nu + \sum_k \mu_k |q_n(z_k)|^2 \rightarrow 1 + 0 + 0 = 1.$$

Now we would like to do something similar in the general case. The main difficulty is that the Taylor series of the function F in general does not converge to F uniformly

on \mathbb{T} . Fortunately, we do not really need the uniform convergence here. Let us find out what kind of convergence it would be appropriate to ask for.

First, in order to have $\int_{\mathbb{T}} |q_n|^2 w dm \rightarrow \int_{\mathbb{T}} |F|^2 w dm$, it suffices to ensure that $S_n \rightarrow F$ in $L^2(w dm)$.

Second, let us estimate the (now possibly infinite) sum $\sum_k \mu_k |q_n(z_k)|^2$. Assuming for a moment that $F \in H^2$, we can try to estimate $|q_n(z_k)|^2$ by $\int_{\mathbb{T}} P_{z_k} |S_n - F|^2 dm$ where P_{z_k} is the Poisson kernel corresponding to the point z_k (instead of the maximum principle, we use the subharmonicity of $|z^n(S_n(z) - F(z))|^2$ in the exterior of the unit disk). Therefore, to conclude that this sum goes to 0, it suffices to ensure that $S_n \rightarrow F$ in $L^2(w_1 dm)$ where $w_1 = \sum_k \mu_k P_{z_k} \in L^1(m)$.

At last, to estimate $\int_{\mathbb{D}} |q_n|^2 d\nu$, let us observe that the assumption $F \in H^2$ is sufficient to ensure that $q_n(z) \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$ (the proof is exactly the same as before). The dominated convergence theorem is somewhat difficult to employ now because it would require an L^2 estimate for $\sup_n |S_n|$ on \mathbb{T} , i.e., a Carleson type theorem, but, fortunately, other boundedness conditions are available to ensure that uniform convergence to 0 on compact subsets of \mathbb{D} implies convergence to 0 in $L^2(\nu)$. The simplest such condition is uniform boundedness of the integrals $\int_{\mathbb{D}} |q_n|^2 d\tilde{\nu}$ where $\tilde{\nu}$ is any finite measure of the kind $d\tilde{\nu} = \varphi(|z|) d\nu$ with some positive function $\varphi(r)$ increasing to $+\infty$ as $r \rightarrow 1-$. Indeed, then, for any $r \in (0, 1)$, we can write

$$\int_{\mathbb{D}} |q_n|^2 d\nu = \int_{r\mathbb{D}} |q_n|^2 d\nu + \int_{\mathbb{D} \setminus r\mathbb{D}} |q_n|^2 d\nu \leq \max_{r\mathbb{D}} |q_n|^2 \nu(\mathbb{D}) + \frac{1}{\varphi(r)} \int_{\mathbb{D}} |q_n|^2 d\tilde{\nu}$$

and observe that the first term tends to 0 for any fixed $r \in (0, 1)$ as $n \rightarrow \infty$ while the second one can be made arbitrarily small by choosing r sufficiently close to 1. Using the subharmonicity of $|q_n|^2$ in \mathbb{D} , we see that to get $\int_{\mathbb{D}} |q_n|^2 d\nu \rightarrow 0$, it would suffice to have a uniform bound for $\int_{\mathbb{T}} |S_n|^2 w_2 dm$ where $w_2(z) = \int_{\mathbb{D}} P_{\zeta}(z) d\tilde{\nu}(\zeta)$ is the “harmonic sweeping” of the measure $\tilde{\nu}$ to the unit circumference \mathbb{T} . Note that, again, we have $w_2 \in L^1(m)$.

The moral of the story is that it would suffice to ensure convergence of S_n to F in $L^2(W dm)$ where $W = 1 + w + w_1 + w_2$ is a certain L^1 function on \mathbb{T} . (we added 1 just to ensure that $L^2(W dm) \subset L^2(m)$). Of course, we cannot hope for that kind of convergence if the function F itself is not in $L^2(W dm)$ and, if we define F exactly as before by $F = \psi B$, most likely, it will fail to belong to that space. So let us see whether we can modify the definition of F . Apparently, we cannot do anything with the second factor: we need F to vanish at all points z_k in order to carry out our trick in the estimate of $\sum_k \mu_k |q_n(z_k)|^2$. On the other hand, after some thought, one can realize that we do not need the first factor to be exactly ψ : any outer function $\tilde{\psi}$ with $|\tilde{\psi}| \leq |\psi|$ and $\tilde{\psi}(\infty) \approx \psi(\infty)$ will do just as well. This freedom allows us to make F belong to any given weighted space $L^2(V dm)$ with $V \geq 1$ (again, this condition is imposed just to get $F \in H^2$ for sure) satisfying $\int_{\mathbb{T}} \log V dm < +\infty$. Indeed, just define $\tilde{\psi}$ by $|\tilde{\psi}|^2 = \min\{\frac{1}{w}, \frac{A}{V}\}$ on \mathbb{T} , $\tilde{\psi}(\infty) > 0$. By choosing A sufficiently large, we can ensure that $\tilde{\psi}(\infty)$ is as close to $\psi(\infty)$ as we wish. On the other hand, we shall always have $\|F\|_{L^2(V dm)} = \|\tilde{\psi}\|_{L^2(V dm)} \leq \sqrt{A} < +\infty$.

Since $W \in L^1(m)$ implies $\int_{\mathbb{T}} \log W \, dm < +\infty$, we, indeed, can make $F \in L^2(W \, dm)$ by choosing V equal to W or any larger weight with integrable logarithm. Unfortunately, this is not enough. We need more, namely, that $S_n \rightarrow F$ in $L^2(W \, dm)$, which, in particular, implies that we must have a uniform bound for the norms $\|S_n\|_{L^2(W \, dm)}$. Since $S_n = z^{-n} \mathcal{P}_+(z^n F)$, where \mathcal{P}_+ is the orthogonal projection from $L^2(m)$ to H^2 , we are naturally led to the question for which integrable weights $W \geq 1$ one can find another weight $V \geq W$ with $\log V \in L^1(m)$ such that \mathcal{P}_+ is bounded as an operator from $L^2(V \, dm)$ to $L^2(W \, dm)$. The answer is given by the celebrated

Theorem of Koosis. *For every integrable weight $W \geq 1$ one can find another weight $V \geq W$ with $\log V \in L^1(m)$ such that \mathcal{P}_+ is bounded as an operator from $L^2(V \, dm)$ to $L^2(W \, dm)$.*

This is a truly remarkable theorem that deserves to be known much better than it currently seems to be. For the reader's convenience, we included its proof in the Appendix. Now let us finish the proof of our theorem. The only remaining difficulty is that we need convergence of S_n to F in $L^2(W \, dm)$ rather than mere boundedness of $\|S_n\|_{L^2(W \, dm)}$, which is all the Koosis theorem provides us with if we apply it directly to the weight W . The (fairly standard) trick is to apply the Koosis theorem to another weight $\widetilde{W} = W\varphi(W)$ where the increasing function $\varphi : [1, +\infty) \rightarrow [1, +\infty)$ is chosen so that $\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$ and the weight \widetilde{W} is still integrable. Let now V be the weight corresponding to \widetilde{W} instead of just W . We claim that $\|S_n - F\|_{L^2(W \, dm)} \rightarrow 0$ as $n \rightarrow \infty$ for all $F \in L^2(V \, dm)$. Indeed, since $V \geq \widetilde{W} \geq 1$, we know that $F \in L^2(m)$ and, thereby, $\|S_n - F\|_{L^2(m)} \rightarrow 0$. On the other hand, the norms $\|S_n - F\|_{L^2(\widetilde{W} \, dm)}$ are uniformly bounded. Hence, for every $M > 0$, we can write

$$\begin{aligned} \int_{\mathbb{T}} |S_n - F|^2 W \, dm &= \int_{\{W \leq M\}} |S_n - F|^2 W \, dm + \int_{\{W > M\}} |S_n - F|^2 W \, dm \\ &\leq M \int_{\mathbb{T}} |S_n - F|^2 \, dm + \frac{1}{\varphi(M)} \int_{\mathbb{T}} |S_n - F|^2 \widetilde{W} \, dm. \end{aligned}$$

Now, the first term tends to 0 as $n \rightarrow \infty$ for any fixed $M > 0$ while the second one can be made arbitrarily small by choosing M large enough.

The theorem is thus completely proved.

3. Proof of Corollary

Again, denote by B_ℓ the partial Blaschke product with zeros z_k , $k \leq \ell$. Consider the integral

$$\int_{\mathbb{T}} \left| 1 - \frac{p_n(z)}{z^n} \frac{1}{\psi(z) B_\ell(z)} \right|^2 dm.$$

Using the residue theorem, we conclude that it equals

$$1 + \|p_n\|_{L^2(w \, dm)}^2 - 2 \frac{\tau_n}{\psi(\infty) B_\ell(\infty)} - 2 \Re \sum_{k: k \leq \ell} \frac{p_n(z_k)}{z_k^{n+1} \psi(z_k) B'_\ell(z_k)}.$$

We have already seen that $\sum_{k:k \leq \ell} \frac{p_n(z_k)}{z_k^{n+1} \psi(z_k) B'_\ell(z_k)} \rightarrow 0$ as $n \rightarrow \infty$. Also,

$$\|p_n\|_{L^2(w \, dm)} \leq \|p_n\|_{L^2(\mu)} = 1.$$

Thus,

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{T}} \left| B_\ell(z) - \frac{p_n(z)}{z^n \psi(z)} \right|^2 dm \leq 2 \left(1 - \frac{B(\infty)}{B_\ell(\infty)} \right),$$

whence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{T}} \left| B(z) - \frac{p_n(z)}{z^n \psi(z)} \right|^2 dm \\ \leq 2 \|B_\ell - B\|_{L^2(m)}^2 + 4 \left(1 - \frac{B(\infty)}{B_\ell(\infty)} \right). \end{aligned}$$

Since the right hand side of the last inequality tends to 0 as $n \rightarrow \infty$, we conclude that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} \left| B(z) - \frac{p_n(z)}{z^n \psi(z)} \right|^2 dm = 0.$$

It remains to recall that, for the analytic functions $g_n(z) = B(z) - \frac{p_n(z)}{z^n \psi(z)}$, convergence to 0 in H^2 is stronger than pointwise convergence to 0 in the exterior of the unit disk.

4. Appendix: Proof of the Koosis theorem.

We shall outline the original proof from [1] here. First of all, note that for any two weights $V \geq W \geq 1$, the boundedness of \mathcal{P}_+ as an operator from $L^2(V)$ to $L^2(W)$ is implied by (actually, equivalent to) its boundedness as an operator from $L^2(w)$ to $L^2(v)$ where $w = \frac{1}{W}$, $v = \frac{1}{V}$. The latter is understood in the sense that there exists a finite constant $C > 0$ such that $\int_{\mathbb{T}} |\mathcal{P}_+ g|^2 v \, dm \leq C \int_{\mathbb{T}} |g|^2 w \, dm$ for any function $g \in L^2(m) \cap L^2(w \, dm)$. This can be seen by a standard duality argument. Using the density of trigonometric polynomials in $L^2(m)$, we also see that it is enough to check this estimate for the case when g is a real trigonometric polynomial. Secondly, let us note that $\mathcal{P}_+ g = \frac{1}{2}(\hat{g}(0) + g + i\tilde{g})$ where $\tilde{\cdot}$ is the operator of harmonic conjugation, i.e., the operator that maps $\sum_k c_k z^k$ to $\frac{1}{i} \sum_k (\operatorname{sgn} k) c_k z^k$. Note that the identity operator is bounded from $L^2(w \, dm)$ to $L^2(v \, dm)$ for any $v \leq w$. Since $|\hat{g}(0)| = |\int_{\mathbb{T}} g \, dm| \leq (\int_{\mathbb{T}} |g|^2 w \, dm)^{\frac{1}{2}} (\int_{\mathbb{T}} W \, dm)^{\frac{1}{2}} = \sqrt{\|W\|_{L^1(m)}} \|g\|_{L^2(w \, dm)}$, we see that the operator that maps g to the constant function $\hat{g}(0)$ is bounded in $L^2(w \, dm)$ and, thereby, from $L^2(w \, dm)$ to $L^2(v \, dm)$ for any $v \leq w$. These two remarks show that it is enough to construct a weight $v \leq w$ with integrable logarithm such that $\int_{\mathbb{T}} |\tilde{g}|^2 v \, dm \leq C \int_{\mathbb{T}} |g|^2 w \, dm$ for any real trigonometric polynomial g with $\hat{g}(0) = 0$. To this end, consider the outer function $\Omega(z)$ with $\Re \Omega = W$ on \mathbb{T} . Note that

$$\left| 1 - \frac{W}{\Omega} \right| = \left| \frac{\Omega - \Re \Omega}{\Omega} \right| = \left| \frac{\Im \Omega}{\Omega} \right| < 1 \text{ almost everywhere on } \mathbb{T}.$$

Let $\rho = 1 - \left|1 - \frac{W}{\Omega}\right|$. Consider the analytic polynomial $P(z) = g(z) + i\tilde{g}(z)$. Since $P(0) = 0$, we have

$$\int_{\mathbb{T}} \frac{P^2}{\Omega} dm = \frac{P(0)^2}{\Omega(0)} = 0.$$

Let us rewrite it as

$$\int_{\mathbb{T}} P^2 w dm = \int_{\mathbb{T}} P^2 \left(1 - \frac{W}{\Omega}\right) w dm$$

and take the real part of the left hand side with minus sign and the absolute value of the right hand side. We shall get the inequality

$$\int_{\mathbb{T}} (\tilde{g}^2 - g^2) w dm \leq \int_{\mathbb{T}} |P|^2 \left|1 - \frac{W}{\Omega}\right| w dm = \int_{\mathbb{T}} (g^2 + \tilde{g}^2)(1 - \rho) w dm,$$

which is equivalent to

$$\int_{\mathbb{T}} \tilde{g}^2 \rho w dm \leq \int_{\mathbb{T}} g^2 (2 - \rho) w dm \leq 2 \int_{\mathbb{T}} g^2 w dm.$$

Thus, we can choose $v = \rho w$. The only thing that remains to check is that $\int_{\mathbb{T}} \log v > -\infty$. To this end, note that

$$\rho = 1 - \left|\frac{\Im \Omega}{\Omega}\right| \geq \frac{1}{2} \left(1 - \left|\frac{\Im \Omega}{\Omega}\right|^2\right) = \frac{|\Re \Omega|^2}{2|\Omega|^2} = \frac{W^2}{2|\Omega|^2}.$$

So $v = \rho w \geq \frac{W}{2|\Omega|^2}$. It remains to note that $W \geq 1$ while $\log |\Omega| \in L^1(m)$.

The Koosis theorem is thus completely proved.

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