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A NOTE ON EQUIVARIANT NORMAL FORMS OF POISSON STRUCTURES

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Abstract. We prove an equivariant version of the local splitting theorem for tame Poisson structures and Poisson actions of compact Lie groups. As a consequence, we obtain an equivariant linearization result for Poisson structures whose transverse structure has semisimple linear part of compact type.

1. Introduction

The main purpose of this note is to prove an equivariant version of Weinstein's splitting theorem for Poisson structures [17]. This theorem asserts that in the neighborhood of any point p in a Poisson manifold (P^n, Π) there is a local coordinate system $(x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_{n-2k})$ in which the Poisson structure Π can be written as

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{1.1}$$

where $2k$ is the rank of Π at p, and f_{ij} are functions which depend only on the variables $(z_1, \ldots, z_{n-2k})$ and which vanish at the origin. Geometrically speaking, locally the Poisson manifold (P^n, Π) can be splitted into the direct product of a $2k$-dimensional symplectic manifold (with the standard nondegenerate Poisson structure $\Pi_1 = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$) and a $(n-2k)$-dimensional Poisson manifold whose Poisson structure $\Pi_2 = \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ vanishes at the origin. We want to show that if there is a (local) action of a compact Lie group G on P^n which fixes the point p and which preserves Π, then this splitting can be made equivariantly.

In the special case when Π is nondegenerate at p (i.e., $2k = n$), one recovers from Weinstein's theorem the classical Darboux theorem about the local existence of canonical (Darboux) coordinates for symplectic manifolds. We know two methods for proving Darboux theorem: 1) the classical coordinate-by-coordinate construction method; and 2) the path method due to Moser [11]. Weinstein's proof of the splitting theorem [17] is also based on the first method (coordinate by coordinate construction). However, this classical method does not seem to work in the equivariant situation, while the path method can be used to prove the equivariant Darboux theorem [16].

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In the same spirit, we will try to use the path method to prove an equivariant version of the splitting theorem for Poisson structures. In doing so, we encounter a technical condition, which we call the *tameness condition*: a smooth Poisson structure Π on a manifold P^n is called *tame* if for any two smooth Poisson vector fields X, Y on P^n (which may depend on some parameters) which are tangent to the symplectic leaves the function $\Pi^{-1}(X, Y)$ is smooth (and depends smoothly on the parameters). We will devote Section 2 of this note to the tameness condition, in order to convince the reader that it is an interesting condition, and many "reasonable" Poisson structures satisfy it. For example, if the linear part of the transverse Poisson structure at a point p has semisimple type, then the Poisson structure is tame near p.

Now we can formulate the main result of this note:

Theorem 1.1. *Let* (P^n, Π) *be a smooth Poisson manifold,* p *a point of* P, $2k =$ rank $\Pi(p)$*, and* G *a compact Lie group which acts on* P *in such a way that the action preserves* Π *and fixes the point* p*. Assume that the Poisson structure* Π *is tame at* p*. Then there is a smooth canonical local coordinate system* $(x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_{n-2k})$ *near* p*, in which the Poisson structure* Π *can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{1.2}$$

and in which the action of G *is linear and preserves the subspaces* $\{x_1 = y_1 = \ldots x_k = y_k = 0\}$ *and* $\{z_1 = \ldots = z_{n-2k} = 0\}$*.*

Remark 1.2. i) We do not know whether the tameness condition is really necessary, or if it is because our method is not good enough. We notice that this condition is also implicitly present in the papers of Ginzburg and Weinstein [8] and of Aleekseev and Meinrenken [1], [2], which involve the path method in Poisson geometry.
ii) The above theorem also holds in the analytic (i.e., real analytic or holomorphic) setting, with basically the same proof. The analytic version of this equivariant theorem is used by Philippe Monnier and the second author in their study of normal forms of vector fields on Poisson manifolds [10]. We hope that our result can be useful in the study of equivariant Hamiltonian systems as well.
iii) If the action of G on (P^n, Π) is Hamiltonian (with an equivariant momentum map), then there is another approach to this equivariant splitting problem, based on the Nash-Moser method, which does not need the tameness condition. We will consider this issue in a separate work.

The above theorem will be proved in Section 3 of this note. In Section 4 we will combine this theorem with linearization results of Conn [4] and Ginzburg [7] to obtain an equivariant linearization theorem (see Theorem 4.1).

2. Tame Poisson structures

We will denote by Π^{-1} the covariant tensor dual to the Poisson tensor Π of a Poisson manifold (P^n, Π), i.e. the symplectic form on symplectic leaves. If X, Y are vector fields on P^n which are tangent to the symplectic leaves, then $\Pi^{-1}(X, Y)$ is well-defined. In particular, if $X = X_h$ is the Hamiltonian vector field of a function h

on (P^n, Π) then $\Pi^{-1}(X, Y) = -Y(h)$. Recall that a Poisson vector field is a vector field which preserves the Poisson structure.

Definition 2.1. Let (P^n, Π) be a smooth Poisson manifold and p a point in P. We will say that Π is *tame* at p if for any pair X_t, Y_t of germs of smooth Poisson vector fields near p which are tangent to the symplectic foliation of (P^n, Π) and which may depend smoothly on a (multi-dimensional) parameter t, then the function $\Pi^{-1}(X_t, Y_t)$ is smooth and depends smoothly on t.

The tameness condition is a kind of homological condition. In particular, if the parametrized germified first Poisson cohomology group, which we will denote by $H^1_\Pi(P^n, p)$, vanishes, then Π is tame at p. Indeed, $H^1_\Pi(P^n, p) = 0$ means that if X_t is a germ of Poisson vector field near p which depends smoothly on a parameter t, then we can write $X_t = X_{h_t}$ where h_t is a germ of smooth function near p which depends smoothly on the parameter t. Hence $\Pi^{-1}(X_t, Y_t) = -Y_t(h_t)$ is smooth.

In particular, it is known that if $\mathfrak{g}$ is a compact semi-simple Lie algebra, and $(\mathfrak{g}^*, \Pi_{lin})$ is the dual of $\mathfrak{g}$ equipped with the corresponding linear Poisson structure then $H^1_{\Pi_{lin}}(\mathfrak{g}^*, 0) = 0$ (see [4]). Hence our first example of tame Poisson structures:

Example 2.2. Any smooth Poisson structure Π, which vanishes at a point p and whose linear part at p corresponds to a compact semisimple Lie algebra $\mathfrak{g}$, is tame at p. Indeed, in this case, according to Conn's smooth linearization theorem [4], (P^n, Π) is locally isomorphic near p to $(\mathfrak{g}^*, \Pi_{lin})$, and therefore $H^1_\Pi(P^n, p) = 0$.

If X is not Hamiltonian (and maybe not even Poisson) but can be written as $X = \sum_{i=1}^m f_i X_{g_i}$ where f_i, g_i are smooth functions, then $\Pi^{-1}(X, Y) = -\sum_{i=1}^m f_i Y(g_i)$ is still smooth. This leads us to:

Definition 2.3. We say that a smooth (resp real analytic) Poisson structure Π satisfies the *smooth division property* (resp *analytic division property*) at a point p if the Hamiltonian vector fields generate the space of vector fields tangent to the associated symplectic foliation near p. More precisely, for any germ of smooth (resp. analytic) vector field Z -which may depend smoothly (resp. analytically) on some parameters-which is tangent to the symplectic foliation there exists a finite number of germs of smooth (resp. analytic) functions $f_1, \ldots, f_m, g_1, \ldots, g_m$ – which depend smoothly (resp. analytically) on the same parameters as Z– such that $Z = \sum f_i X_{g_i}$.

Clearly, if Π satisfies the division property at a point p, then it is tame at p. A natural question is to know which Poisson structures satisfy the division property. In particular, is it true that all linear Poisson structures satisfy the division property at the origin? In the appendix we prove that low-dimensional Lie algebras satisfy the division property at the origin. Namely

Proposition 2.4. *Any linear Poisson structure in dimension 2 or 3 has the division property at the origin.*

In the higher-dimensional case, a result of Dixmier [5] says (in our language) that if Π is a linear Poisson structure which corresponds to a semisimple Lie algebra then it has the analytic division property at the origin (mainly due to the fact that the singular set has codimension 3 in this case). We would conjecture that Dixmier's result also holds in the smooth case. On the other hand, one can probably produce linear Poisson (non semisimple) structures which do not satisfy the division property (similiar to Dixmier's counterexample 3.3 in [5]).

It is not difficult to construct examples of Poisson structures with a trivial 1-jet which are not tame.

Example 2.5. Consider the Poisson structure $\Pi = x^4 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{R}^2$. The following vector fields are Poisson and tangent to the symplectic foliation:

$$X = x^2 \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}, \quad Y = x \frac{\partial}{\partial y},$$

but $\Pi^{-1}(X,Y) = \dfrac{1}{x}$ is not smooth at the origin. So this Poisson structure is not tame.

Recall that if $\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ in a local canonical coordinate system in the neighborhood of a point p, then $\Pi_2 = \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ is called the *transverse* Poisson structure of Π at p. Up to local Poisson isomorphisms, this Poisson transverse structure is unique, i.e. it does not depend on the choice of local canonical coordinates, see, e.g., [6, 17]. The following lemma shows that, to verify the tameness condition, it is sufficient to check it in the transverse direction to the symplectic leaf:

Lemma 2.6. *A smooth Poisson structure* Π *is tame at a point* p *if and only if the transverse Poisson structure of* Π *at* p *is tame at* p*.*

Proof. Write $\Pi = \Pi_1 + \Pi_2 = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ in a local canonical coordinate system near p. For each germ of vector field X near p write $X = X_{hor} + X_{vert}$, where X_{hor} is the "horizontal part" of X, i.e. is a combination of the vector fields $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$, and X_{vert} is the "vertical part" of X, i.e. is a combination of the vector fields $\frac{\partial}{\partial z_i}$. If X is a smooth Poisson vector field for Π, then X_{hor} (resp. X_{vert}) may be viewed as a Poisson vector field for Π_1 (resp., Π_2) which depends smoothly on parameters z_i (resp., x_i, y_i). We have $\Pi^{-1}(X,Y) = \Pi_1^{-1}(X_{hor}, Y_{hor}) + \Pi_2^{-1}(X_{vert}, Y_{vert})$. The term $\Pi_1^{-1}(X_{hor}, Y_{hor})$ is always smooth (provided that X and Y are smooth), and so the smoothness of $\Pi^{-1}(X,Y)$ is equivalent to the smoothness of $\Pi_2^{-1}(X_{vert}, Y_{vert})$. The lemma then follows easily. □

3. Proof of the equivariant splitting theorem

In this section we will give a proof of Theorem 1.1. It uses coupling tensors for Poisson manifolds, so we will first recall a result of Yu. Vorobiev about coupling tensors (see, e.g., [6, 14]). The proof of the theorem consists of three steps. In the first step we prove that we can assume that the action of our compact Lie group G is linear and that the symplectic foliation is normalized (i.e. is the same as in the splitting theorem). In the second step we construct a path of G-invariant Poisson structures connecting the initial Poisson structure to the splitted one. Finally, in the last step, we use this path of Poisson structures and the averaging method to construct a flow which intertwines with the action of G and whose time-1 map moves the initial Poisson structure to the splitted one, thus giving an equivariant splitting of our Poisson structure.

3.1. Preliminaries: coupling tensors.

Let $\pi : E \longrightarrow S$ be a submersion over a manifold S and let $T_V E = \ker d\pi$. An Ehresmann connection on E is a splitting of the tangent bundle of E as $TE = T_V E \oplus T_H E$. We call $T_H E$ the horizontal space. Denote by $\mathcal{V}^1_V(E)$ the set of vertical vector fields. We can associate to this splitting a $\mathcal{V}^1_V(E)$-valued 1-form $\Gamma \in \Omega^1(E) \otimes \mathcal{V}^1_V(E)$ such that $\Gamma(Z) = Z$ for any vertical vector field. Then the horizontal space can be written as $T_H E = \{X \in TE, \quad \Gamma(X) = 0\}$. We can define the horizontal lifting of vector fields from S to E. In the same way, we may associate a parallel transport to Γ which is smooth, a curvature form and a covariant derivative (for details see for example [6]).

Consider now the case when S is a symplectic leaf of a Poisson manifold (P, Π). We can consider a neighbourhood E of S and submersion $\pi : E \longrightarrow S$ whose restriction to S is the identity.

There is a natural smooth Ehresmann connection where the horizontal subbundle is spanned by the Hamiltonian vector fields $X_{f\circ\pi}$.

We can also associate to it a 2-form $\mathbb{F} \in \Omega^2(S) \otimes \mathcal{C}^\infty(E)$ defined as

$$\mathbb{F}(X_{f\circ\pi}, X_{g\circ\pi}) = \langle \Pi, \pi^* df \wedge \pi^* dg \rangle.$$

Recall that we have an induced transverse Poisson structure Π_{Vert} on the vertical space.

The triple $(\Pi_{Vert}, \Gamma, \mathbb{F})$ is called the geometric data associated to the Poisson manifold (P, Π) in a neighbourhood of a symplectic leaf. In [14], Vorobjev studies the reconstruction problem from given geometric data. That is, given a triple of smooth geometric data he gives compatibility conditions that guarantee the existence of a Poisson structure with those geometric data. Those compatibility conditions come from the Schouten condition $[\Pi, \Pi] = 0$ imposed on the bivector field Π reconstructed from the geometric data.

Assume that we are given $(\Pi_{Vert}, \Gamma, \mathbb{F})$ on a fibration $\pi : E \longrightarrow S$, where Γ is an Ehresmann connection on E, Π_{Vert} a vertical bivector field, and $\mathbb{F} \in \Omega^2(S) \otimes \mathcal{C}^\infty(E)$ a nondegenerate $\mathcal{C}^\infty(E)$-valued 2-form on S.

We will need the following characterization of geometrical data which come from a Poisson structure:

Theorem 3.1 (Vorobiev [14]). *The triple $(\Pi_{Vert}, \Gamma, \mathbb{F})$ on a fibration $\pi : E \longrightarrow S$ determines a Poisson structure on E if and only if $\mathbb{F}$ is nondegenerate and the following four compatibility conditions are satisfied:*

$$[\Pi_{Vert}, \Pi_{Vert}] = 0, \tag{3.1}$$

$$L_{Hor(u)}(\Pi_{Vert}) = 0 \quad \forall\, u \in \mathcal{V}^1_V(E), \tag{3.2}$$

$$\partial_\Gamma \mathbb{F} = 0, \tag{3.3}$$

$$Curv_\Gamma(u, v) = \nu^\sharp(d(\mathbb{F}(u, v))), \tag{3.4}$$

where ∂_Γ *stands for the covariant derivative and* $\nu^\sharp$ *stands for the map from* T^*E *to* TE *defined by* $\langle \nu^\sharp(\alpha), \beta \rangle = \langle \nu, \alpha \wedge \beta \rangle$.

Remark 3.2. We may think of Π as the coupling of Π_{Vert} with $\mathbb{F}$ by Γ. This so-called coupling method is a generalization of the minimal coupling procedure established for symplectic fibrations by Guillemin, Lerman and Sternberg [9], [13].

3.2. First step of the proof: linearization of the group action.

Consider an action $\rho : G \times P^n \to P^n$ of a compact Lie group G on a Poisson manifold (P^n, Π), which fixes a point $p \in P^n$ and preserves the Poisson structure Π. Denote by S the local symplectic leaf through p. Note that S is invariant under the action of G. According to Bochner's theorem [3], the action of G is linearizable near p, i.e., there is a local coordinate system in which the action is linear. Moreover, we may assume that S is linear in these cordinates. Since linear representations of compact Lie groups are completely reducible, there is a local submanifold N (which is also linear in these coordinates), which is invariant under the action of G and which is transverse to S at p. The following lemma says that we can choose this coordinate system in such a way that the symplectic foliation of (P^n, Π) will also be the same as in the splitting theorem.

Lemma 3.3. *With the above notations, there is a local system of coordinates near* p *in which the action of* G *is linear, the submanifolds* S, N *are linear, and the local symplectic leaves near* p *are direct products of* S *with symplectic leaves of the transverse Poisson structure on* N.

Proof. We can start with a first coordinate system in which the action of G is linear and the submanifolds S, N are linear. Denote by p_1 the linear projection from a sufficiently small neighborhood U of p in P^n to S which projects N to p. Define another (a-priori nonlinear) projection p_2, from U to N, as follows: Denote by Γ the Ehresmann connection associated to the Poisson structure Π and the projection p_1. For each $x \in U$, let $\alpha_x(t)$ be the linear path joining $p_1(x)$ to the origin p in S, with $\alpha_x(0) = p_1(x)$ and $\alpha_x(1) = p$. Denote by $\hat{\alpha}_x$ the horizontal lift of α_x through x with respect to Γ. Then we take $p_2(x) = \hat{\alpha}_x(1) \in N$.

By construction both projections are smooth and G-equivariant: The projection p_1 is equivariant since N is G-invariant and p_2 is equivariant because the action of G preserves Π and therefore the parallel transport is equivariant.

Now consider the G-equivariant local diffeomorphism

$$\begin{aligned} \phi : \quad U &\longrightarrow \quad S \times N \\ x &\longmapsto \quad (p_1(x), p_2(x)) \end{aligned}$$

Since the parallel transport preserves the Poisson structure, ϕ takes the Poisson structure on U to a Poisson structure on $S \times N$ which has as symplectic leaves the product of the symplectic leaves on N with S. This ends the proof of the lemma. □

3.3. Second step: constructing a path of Poisson structures.

After the first step, we can now assume that $P = N \times S$, and the Poisson structure Π has the same symplectic leaves as the splitted Poisson structure $\tilde{\Pi} = \Pi_S + \Pi_N$, where Π_S is the standard nondegenerate Poisson structure on S and Π_N is the transverse

Poisson structure on N, and both Π and $\tilde{\Pi}$ are invariant under our linear action of G. We will assume that Π is tame at p, or equivalently, the transverse Poisson structure Π_N is tame at the origin.

Lemma 3.4. *With the above notations and assumptions, there is a smooth path of G-invariant Poisson structures Π_t, $t \in [0,1]$, on (a neighborhood of the origin in) $N \times S$, such that $\Pi_0 = \Pi$, $\Pi_1 = \Pi_S + \Pi_N$, and which have the same symplectic foliation for all $t \in [0,1]$.*

Proof. We denote by ω_0 the symplectic structure induced on the symplectic leaves by $\Pi_0 = \Pi$. In the same way we denote by ω_1 the symplectic structure induced by $\Pi_1 = \Pi_S + \Pi_N$ on the same symplectic foliation. Consider the linear path of 2-forms

$$\omega_t = t\omega_1 + (1-t)\omega_0. \tag{3.5}$$

This is a path of smooth closed 2-forms on each symplectic leaf of the common symplectic foliation. We want to show that, for each t there is a smooth bivector field Π_t which corresponds to ω_t. Then, automatically, Π_t is a Poisson structure because of the closedness of ω_t, has the same symplectic foliation as Π_0 and Π_1, and is G-invariant.

Denote by $(\Pi_N, \Gamma_0, \mathbb{F}_0)$ and $(\Pi_N, \Gamma_1, \mathbb{F}_1)$, the geometric data associated to the Poisson structures $\Pi_0 = \Pi$ and $\Pi_1 = \Pi_N + \Pi_S$ with respect to the projection $p_1 : N \times S \to S$ (remark that, by construction, they have the same vertical component, which is equal to Π_N). We will use Vorobjev's Theorem 3.1 to construct Π_t and to prove its smoothness. In other words, we will construct geometric data $(\Pi_N, \Gamma_t, \mathbb{F}_t)$, which will be shown to be smooth and satisfy the compatibility conditions of Theorem 3.1, so they will give rise to a smooth Poisson structure Π_t.

Construction and smoothness of Γ_t:

In order to construct the connection Γ_t, it is enough to show how to lift each vector field X on S horizontally with respect to Γ_t. The horizontal lift X_t of X with respect to Γ_t is uniquely characterized by ω_t (the would-be associated symplectic form on the symplectic leaves) and by the following two conditions:

(1) The vector field X_t is tangent to the common symplectic foliation of Π_0 and Π_1, and its projection to S by p_1 is X.
(2) $\omega_t(X_t, Z) = 0$ for any vertical vector field Z.

Denote by X_0 and X_1 the horizontal lift of X with respect to Γ_0 and Γ_1 respectively. We will show that

$$X_t = (1-t)X_0 + tX_1. \tag{3.6}$$

(Then the smoothness of X_t, and hence of Γ_t, is automatic). It is clear that $(1-t)X_0 + tX_1$ is tangent to the symplectic foliation and projects to X under p_1. It remains to show that

$$\omega_t((1-t)X_0 + tX_1, Z) = 0$$

for any vertical vector field Z on $N \times S$. Indeed, denoting $W = X_0 - X_1$, we have

$$\begin{array}{rl} & \omega_t((1-t)X_0 + tX_1, Z) \\ = & t\omega_1((1-t)X_0 + tX_1, Z) + (1-t)\omega_0((1-t)X_0 + tX_1, Z) \\ = & t\omega_1(X_1 + (1-t)W, Z) + (1-t)\omega_0(X_0 - tW, Z) \\ = & t\omega_1(X_1, Z) + (1-t)\omega_0(X_0, Z) + t(1-t)[\omega_1(W, Z) - \omega_0(W, Z)]. \end{array}$$

Since X_1 and X_0 are the horizontal lifts of X with respect to Γ_1 and Γ_0, the terms $\omega_1(X_1, Z)$ and $\omega_0(X_0, Z)$ vanish. Since the Poisson structures Π_0 and Π_1 have the same transverse component, and W and Z are vertical vector fields, we have $\omega_1(W, Z) = \omega_0(W, Z) = \Pi_N^{-1}(W, Z)$. Hence $\omega_t((1-t)X_0 + tX_1, Z) = 0$ as desired.

Construction and smoothness of $\mathbb{F}_t$*:*

If X is a vector field on S then we will denote by $X_t = (1-t)X_0 + tX_1$ the horizontal lift of X to $N \times S$ via Γ_t as above. For any two smooth vector fields X, Y on S and a point $q \in N \times S$, put

$$\mathbb{F}_t(X, Y)(q) = \omega_t(X_t, Y_t)(q). \tag{3.7}$$

The main point here is to check the smoothness of the function $\mathbb{F}_t(X, Y)$ defined by the above formula, in a neighborhood of the origin in $N \times S$. Denote $Z^X = X_0 - X_1$ and $Z^Y = Y_0 - Y_1$; they are vertical vector fields. Since the Ehresmann connection Γ_i $(i = 0, 1)$ preserves the transverse Poisson structures, the vector fields $\hat{X}_i$ and $\hat{Y}_i$ preserve the transverse Poisson structure Π_N. Therefore the vertical vector fields Z^X and Z^Y also preserve the transverse Poisson structure. (They may be viewed as Poisson fields on (N, Π_N) parametrized by S).

We can write $X_t = X_0 - tZ^X = X_1 + (1-t)Z^X$ and $Y_t = Y_0 - tZ^Y = Y_1 + (1-t)Z^Y$. Recall that if X_t is horizontal with respect to Γ_t and Z is vertical then $\omega_t(X_t, Z) = 0$. We have:

$$\begin{array}{rl} & \mathbb{F}_t(X, Y) \\ = & t\omega_1(X_1 + (1-t)Z^X, Y_1 + (1-t)Z^Y) + (1-t)\omega_0(X_0 - tZ^X, Y_0 - tZ^Y) \\ = & t\omega_1(X_1, Y_1) + (1-t)\omega_0(X_0, Y_0) + \\ & +t(1-t)^2\omega_1(Z^X, Z^Y) + t^2(1-t)\omega_0(Z^X, Z^Y) \\ = & t\omega_1(X_1, Y_1) + (1-t)\omega_0(X_0, Y_0) + t(1-t)\Pi_N^{-1}(Z^X, Z^Y) \end{array}$$

By our tameness hypothesis, $\Pi_N^{-1}(Z^X, Z^Y)$ is smooth, and so $\mathbb{F}_t(X, Y)$ is smooth (and depends smothly on t).

Remark that $\mathbb{F}_t$ coincides with $\mathbb{F}_0$ and $\mathbb{F}_1$ at the origin p. Since $\mathbb{F}_0$ is nodegenerate, $\mathbb{F}_t$ is also nondegenerate in a neighborhood of p in $N \times S$.

Since the form ω_t used in the construction of $(\Pi_N, \Gamma_t, \mathbb{F}_t)$ is closed on each symplectic leaf, the four compatibility conditions for the triple $(\Pi_N, \Gamma_t, \mathbb{F}_t)$ are automatically satisfied. Hence the triple $(\Pi_N, \Gamma_t, \mathbb{F}_t)$ corresponds to a smooth Poisson structure Π_t in a neighborhood of p in $N \times S$.

Moreover, by construction, $\Pi_0 = \Pi$, $\Pi_1 = \Pi_N + \Pi_S$, and Π_t depends smoothly on t. Lemma 3.4 is proved. □

3.4. End of the proof.

According to Lemma 3.4, we now have a smooth path of G-invariant Poisson structures Π_t, where Π_0 is our initial Poisson structure, and $\Pi_1 = \Pi_N + \Pi_S$ is the splitted one. (The action of G is already linearized, and by the equivariant Darboux theorem we may assume that Π_S is already equivariantly normalized, i.e. has Darboux form). In order to finish the proof of the theorem, it suffices to find a local diffeomorphism of $N \times S$ which commutes with the action of G and which moves Π_0 to Π_1.

According to Weinstein's splitting theorem (or rather its parametrized version, whose proof is the same), there is a smooth family of local diffeomorphisms $\phi_t, t \in [0,1]$ such that $\phi_{t*}(\Pi_0) = \Pi_t$ and $\phi_0 = Id$. Note that, a-priori, ϕ_t does not commute with the action of G. Denote by X_t the time-dependent vector field whose flow generates ϕ_t, i.e.,

$$X_t(\phi_t(q)) = \frac{\partial \phi_t}{\partial t}(q). \tag{3.8}$$

By derivation of the condition

$$\phi_{t*}(\Pi_0) = \Pi_t \tag{3.9}$$

we get the following equation for X_t:

$$L_{X_t}(\Pi_t) = -\frac{d\Pi_t}{dt} \tag{3.10}$$

Denote by X_t^G the averaging of X with respect to the action of G, i.e.,

$$X_t^G = \int_G \rho_{g*}(X_t) d\mu, \tag{3.11}$$

where $d\mu$ is the probabilistic Haar measure on G, and ρ_g denotes the action of $g \in G$. Then X_t^G is a G-invariant time-dependent vector field. Since Π_t is invariant under the action of G, it follows from Equation (3.10) that we also have

$$L_{X_t^G} \Pi_t = -\frac{d\Pi_t}{dt}. \tag{3.12}$$

Denote by ϕ_t^G the flow X_t^G. Then ϕ_t^G commutes with the action of G. Equation (3.12) implies that $\phi_t^G{}_*(\Pi_0) = \Pi_t$. In particular, ϕ_1^G is a G-equivariant local diffeomorphism such that $\phi_1^G{}_*(\Pi_0) = \Pi_1 = \Pi_N + \Pi_S$. This concludes the proof of Theorem 1.1.

4. Equivariant linearization of Poisson structures

Theorem 4.1. *Let (P^n, Π) be a smooth Poisson manifold, p a point of P, $2r =$ rank $\Pi(p)$, and G a compact Lie group which acts on P in such a way that the action preserves Π and fixes the point p. Assume that the linear part of transverse Poisson structure of Π at p corresponds to a semisimple compact Lie algebra $\mathfrak{k}$. Then there is a smooth canonical local coordinate system $(x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_{n-2r})$ near p, in which the Poisson structure Π can be written as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{4.1}$$

where c_{ij}^k *are structural constants of* $\mathfrak{k}$*, and in which the action of* G *is linear and preserves the subspaces* $\{x_1 = y_1 = \dots x_r = y_r = 0\}$ *and* $\{z_1 = \dots = z_{n-2r} = 0\}$.

Proof. Invoking Theorem 1.1, we may assume that Π is already equivariantly splitted, i.e. $\Pi = \sum_{i=1}^r \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$. It remains to linearize the transverse Poisson structure $\Pi_N = \sum_{i,j} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ on N in an equivariant way. But this last step is provided by the following results of Conn and Ginzburg:

Theorem 4.2 (Conn [4])**.** *Any smooth Poisson structure, which vanishes at a point and whose linear part at that point is of semisimple compact type, is locally smoothly linearizable.*

Theorem 4.3 (Ginzburg [7])**.** *Assume that a Poisson structure* Π *vanishes at a point* p *and is smoothly linearizable near* p*. If there is an action of a compact Lie group* G *which fixes* p *and preserves* Π*, then* Π *and this action of* G *can be linearized simultaneously.*

Indeed, by Theorem 4.2, the transverse Poisson structure Π_N is smoothly linearizable because its linear part is compact semisimple. As a consequence, by Theorem 4.3, Π_N can be linearized in a G-equivariant way. □

5. Appendix

In this appendix we will give a proof of Proposition 2.4. We will assume that our linear Poisson structure corresponds to a 3-dimensional Lie algebra $\mathfrak{g}$ (the case of dimension 2 is similar and simpler and can be reduced from the 3-dimensional case). Recall that any 3-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ belongs to one of the following types:

(1) Solvable: $\mathfrak{g} = \mathbb{R} \ltimes_A \mathbb{R}^2$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2-by-2 matrix, i.e. with Lie brackets $[x, y] = ay + bz$, $[x, z] = cy + dz$, $[y, z] = 0$.
(2) Simple: $\mathfrak{so}(3, \mathbb{R})$ or $\mathfrak{sl}(2, \mathbb{R})$.

We will prove that any vector field X tangent to the symplectic foliation of $\mathfrak{g}^*$ (i.e. the foliation by coadjoint orbits on $\mathfrak{g}^*$) can be expressed as a smooth combination of the Hamiltonian vector fields X_x, X_y and X_z, where (x, y, z) is a basis of $\mathfrak{g}$.

Let us first consider the case when $\mathfrak{g} = \mathbb{R} \ltimes_A \mathbb{R}^2$. In this case, our linear Poisson structure Π can be written as:

$$\Pi = \frac{\partial}{\partial x} \wedge ((ay + bz)\frac{\partial}{\partial y} + (cy + dz)\frac{\partial}{\partial z}). \tag{5.1}$$

We distinguish two subcases.

1) The matrix A has non-zero determinant.

A vector field tangent to the symplectic foliation can be written as $Z = f\frac{\partial}{\partial x} + g((ay+bz)\frac{\partial}{\partial y} + (cy+dz)\frac{\partial}{\partial z})$ where the function f vanishes for $(ay+bz, cy+dz) = (0, 0)$.

Since the mapping $(x, y, z) \mapsto (x, ay + bz, cy + dz)$ defines new smooth coordinates, we may write $f = (ay + bz)f_1 + (cy + dz)f_2$ for smooth functions f_1 and f_2.

Finally we obtain $Z = f_1X_y + f_2X_z - gX_x$ for smooth functions f_1, f_2 and g as desired.

2) The determinant of A is equal to zero. In the case $a = b = c = d = 0$, the Lie algebra considered is abelian and the Poisson structure is trivial so in this case there is nothing to prove.

In the nontrivial subcase we may write,

$$\Pi = \frac{\partial}{\partial x} \wedge (B\frac{\partial}{\partial y} + \lambda B\frac{\partial}{\partial z})$$

B being a linear function in y and z. After a linear change we may assume that $\Pi = \overline{B}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \overline{y}}$.

A vector field tangent to the symplectic foliation is of the form $Z = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial \overline{y}}$ where the functions f and g vanish when $\overline{B} = 0$. Since $\overline{B}$ is a non-trivial linear function in $\overline{y}$ and z, we may write $f = \overline{B}f_1$ and $g = \overline{B}g_1$. Therefore we may write $Z = f_1X_x + g_1X_{\overline{y}}$.

Consider now the case when $\mathfrak{g}$ is simple. We will use the following lemma, which is a smooth version of de Rham's division lemma, due to Moussu [12]:

Lemma 5.1. *Let α be a smooth (resp. analytic) 1-form on a neighbourhood of the origin in $\mathbb{R}^n$ for which the origin is an algebraically isolated singularity, then for any smooth (resp. analytic) p-form ω such that $\omega \wedge \alpha = 0$ we can factorize ω as $\omega = \beta \wedge \alpha$ for a smooth (resp. analytic) $(p-1)$-form β.*

Denote by Π the linear Poisson structure, it can be written as $\Pi = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ (in the case of $\mathfrak{so}(3,\mathbb{K})$) or as $\Pi = z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$ (in the case of $\mathfrak{sl}(2,\mathbb{K})$).

Let Ω be the volume form $\Omega = dx \wedge dy \wedge dz$, then the map $\Omega^b : \mathcal{V}^p(\mathfrak{g}^*) \longrightarrow \Omega^{3-p}(\mathfrak{g}^*)$ from the space of multivector fields to the space of forms defined by $\Omega^b(A) = i_A\Omega$ is an isomorphism.

Let X be a vector field tangent to the symplectic foliation. The condition of tangency to the symplectic foliation implies the relation $X \wedge \Pi = 0$. Under the above linear isomorphism this condition becomes $i_X\Omega \wedge i_\Pi\Omega = 0$. Since $i_\Pi\Omega$ has isolated singularities at the origin, we can now apply lemma 5.1 to write $i_X\Omega = \beta \wedge i_\Pi\Omega$ for a smooth one-form β.

Finally, we make convenient substitutions to obtain $X = i_X\Omega \lrcorner (\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}) = (\beta \wedge i_\Pi\Omega) \lrcorner (\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}) = \beta \lrcorner \Pi$. From this we conclude the proof of proposition 2.4 since from this equality if $\beta = fdx + gdy + hdz$ then $X = fX_x + gX_y + hX_z$ as desired.

□

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