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SHARP L^q BOUNDS ON SPECTRAL CLUSTERS FOR HOLDER METRICS

Herbert Koch, Hart F. Smith, and Daniel Tataru

Abstract. We establish L^q bounds on eigenfunctions, and more generally on spectrally localized functions (spectral clusters), associated to a self-adjoint elliptic operator on a compact manifold, under the assumption that the coefficients of the operator are of regularity C^s, where $0 \le s \le 1$. We also produce examples which show that these bounds are best possible for the case $q = \infty$, and for $2 \le q \le q_n$.

1. Introduction

Let M be a compact manifold without boundary, on which we fix a smooth volume form dx. Let a be a section of real, symmetric quadratic forms on $T^*(M)$, with associated linear transforms $a_x : T^*_x(M) \to T_x(M)$, and let ρ be a real valued function on M. We assume both a and ρ are strictly positive, with uniform bounds above and below.

Consider the eigenfunction problem, with d^* the adjoint of d relative to dx,

$$d^*(a\, df) + \lambda^2 \rho f = 0\,. \tag{1}$$

Under the condition that a and ρ are bounded, measurable, and uniformly bounded from below, there exists an orthonormal basis ϕ_j of eigenfunctions for $L^2(M, \rho\, dx)$ with frequencies $\lambda_j \to \infty$. In this paper we establish the following theorem.

Theorem 1. *Suppose that $a, \rho \in C^s(M)$, where $0 \le s \le 1$. Assume that the frequencies λ_j of f are contained in the interval $[\lambda, \lambda + \lambda^{1-s}]$, so that*

$$f = \sum_{j:\lambda_j \in [\lambda, \lambda+\lambda^{1-s}]} c_j\, \phi_j \tag{2}$$

Then for $2 \le q \le q_n = \frac{2(n+1)}{n-1}$

$$\|f\|_{L^q(M)} \le C\, \lambda^{\left(\frac{2(n-1)}{2+s}+1-s\right)\left(\frac{1}{2}-\frac{1}{q}\right)} \|f\|_{L^2(M)}\,. \tag{3}$$

Furthermore

$$\|f\|_{L^\infty(M)} \le C\, \lambda^{\frac{n-s}{2}}\, \|f\|_{L^2(M)}\,. \tag{4}$$

The constant C depends only on the C^s norm and lower bounds of a and ρ. In particular, C is uniform under small C^s perturbations of a and ρ.

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We also produce examples to show that, for general C^s metrics, (4) and (3) are best possible, in the latter case for all q in the given range. The examples are exponentially localized eigenfunctions on open sets, and show that the exponents in Theorem 1 cannot be improved in general even for functions f with frequency spread $O(\lambda^{-N})$. It is not known what the sharp bounds are for $q_n < q < \infty$.

We compare Theorem 1 to the bounds in the case a and ρ belong to C^2, or more generally $C^{1,1}$, and where the frequencies of f lie in the interval $[\lambda, \lambda+1]$. In that case, by [8],

$$\|f\|_{L^q(M)} \le C\,\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})}\,\|f\|_{L^2(M)}\,, \qquad 2 \le q \le q_n\,,$$

$$\|f\|_{L^q(M)} \le C\,\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}\,\|f\|_{L^2(M)}\,, \qquad q_n \le q \le \infty\,.$$

These estimates were established by Sogge [11] for smooth a and ρ, and are best possible at all q for unit width spectral clusters. Semiclassical generalizations were obtained by Koch-Tataru-Zworksi [5]. The case $q = \infty$ is related to the spectral counting remainder estimate of Avakumović-Levitan-Hörmander. Recently, Bronstein and Ivrii [2] have obtained spectral counting remainder estimates in the case of Hölder coefficients. A direct corollary of their estimates is an upper bound of the form $O((\log\lambda)^\sigma \lambda^{n-s})$ for the number of frequencies λ_j (counting multiplicity) in the interval $[\lambda, \lambda+\lambda^{1-s}]$. A corollary of (4) is that this upper bound holds with $\sigma = 0$. Indeed, (4) implies pointwise bounds on the kernel $\chi_\lambda(x,y)$ of the spectral projection onto frequencies in the range $[\lambda, \lambda+\lambda^{1-s}]$,

$$|\chi_\lambda(x,y)| \le C^2\,\lambda^{n-s}\,,$$

and integrating over the diagonal yields the desired trace bounds.

Estimates for C^s metrics are derived from the C^2 result together with a frequency dependent scaling argument. Our work shows that there are two distinct spatial scales that enter into the estimates when $s < 1$,

$$R = \lambda^{-\frac{2-s}{2+s}}\,, \qquad T = \lambda^{s-1}\,.$$

The scale R is the size of a ball on which, when working with solutions f with frequencies of magnitude λ, the coefficients a and ρ are well approximated by C^2 functions, in the sense that the errors can be absorbed as an appropriate source term. The larger scale T is the size of a ball on which the coefficients are well approximated by C^1 functions.

Compared to the C^2 case, there is in Theorem 1 a loss of $(TR^{-1})^{\frac{1}{q}}T^{-\frac{1}{2}}$ at the indices $q = q_n$ and $q = \infty$. (The sharp bounds for $2 \le q \le q_n$ are obtained by interpolation.) The loss of $T^{-\frac{1}{2}}$ arises in spatially localizing solutions to balls of size T in order to have energy flux bounds. The loss of $(TR^{-1})^{\frac{1}{q}}$ arises from the fact that we have good L^q bounds on sets of size R, and need to sum over a total of TR^{-1} sets to obtain bounds on sets of size T. For $1 \le s \le 2$ only the scale R enters, and the loss relative to the C^2 case is $R^{-\frac{1}{q}}$ for $q = q_n$ and $q = \infty$, as shown by the second author in [9].

One can establish better bounds on the L^q norm of f over balls of size R. In that case, there is only the loss of $T^{-\frac{1}{2}}$ relative to the C^2 case. Precisely, in the process of proving Theorem 1 we also establish the following.

Theorem 2. *Let $B_R \subset M$ be a ball of radius $R = \lambda^{-\frac{2-s}{2+s}}$. Then under the same conditions as Theorem 1, and with a constant C uniform over such balls B_R,*

$$\|f\|_{L^q(B_R)} \leq C\,\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}}\,\|f\|_{L^2(M)}\,, \qquad q_n \leq q \leq \infty\,. \tag{5}$$

The examples we produce to show that (4) is sharp also show that (5) cannot be improved for any $q_n \leq q \leq \infty$. It is expected, though, that the bound obtained by interpolating (3) at $q = q_n$ with (4) is not sharp for $q_n < q < \infty$. That is, the additional loss of $(TR^{-1})^{\frac{1}{q}}$ which would be obtained by adding (5) over TR^{-1} disjoint sets is sharp only for q_n and ∞.

Since the constant C in (5) is independent of the center of B_R, estimate (4) is an immediate consequence of the case $q = \infty$ of (5). We also remark that all cases of (5) for $q_n \leq q \leq \infty$ follow from the case $q = q_n$ of (5). This was noted in [9], using heat kernel estimates. Briefly, by Theorem 6.3 of Saloff-Coste [7], the heat kernel $h_\lambda(x,y)$ at time $t = \lambda^{-2}$ for the diffusion system associated to (1) satisfies

$$|h_\lambda(x,y)| \leq C\,\lambda^n \exp(-c\,\lambda^2 d(x,y)^2)\,.$$

By Young's inequality, then for $q_n \leq q \leq \infty$

$$\begin{aligned}\|f\|_{L^q(B_R)} &\leq C\,\lambda^{n(\frac{1}{q_n}-\frac{1}{q})}\|H_\lambda^{-1}f\|_{L^{q_n}(B_R^*)} + C_N\,\lambda^{-N}\|H_\lambda^{-1}f\|_{L^2(M\setminus B_R^*)}\\ &\leq C\,\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{s}{2}}\|f\|_{L^2(M)}\end{aligned}$$

where we use (5) at $q = q_n$ with B_R replaced by its double B_R^*, and the fact that $\|H_\lambda^{-1}f\|_{L^2} \approx \|f\|_{L^2}$ since $\exp(\lambda_j^2/\lambda^2) \approx 1$ for $\lambda_j \in [\lambda, \lambda+1]$.

Since (3) follows from the case $q = q_n$ (by interpolating with the trivial case $q = 2$) we are thus reduced to establishing the estimates (3) and (5) at $q = q_n$.

Notation. By the C^s norm on $\mathbb{R}^n$ for $0 < s \leq 1$ we mean

$$\|f\|_{C^s} = \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{|h|<1} |h|^{-s}\,\|f(\,\cdot + h) - f(\cdot)\|_{L^\infty(\mathbb{R}^n)}\,.$$

Thus, C^s coincides with Lipschitz for $s = 1$.

We use d to denote the differential taking functions to covector fields, and d^* its adjoint with respect to dx. When working on $\mathbb{R}^n$, $d = (\partial_1, \ldots, \partial_n)$, and d^* is the standard divergence operator.

The notation $A \lesssim B$ means $A \leq C\,B$, where C is a constant that depends only on the C^s norm of a and ρ, as well as on universally fixed quantities, such as the manifold M and the non-degeneracy of a and ρ. In particular, C will depend continuously on a and ρ in the C^s norm.

2. Proof of Theorems 1 and 2

For $s = 0$ both theorems are a result of Sobolev embedding, for example using heat kernel estimates, so we restrict to the case $s > 0$. Let ϕ_T denote a smooth cutoff to a ball $B_T \subset M$ of diameter $T = \lambda^{s-1}$. We then write

$$d^*(a\,d\,(\varphi_T f)) + \lambda^2\rho\,(\varphi_T f) = T^{-1}\big(\,d^*g_1 + g_2\big)$$

where $g_1\,, g_2$ are supported in B_T^* (the double of B_T), with

$$g_1 = a\,(Td\varphi_T)f \qquad g_2 = a(Td\varphi_T, df) + T\,\varphi_T\big(d^*(a\,df) + \lambda^2\rho\,f\big)$$

and hence

$$\|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2} \lesssim \|f\|_{L^2(B_T^*)} + \lambda^{-1}\|df\|_{L^2(B_T^*)} + \lambda^{s-2}\|d^*(a\,df) + \lambda^2\rho f\|_{L^2(B_T^*)}$$

Note that if f is of the form (2), and $\lambda \geq 1$, then

$$\|d^*(a\,df) + \lambda^2\rho f\|_{L^2(M)} \lesssim \lambda^{2-s}\|f\|_{L^2(M)}\,,$$
$$\|df\|_{L^2(M)} \lesssim \lambda\,\|f\|_{L^2(M)}\,.$$

Consequently, if we prove that for f satisfying

$$d^*(a\,df) + \lambda^2\rho = T^{-1}\big(\,d^*g_1 + g_2\big) \tag{6}$$

we have

$$\|f\|_{L^{q_n}(B_T)} \lesssim \lambda^{\frac{1}{q_n}}\,(TR^{-1})^{\frac{1}{q_n}}\,T^{-\frac{1}{2}}\big(\,\|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}\big) \tag{7}$$

as well as

$$\|f\|_{L^{q_n}(B_R)} \lesssim \lambda^{\frac{1}{q_n}}\,T^{-\frac{1}{2}}\,\big(\,\|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}\big) \tag{8}$$

then summing over a cover of M by balls B_T with bounded overlap yields (3) and (5) at $q = q_n$, and hence all cases of Theorems 1 and 2 as remarked above.

By choosing local coordinates we may assume we are working with an equation of the form (6) on $\mathbb{R}^n$ with Lebesgue measure, and with f supported in a ball of radius T. After making a linear change of coordinates and multiplying ρ by a harmless constant, we may additionally assume (for $s > 0$) that

$$\|a - I\|_{C^s(\mathbb{R}^n)} + \|\rho - 1\|_{C^s(\mathbb{R}^n)} \leq c_0\,, \tag{9}$$

for c_0 a suitably small constant to be fixed depending on the dimension n.

Let $S_r = S_r(D)$ denote a smooth cutoff on the Fourier transform side to frequencies of size $|\xi| \leq r$. Let $a_\lambda = S_{c^2\lambda}a$, for c to be chosen suitably small. Then

$$\|(a - a_\lambda)df\|_{L^2} \lesssim \lambda^{-1}T^{-1}\|df\|_{L^2}\,, \qquad \lambda^2\|(\rho - \rho_\lambda)f\|_{L^2} \lesssim \lambda\,T^{-1}\|f\|_{L^2}\,,$$

and thus we may replace a and ρ by a_λ and ρ_λ at the expense of absorbing the above two terms into g_1 and g_2, which does not change the size of the right hand side of (7) and (8).

Next, let $f_{<\lambda} = S_{c\lambda}f$. Then

$$\big\|[S_{c\lambda}, a_\lambda]\big\|_{L^2\to L^2} \lesssim \lambda^{-1}\|a_\lambda\|_{C^1} \lesssim \lambda^{-1}T^{-1}\,, \tag{10}$$

and similarly for $[S_{c\lambda}, \rho_\lambda]$, hence we can absorb the commutator terms into g_1 and g_2, and since all terms are localized to frequencies less than λ we can write

$$d^*(a_\lambda\, df_{<\lambda}) + \lambda^2\rho_\lambda\, f_{<\lambda} = T^{-1}g_{<\lambda}\,, \tag{11}$$

where

$$\|g_{<\lambda}\|_{L^2} \lesssim \lambda\,\|f\|_{L^2} + \|df\|_{L^2} + \lambda\,\|g_1\|_{L^2} + \|g_2\|_{L^2}\,.$$

Since $\|d^*(a_\lambda df_{<\lambda})\|_{L^2} \lesssim (c\lambda)^2\|f_{<\lambda}\|_{L^2}$, for c suitably small the L^2 norm of the left hand side of (11) is comparable to $\lambda^2\|f_{<\lambda}\|_{L^2}$, hence we have

$$\|f_{<\lambda}\|_{L^2} \lesssim \lambda^{-1}T^{-1}\big(\,\|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}\big)\,.$$

Since $\frac{1}{q_n} = n(\frac{1}{2} - \frac{1}{q_n}) - \frac{1}{2}$, Sobolev embedding implies

$$\|f_{<\lambda}\|_{L^{q_n}} \lesssim \lambda^{\frac{1}{q_n}-\frac{1}{2}}T^{-1}\big(\,\|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}\big)\,, \tag{12}$$

which implies (7) and (8) for this term since $R, T \geq 1$ and $\lambda^{-\frac{1}{2}} \leq T^{\frac{1}{2}}$.

If we let $f_{>\lambda} = f - S_{c^{-1}\lambda} f$, then similar arguments let us write

$$d^*(a_\lambda\, df_{>\lambda}) + \lambda^2 \rho_\lambda\, f_{>\lambda} = T^{-1} d^* g_{>\lambda} \tag{13}$$

where now $g_{>\lambda}$, like $f_{>\lambda}$, is frequency localized to frequencies larger than $c^{-1}\lambda$, and

$$\|g_{>\lambda}\|_{L^2} \lesssim \|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}\,.$$

Taking the inner product of both sides of (13) against $f_{>\lambda}$ yields

$$\|df_{>\lambda}\|_{L^2}^2 - 4\lambda^2 \|f_{>\lambda}\|_{L^2}^2 \lesssim T^{-1}\|g_{>\lambda}\|_{L^2}\|df_{>\lambda}\|_{L^2}\,,$$

and by the frequency localization of $f_{>\lambda}$ we obtain

$$\|f_{>\lambda}\|_{H^1} \lesssim T^{-1}\big(\,\|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}\big)\,.$$

Since $n(\frac{1}{2} - \frac{1}{q_n}) = \frac{1}{q_n} + \frac{1}{2} \leq 1$, Sobolev embedding yields (12) for the term $f_{>\lambda}$.

It remains to establish (7) and (8) for $f_\lambda = S_{c^{-1}\lambda} f - S_{c\lambda} f$. We decompose f_λ using a partition of unity in the Fourier transform variable ξ to cones of small angle. We may thus assume that f_λ is frequency localized to a small cone about the ξ_1 axis. Since the localization is by means of an order 0 multiplier at frequency λ, the commutator satisfies the same bounds (10), and we may write

$$d^*(a_\lambda\, df_\lambda) + \lambda^2 \rho_\lambda\, f_\lambda = T^{-1} g_\lambda \tag{14}$$

where

$$\|g_\lambda\|_{L^2} \lesssim \lambda\,\|f\|_{L^2} + \|df\|_{L^2} + \lambda\,\|g_1\|_{L^2} + \|g_2\|_{L^2}\,.$$

The functions a_λ and ρ_λ satisfy the condition (9). By Corollary 7 of [9], we thus have the localized estimate, uniformly over cubes Q_R of sidelength R,

$$\|f_\lambda\|_{L^{q_n}(Q_R)} \lesssim R^{-\frac{1}{2}} \lambda^{\frac{1}{q_n}} \big(\,\|f_\lambda\|_{L^2(Q_R^*)} + \lambda^{-1}\|df_\lambda\|_{L^2(Q_R^*)} + R\,T^{-1}\lambda^{-1}\|g_\lambda\|_{L^2(Q_R^*)}\big)\,.$$

Summing over disjoint cubes contained in a slab S_R of the form $\{x \in \mathbb{R}^n : |x_1 - c\,| \leq R\,\}$ yields

$$\|f_\lambda\|_{L^{q_n}(S_R)} \lesssim R^{-\frac{1}{2}} \lambda^{\frac{1}{q_n}} \big(\,\|f_\lambda\|_{L^2(S_R^*)} + \lambda^{-1}\|df_\lambda\|_{L^2(S_R^*)} + R\,T^{-1}\lambda^{-1}\|g_\lambda\|_{L^2(S_R^*)}\big)\,.$$

We will show that

$$\|f_\lambda\|_{L^2(S_R^*)} + \lambda^{-1}\|df_\lambda\|_{L^2(S_R^*)} \lesssim R^{\frac{1}{2}} T^{-\frac{1}{2}} \big(\,\|f_\lambda\|_{L^2} + \lambda^{-1}\|df_\lambda\|_{L^2} + \lambda^{-1}\|g_\lambda\|_{L^2}\big)\,. \tag{15}$$

Since $R^{\frac{1}{2}} T^{-1} \leq T^{-\frac{1}{2}}$, this yields

$$\|f\|_{L^{q_n}(S_R)} \lesssim \lambda^{\frac{1}{q_n}}\, T^{-\frac{1}{2}} \,\big(\,\|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}\big)\,.$$

The estimate (8) follows immediately; the estimate (7) follows after summing over the TR^{-1} disjoint slabs S_R that intersect B_T.

The bound (15) follows by energy methods. Let V denote the vector field

$$V = 2(\partial_1 f_\lambda)\, a_\lambda\, df_\lambda + \big(\lambda^2 \rho_\lambda\, f_\lambda^2 - \langle a_\lambda\, df_\lambda, df_\lambda\rangle\big)\overrightarrow{e_1}\,.$$

Then

$$d^*V = 2T^{-1}(\partial_1 f_\lambda)\, g_\lambda + \lambda^2(\partial_1 \rho_\lambda) f_\lambda^2 - \langle(\partial_1 a_\lambda) df_\lambda, df_\lambda\rangle\,.$$

Since $|\partial_1 a_\lambda| + |\partial_1 \rho_\lambda| \lesssim T^{-1}$, applying the divergence theorem on the set $x_1 \leq r$ yields

$$\int_{x_1 = r} V_1\, dx' \lesssim T^{-1}\big(\,\lambda^2 \|f_\lambda\|_{L^2(\mathbb{R}^n)}^2 + \|df_\lambda\|_{L^2(\mathbb{R}^n)}^2 + \|g_\lambda\|_{L^2(\mathbb{R}^n)}^2\big)\,.$$

Since a_λ and ρ are pointwise close to the flat metric, we have pointwise that

$$V_1 \geq \tfrac{3}{4}|\partial_1 f_\lambda|^2 + \tfrac{3}{4}\lambda^2|f_\lambda|^2 - |\partial_{x'} f_\lambda|^2 \,.$$

The frequency localization of $\widehat{f_\lambda}$ to $|\xi'| \leq c\lambda$ yields

$$2\int_{x_1=r} V_1 \, dx' \geq \int_{x_1=r} |df_\lambda|^2 + \lambda^2|f_\lambda|^2 \, dx' \,.$$

Integrating this over r in an interval of size R yields (15). □

3. Examples to show (4) and (5) are sharp

In this section we produce examples of Hölder metrics and associated eigenfunctions which show that the estimates (4) of Theorem 1 and (5) of Theorem 2 are best possible. Our example is a radial version of the 1-dimensional example of Castro-Zuazua [3], which in turn is based on a calculation of Colombini-Spagnolo [4]. In our example the metric depends on the frequency λ (with uniform bounds on its Hölder norm), but because of the exponential localization of the eigenfunctions one may easily cut and paste together a sequence of examples to produce a metric for which these estimates fail for a sequence of λ tending to ∞, as in [3]. Our example is also global on $\mathbb{R}^n$, but again by its exponential localization it may be truncated and placed on a compact manifold, with the truncation errors small enough to show that the estimates are still best possible for spectral clusters, even those of spectral width $O(\lambda^{-N})$.

We start by producing smooth radial functions Φ, q_1, q_2, all of which vanish near 0, with $\Phi(r) = r + O(1)$, such that for all $\varepsilon > 0$:

$$\big(\Delta + 1 + \varepsilon q_1 + \varepsilon^2 q_2\big) e^{-\varepsilon\Phi}\, \widehat{d\sigma} = 0 \,,$$

where $d\sigma$ is surface measure on the unit sphere S^{n-1}. Furthermore, Φ', q_1, q_2 are globally bounded, together with their derivatives of all order.

For this, we write

$$\widehat{d\sigma}(r) = r^{\frac{1-n}{2}} F_n(r) \,.$$

Then (see [12] p. 338–348)

$$F_n(r) = a_n \cos(r - \tfrac{n-1}{4}\,\pi) + b_n\, r^{-1} \sin(r - \tfrac{n-1}{4}\,\pi) + O(r^{-2}) \,.$$

We normalize $d\sigma$ so $a_n = 1$, and set

$$\Phi(r) = 2\int_0^r h(t)\, F_n(t)^2 \, dt \,,$$

where h is a smooth non-negative function which vanishes near 0 and equals 1 for $s > \frac{1}{2}$. By the asymptotics of F_n, we have $\Phi(r) = r + O(1)$.

Consider the radial Laplacian $\Delta = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}$. Then

$$r^{\frac{n-1}{2}}\, \Delta\, r^{\frac{1-n}{2}} = \frac{d^2}{dr^2} - \frac{(n-1)(n-3)}{4r^2}$$

and hence we may expand $\Delta(e^{-\varepsilon\Phi}\widehat{d\sigma})$ as

$$r^{\frac{1-n}{2}} \Big[F_n'' - \frac{(n-1)(n-3)}{4r^2}\, F_n - 4\,\varepsilon F_n' h F_n^2 - 2\,\varepsilon F_n \big(hF_n^2\big)' + 4\,\varepsilon^2 F_n\, h^2 F_n^4 \Big] e^{-\varepsilon\Phi}$$

where $'$ denotes $\frac{d}{dr}$. This in turn may be written as

$$-\big(1+\varepsilon\, q_1+\varepsilon^2\, q_2\big)\, e^{-\varepsilon\Phi}\widehat{d\sigma}$$

with

$$q_1 = 4\, hF_nF_n' + 2\big(hF_n^2\big)', \qquad q_2 = -4\, h^2 F_n^4\,.$$

These functions are smooth since h vanishes near 0, and the global boundedness of their derivatives follows easily from boundedness of $F_n^{(k)}$ for r bounded away from 0.

To construct the example, we set $\varepsilon = \lambda^{-s}$ and change variables $r \to \lambda r$, and let

$$\psi_\lambda(r) = e^{-\lambda^{-s}\Phi(\lambda r)}\widehat{d\sigma}(\lambda r)\,, \qquad \rho_\lambda(r) = 1+\lambda^{-s}q_1(\lambda r)+\lambda^{-2s}q_2(\lambda r)\,,$$

so that

$$\Delta\psi_\lambda + \lambda^2\rho_\lambda\psi_\lambda = 0\,.$$

Note that $\rho_\lambda \in C^s$, since

$$\|1-\rho_\lambda\|_{L^\infty} \le \lambda^{-s}\,, \qquad \|1-\rho_\lambda\|_{\mathrm{Lip}} \le \lambda^{1-s}\,,$$

and $\min(\lambda^{-s}, \lambda^{1-s}|x-y|) \le |x-y|^s$. On the other hand,

$$|\psi_\lambda(r)| \approx e^{-\lambda^{1-s}r}|\widehat{d\sigma}|(\lambda r)\,.$$

Precisely,

$$\|\psi_\lambda\|_{L^2}^2 = \lambda^{1-n}\int_0^\infty |F_n(\lambda r)|^2\, e^{-2\lambda^{-s}\Phi(\lambda r)}\, dr \approx \lambda^{s-n}\int_0^\infty |F_n(\lambda^s r)|^2 e^{-2r}\, dr \approx \lambda^{s-n}\,.$$

The maximum of $|\widehat{d\sigma}|$ occurs at $r=0$, so $\|\psi_\lambda\|_{L^\infty}$ is independent of λ. Thus

$$\frac{\|\psi_\lambda\|_{L^\infty}}{\|\psi_\lambda\|_{L^2}} \approx \lambda^{\frac{n-s}{2}}$$

showing that (4) is sharp.

We also note that $|\psi_\lambda| \approx 1$ for $r \le \lambda^{-1}$, hence $\|\psi_\lambda\|_{L^p(B_{1/\lambda})} \gtrsim \lambda^{-\frac{n}{q}}$. Thus

$$\frac{\|\psi_\lambda\|_{L^q(B_{1/\lambda})}}{\|\psi_\lambda\|_{L^2}} \gtrsim \lambda^{n(\frac12-\frac1q)-\frac{s}{2}}$$

showing that (5) is sharp. □

4. Examples to show (3) is sharp

In this section we produce examples to show that (3) is similarly sharp, in the range $2 \le q \le q_n$. These examples are exponentially localized to a tube of diameter $\lambda^{\frac{2}{2+s}}$ and length λ^{s-1}. They are essentially a product of the examples of Smith-Sogge [10], where the metric depends on $n-1$ variables, with the 1-dimensional example of Castro-Zuazua [3].

For the examples, fix $0<s<1$. Let (x,y) denote variables on $\mathbb{R}^n$ with $x\in\mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. We take $A(y)$ to be the ground state solution of

$$-\Delta_y A(y) + |y|^s A(y) = cA(y)\,.$$

Here $c>0$ and $A(y)$ is radial and of class $C^{2,s}$, with

$$|A(y)| \lesssim e^{-N|y|}$$

for all $N > 0$. This may be seen, for example, by Theorem XIII.47 of [6] and Theorems 4.1 and 5.1 of [1]. The regularity follows by examining the ordinary differential equation in r near 0.

Given $\kappa > 0$, let $\lambda = \lambda(\kappa)$ solve

$$\lambda^2 - \kappa^2 = c\,\lambda^{2\delta}\,, \qquad \delta = \frac{2}{2+s}\,.$$

For $s > 0$ there is a unique positive solution for large κ, which satisfies $\lambda \approx \kappa$.

We then have

$$\big(\Delta_y + \lambda^2(1-|y|^s)\,\big)A(\lambda^\delta y) = \kappa^2 A(\lambda^\delta y)\,.$$

Next consider Φ, q_1, q_2, as in the preceeding section, with $n = 1$ and $F_1 = \cos(x)$. We set

$$\rho_\kappa(x) = 1 + \kappa^{-s} q_1(\kappa\,|x|\,) + \kappa^{-2s} q_2(\kappa\,|x|\,)\,.$$

Then

$$\big(\partial_x^2 + \kappa^2 \rho_\kappa(x)\,\big)e^{-\kappa^{-s}\Phi(\kappa\,|x|\,)}\cos(\kappa x) = 0\,,$$

and consequently

$$\Big(d_x^2 + \rho_\kappa(x)\Delta_y + \lambda^2(1-|y|^s)\rho_\kappa(x)\Big)e^{-\kappa^{-s}\Phi(\kappa\,|x|)}A(\lambda^\delta y)\,\cos(\kappa x) = 0\,.$$

This equation takes the form (1). Since $\kappa \approx \lambda$, the eigenfunction is exponentially concentrated in the set

$$|y| \le \lambda^{-\frac{2}{2+s}}\,, \qquad |x| \le \lambda^{s-1}\,.$$

By Hölder's inequality, this implies that

$$\frac{\|\psi_\lambda\|_{L^p}}{\|\psi_\lambda\|_{L^2}} \gtrsim \lambda^{\left(\frac{2}{2+s}(n-1)+(1-s)\right)\left(\frac{1}{2}-\frac{1}{p}\right)}$$

showing that (3) is sharp. □

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References

[1] S. Agmon, *Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations*, Mathematical notes **29** Princeton University Press, Princeton, N.J. (1982)

[2] M. Bronstein and V. Ivrii, *Sharp spectral asymptotics for operators with irregular coefficients. I. Pushing the limits*, Comm. Partial Differential Equations **28** (2003), 83–102.

[3] C. Castro and E. Zuazua, *Concentration and lack of observability of waves in highly heterogeneous media*, Arch. Ration. Mech. Anal. **164** (2002), no. 1, 39–72.

[4] F. Colombini and S. Spagnolo, *Some examples of hyperbolic equations without local solvability*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 109–125.

[5] H. Koch, D. Tataru and M. Zworski, *Semiclassical L^p estimates*, to appear in Annales Henri Poincaré.

[6] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of Operators*, Academic Press, New York-London (1978)

[7] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices (1992), No. 2, 27–38.

[8] H. Smith, *Spectral cluster estimates for $C^{1,1}$ metrics*, Amer. Jour. Math. **128** (2006), 1069–1103.

[9] ———, *Sharp $L^2 \to L^q$ bounds on spectral projectors for low regularity metrics*, Math. Res. Lett. **13** (2006), no. 6, 965–972.

[10] H. Smith and C. Sogge, *On Strichartz and eigenfunction estimates for low regularity metrics*, Math. Res. Lett. **1** (1994), no. 6, 729–737.

[11] C. Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), no. 1, 123–134.

[12] E. M. Stein, *Harmonic analysis: real variable methods, orthogonality, and oscillatory integrals,* Princeton University Press, Princeton, NJ (1993)

Mathematisches Institut, Universität Bonn, 53113 Bonn, Germany
E-mail address: koch@math.uni-bonn.de

Department of Mathematics, University of Washington, Seattle, WA 98195
E-mail address: hart@math.washington.edu

Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: tataru@math.berkeley.edu