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A REMARK ON 'SOME NUMERICAL RESULTS IN COMPLEX DIFFERENTIAL GEOMETRY'

Kefeng Liu and Xiaonan Ma

Abstract. In this note we verify certain statement about the operator Q_K constructed by Donaldson in [3] by using the full asymptotic expansion of Bergman kernel obtained in [2] and [4].

In order to find explicit numerical approximation of Kähler-Einstein metric of projective manifolds, Donaldson introduced in [3] various operators with good properties to approximate classical operators. See the discussions in Section 4.2 of [3] for more details related to our discussion. In this note we verify certain statement of Donaldson about the operator Q_K in Section 4.2 by using the full asymptotic expansion of Bergman kernel derived in [2, Theorem 4.18] and [4, §3.4]. Such statement is needed for the convergence of the approximation procedure.

As a warm up, we explain first the classical Bergman kernel on $\mathbb{C}^n$ [4, Remark 1.14] which will serve as a model for our problem.

Let $F = \mathbb{C}$ be the trivial holomorphic line bundle on $\mathbb{C}^n$ with the canonical section $\mathbf{1}$. Let h^F be the metric on F defined by $|\mathbf{1}|_{h^F}(z) := e^{-\frac{\pi}{2}|Z|^2}$ for $z \in \mathbb{C}^n$ with $|Z|^2 = \sum_{j=1}^n |z_j|^2$. Let $g^{T\mathbb{C}^n}$ be the Euclidean metric on $\mathbb{C}^n$. Let P be the orthogonal projection from $(L^2(\mathbb{C}^n, F), \| \quad \|_{L^2})$ onto the space of L^2-holomorphic sections of F, and let $P(z, z')$ $(z, z' \in \mathbb{C}^n)$ be the smooth kernel of P with respect to the Euclidean volume form dZ. We trivialize F by using the unit section $e^{\frac{\pi}{2}|Z|^2}\mathbf{1}$. Then an orthonormal basis of L^2-holomorphic sections of F under this trivialization is

$$\left(\frac{(2\pi)^\beta}{2^{|\beta|}\beta!}\right)^{1/2} z^\beta \exp\Big(-\frac{\pi}{2}|Z|^2\Big), \quad \beta \in \mathbb{N}^n, \tag{1}$$

and the classical Bergman kernel $P(z, z')$ (cf. [2, (4.114)], [4, (1.91)]), is

$$P(z, z') = \exp\Big(-\frac{\pi}{2}\sum_{i=1}^n \big(|z_i|^2 + |z_i'|^2 - 2z_i\overline{z}_i'\big)\Big). \tag{2}$$

Recall that the classical heat kernel on $\mathbb{C}^n$ is $e^{-u\Delta}(z, z') = (4\pi u)^{-n} e^{-\frac{1}{4u}|Z-Z'|^2}$. Thus from (2), we get

$$|P(z, z')|^2 = e^{-\pi|Z-Z'|^2} = e^{-\frac{\Delta}{4\pi}}(z, z'). \tag{3}$$

In this note, we will establish an asymptotic version of (3) in the general case.

Let (X, ω, J) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$, and let (L, h^L) be a holomorphic Hermitian line bundle on X. Let ∇^L be the holomorphic Hermitian

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connection on (L, h^L) with curvature R^L. We assume that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{4}$$

Let $g^{TX}(\cdot,\cdot) := \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω, J. Let dv_X be the Riemannian volume form of (TX, g^{TX}), then $dv_X = \omega^n/n!$. Let $d\nu$ be any volume form on X. Let η be the positive function on X defined by

$$dv_X = \eta\, d\nu. \tag{5}$$

The L^2–scalar product $\langle\ \ \rangle_\nu$ on $\mathscr{C}^\infty(X, L^p)$, the space of smooth sections of L^p, is given by

$$\langle \sigma_1, \sigma_2\rangle_\nu := \int_X \langle \sigma_1(x), \sigma_2(x)\rangle_{h^{L^p}}\, d\nu(x)\,. \tag{6}$$

Let $P_{\nu,p}(x,x')$ $(x, x' \in X)$ be the smooth kernel of the orthogonal projection from $(\mathscr{C}^\infty(X, L^p), \langle\ \ \rangle_\nu)$ onto $H^0(X, L^p)$, the space of the holomorphic sections of L^p on X, with respect to $d\nu(x')$. Note that $P_{\nu,p}(x,x') \in L^p_x \otimes L^{p*}_{x'}$. Following [3, §4], set

$$K_p(x,x') := |P_{\nu,p}(x,x')|^2_{h^{L^p}_x \otimes h^{L^{p*}}_{x'}}, \quad R_p := (\dim H^0(X, L^p))/\mathrm{Vol}(X,\nu), \tag{7}$$

here $\mathrm{Vol}(X,\nu) := \int_X d\nu$. Set $\mathrm{Vol}(X, dv_X) := \int_X dv_X$.

Let Q_{K_p} be the integral operator associated to K_p which is defined for $f \in \mathscr{C}^\infty(X)$,

$$Q_{K_p}(f)(x) := \frac{1}{R_p}\int_X K_p(x,y) f(y) d\nu(y). \tag{8}$$

Let Δ be the (positive) Laplace operator on (X, g^{TX}) acting on the functions on X. We denote by $|\ \ |_{L^2}$ the L^2-norm on the function on X with respect to dv_X.

Theorem 0.1. *There exists a constant $C > 0$ such that for any $f \in \mathscr{C}^\infty(X)$, $p \in \mathbb{N}$,*

$$\begin{aligned} &\left|\Big(Q_{K_p} - \frac{\mathrm{Vol}(X,\nu)}{\mathrm{Vol}(X, dv_X)}\eta \exp\Big(-\frac{\Delta}{4\pi p}\Big)\Big) f\right|_{L^2} \leq \frac{C}{p}\, |f|_{L^2}\,, \\ &\left|\Big(\frac{\Delta}{p} Q_{K_p} - \frac{\mathrm{Vol}(X,\nu)}{\mathrm{Vol}(X, dv_X)}\frac{\Delta}{p}\eta \exp\Big(-\frac{\Delta}{4\pi p}\Big)\Big) f\right|_{L^2} \leq \frac{C}{p}\, |f|_{L^2}\,. \end{aligned} \tag{9}$$

Moreover, (9) *is uniform in that there is an integer s such that if all data h^L, $d\nu$ run over a set which is bounded in $\mathscr{C}^s$-topology and that g^{TX}, dv_X are bounded from below, then the constant C is independent of h^L, $d\nu$.*

Proof. We explain at first the full asymptotic expansion of $P_{\nu,p}(x,x')$ from [2, Theorem 4.18′] and [4, §3.4]. For more details on our approach we also refer the readers to the recent book [5].

Let $E = \mathbb{C}$ be the trivial holomorphic line bundle on X. Let h^E the metric on E defined by $|\mathbf{e}|^2_{h^E} = 1$, here $\mathbf{e}$ is the canonical unity element of E. We identify canonically L^p to $L^p \otimes E$ by section $\mathbf{e}$.

As in [4, §3.4], let h^E_ω be the metric on E defined by $|\mathbf{e}|^2_{h^E_\omega} = \eta^{-1}$. Let $\langle\ \ \rangle_\omega$ be the Hermitian product on $\mathscr{C}^\infty(X, L^p \otimes E) = \mathscr{C}^\infty(X, L^p)$ induced by h^L, h^E_ω, dv_X as in (6). Then by (5),

$$(\mathscr{C}^\infty(X, L^p \otimes E), \langle\ \ \rangle_\omega) = (\mathscr{C}^\infty(X, L^p), \langle\ \ \rangle_\nu). \tag{10}$$

Observe that $H^0(X, L^p \otimes E)$ does not depend on g^{TX}, h^L or h^E. If $P_{\omega,p}(x,x')$, $(x,x' \in X)$ denotes the smooth kernel of the orthogonal projection $P_{\omega,p}$ from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle \cdot,\cdot \rangle_\omega)$ onto $H^0(X, L^p \otimes E) = H^0(X, L^p)$ with respect to $dv_X(x)$, from (5), as in [4, (3.38)], we have

$$P_{\nu,p}(x,x') = \eta(x')\, P_{\omega,p}(x,x'). \tag{11}$$

For $f \in \mathscr{C}^\infty(X)$, set

$$\begin{aligned} K_{\omega,p}(x,x') &= |P_{\omega,p}(x,x')|^2_{(h^{L^p}\otimes h^E_\omega)_x \otimes (h^{L^{p*}} \otimes h^{E^*}_\omega)_{x'}}, \\ (K_{\omega,p} f)(x) &= \int_X K_{\omega,p}(x,y) f(y) dv_X(y). \end{aligned} \tag{12}$$

By the definition of the metric h^E, h^E_ω, if we denote by $\mathbf{e}^*$ the dual of the section $\mathbf{e}$ of E, we know

$$1 = |\mathbf{e}\otimes \mathbf{e}^*|^2_{h^E \otimes h^{E^*}}(x,x') = |\mathbf{e}\otimes \mathbf{e}^*|^2_{h^E_\omega \otimes h^{E^*}_\omega}(x,x')\eta(x)\eta^{-1}(x'). \tag{13}$$

Recall that we identified (L^p, h^{L^p}) to $(L^p \otimes E, h^{L^p} \otimes h^E)$ by section $\mathbf{e}$. Thus from (7), (11) and (13), we get

$$K_p(x,x') = |P_{\nu,p}(x,x')|^2_{(h^{L^p}\otimes h^E)_x \otimes (h^{L^{p*}} \otimes h^{E^*})_{x'}} = \eta(x)\,\eta(x')\,K_{\omega,p}(x,x'), \tag{14}$$

and from (5), (8) and (14),

$$Q_{K_p}(f)(x) = \frac{1}{R_p}\int_X K_{\omega,p}(x,y)\eta(x) f(y) dv_X(y). \tag{15}$$

Now for the kernel $P_{\omega,p}(x,x')$, we can apply the full asymptotic expansion [2, Theorem 4.18′]. In fact let $\overline{\partial}^{L^p\otimes E, *_\omega}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^p\otimes E)$ with the scalar product induced by g^{TX}, h^L, h^E_ω, dv_X as in (6), and set

$$D_p = \sqrt{2}(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E, *_\omega}). \tag{16}$$

Then $H^0(X, L^p\otimes E) = \operatorname{Ker}(D_p)$ for p large enough, and D_p is a Dirac operator, as $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX.

Let ∇^E be the holomorphic Hermitian connection on (E, h^E_ω). Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}). Let R^E, R^{TX} be the corresponding curvatures.

Let $d(x,y)$ be the Riemannian distance from x to y on (X, g^{TX}). Let a^X be the injectivity radius of (X, g^{TX}). We fix $\varepsilon \in]0, a^X/4[$. We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε. We identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ by using the exponential map of (X, g^{TX}).

We fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}(0,\varepsilon)$ we identify (L_Z, h^L_Z), (E_Z, h^E_Z) and $(L^p\otimes E)_Z$ to $(L_{x_0}, h^L_{x_0})$, $(E_{x_0}, h^E_{x_0})$ and $(L^p\otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p\otimes E}$ along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$. Then under our identification, $P_{\omega,p}(Z,Z')$ is a function on $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \le \varepsilon$, we denote it by $P_{\omega,p,x_0}(Z,Z')$.

Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Then we can view $P_{\omega,p,x_0}(Z,Z')$ as a smooth function on $TX \times_X TX$ with complex values (which is defined for $|Z|, |Z'| \le \varepsilon$) by identifying a section $S \in$

$\mathscr{C}^\infty(TX \times_X TX, \pi^*\mathrm{End}(E))$ with the family $(S_x)_{x\in X}$, where $S_x = S|_{\pi^{-1}(x)}$, since $\mathrm{End}(E) = \mathbb{C}$.

We choose $\{w_i\}_{i=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)}X$, then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)\,, j = 1, \ldots, n$ forms an orthonormal basis of $T_{x_0}X$. We use the coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ where the identification is given by

$$(Z_1, \cdots, Z_{2n}) \in \mathbb{R}^{2n} \longrightarrow \sum_{i=1}^{2n} Z_i e_i \in T_{x_0}X. \tag{17}$$

In what follows we also introduce the complex coordinates $z = (z_1, \cdots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

By [2, Proposition 4.1], for any $l, m \in \mathbb{N}$, $\varepsilon > 0$, there exists $C_{l,m,\varepsilon} > 0$ such that for $p \geq 1$, $x, x' \in X$,

$$|P_{\omega,p}(x, x')|_{\mathscr{C}^m(X\times X)} \leq C_{l,m,\varepsilon}\, p^{-l} \quad \text{if } d(x, x') \geq \varepsilon. \tag{18}$$

Here the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E, ∇^{TX} and h^L, h^E, g^{TX}.

By [2, Theorem 4.18′], there exist $J_r(Z, Z')$ polynomials in Z, Z', such that for any $k, m, m' \in \mathbb{N}$, there exist $N \in \mathbb{N}, C > 0, C_0 > 0$ such that for $\alpha, \alpha' \in \mathbb{N}^n$, $|\alpha|+|\alpha'| \leq m$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leq \varepsilon$, $x_0 \in X$, $p > 1$,

$$\begin{aligned} &\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_{\omega,p,x_0}(Z, Z') - \sum_{r=0}^{k} (J_r P)(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \right) \right|_{\mathscr{C}^{m'}(X)} \\ &\qquad \leq C p^{-(k+1-m)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p}|Z - Z'|) + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{19}$$

Here $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$ norm for the parameter $x_0 \in X$. The term $\mathscr{O}(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1} p^{-l}$.

Now we claim that in (19),

$$J_0 = 1, \quad J_1(Z, Z') = 0. \tag{20}$$

In fact, let $dv_{T_{x_0}X}$ be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, and κ_{x_0} be the function defined by

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{T_{x_0}X}(Z). \tag{21}$$

Then (also cf. [4, (1.31)])

$$\kappa_{x_0}(Z) = 1 + \frac{1}{6} \left\langle R^{TX}_{x_0}(Z, e_i)Z, e_i \right\rangle_{x_0} + \mathscr{O}(|Z|^3). \tag{22}$$

As we only work on $\mathscr{C}^\infty(X, L^p \otimes E)$, by [2, (4.115)], we get the first equation in (20).

Recall that in the normal coordinate, after the rescaling $Z \to Z/t$ with $t = \frac{1}{\sqrt{p}}$, we get an operator $\mathscr{L}_t$ from the restriction of D_p^2 on $\mathscr{C}^\infty(X, L^p \otimes E)$ which has the following formal expansion (cf. [2, (4.104)], [4, Theorem 1.4]),

$$\mathscr{L}_t = \mathscr{L} + \sum_{r=1}^{\infty} \mathscr{Q}_r t^r. \tag{23}$$

Now, from [2, Theorem 5.1] (or [4, (1.87), (1.98)]),

$$\mathscr{L}=\sum_{j=1}^{n}(-2\tfrac{\partial}{\partial z_i}+\pi\overline{z}_i)(2\tfrac{\partial}{\partial \overline{z}_i}+\pi z_i),\quad \mathscr{Q}_1=0. \tag{24}$$

(In fact, $P(Z,Z')$ is the smooth kernel of the orthogonal projection from $L^2(\mathbb{C}^n)$ onto $\mathrm{Ker}(\mathscr{L})$). Thus from [2, (4.107)] (cf. [4, (1.111)]), (22) and (24) we get the second equation of (20).

Note that $|P_{\omega,p,x_0}(Z,Z')|^2=P_{\omega,p,x_0}(Z,Z')\overline{P_{\omega,p,x_0}(Z,Z')}$, thus from (12), (19) and (20), there exist $J_r'(Z,Z')$ polynomials in Z,Z' such that

(25)

$$\begin{aligned}&\left|\frac{1}{p^{2n+1}}\Delta_Z\Big(K_{\omega,p,x_0}(Z,Z')-\Big(1+\sum_{r=2}^{k}p^{-r/2}J_r'(\sqrt{p}Z,\sqrt{p}Z')\Big)e^{-\pi p|Z-Z'|^2}\Big)\right|_{\mathscr{C}^0(X)}\\ &\qquad\leq Cp^{-(k+1)/2}(1+|\sqrt{p}Z|+|\sqrt{p}Z'|)^N\exp(-C_0\sqrt{p}|Z-Z'|)+\mathscr{O}(p^{-\infty}).\end{aligned}$$

For a function $f\in\mathscr{C}^\infty(X)$, we denote it as $f_{x_0}(Z)$ a family (with parameter x_0) of function of Z in the normal coordinate near x_0. Now, for any polynomial $A_{x_0}(Z')$, we define the operator

$$(\mathcal{A}_pf)(x_0)=p^n\int_{|Z'|\leq\varepsilon}A_{x_0}(\sqrt{p}Z')e^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z'). \tag{26}$$

Then we observe that there exists $C_1>0$ such that for any $p\in\mathbb{N},f\in\mathscr{C}^\infty(X)$, we have

$$|\mathcal{A}_pf|_{L^2}\leq C_1|f|_{L^2}. \tag{27}$$

In fact, there exist $C',C_1>0$ independent on p such that

$$\begin{aligned}|\mathcal{A}_pf|^2_{L^2}&\leq\int_X dv_X(x_0)\Big\{p^n\Big(\int_{|Z'|\leq\varepsilon}|A_{x_0}(\sqrt{p}Z')|e^{-\pi p|Z'|^2}dv_X(Z')\Big)\\ &\qquad\times p^n\Big(\int_{|Z'|\leq\varepsilon}|A_{x_0}(\sqrt{p}Z')|e^{-\pi p|Z'|^2}|f_{x_0}(Z')|^2dv_X(Z')\Big)\Big\}\\ &\leq C'\int_X dv_X(x_0)p^n\int_{|Z'|\leq\varepsilon}|A_{x_0}(\sqrt{p}Z')|e^{-\pi p|Z'|^2}|f_{x_0}(Z')|^2dv_X(Z')\\ &\leq C_1|f|^2_{L^2}.\end{aligned} \tag{28}$$

Observe that in the normal coordinate, at $Z=0$, $\Delta_Z=-\sum_{j=1}^{2n}\frac{\partial^2}{\partial Z_j^2}$. Thus

$$(\Delta_Z e^{-\pi p|Z-Z'|^2})|_{Z=0}=4\pi p(n-\pi p|Z'|^2)e^{-\pi p|Z'|^2}. \tag{29}$$

Thus from (3), (18), (19), (20), (25) and (27), we get

(30)

$$\begin{aligned}&\left|p^{-n}K_{\omega,p}f-p^n\int_{|Z'|\leq\varepsilon}e^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')\right|_{L^2}\leq\frac{C}{p}\,|f|_{L^2},\\ &\left|p^{-n-1}\Delta K_{\omega,p}f-4\pi p^n\int_{|Z'|\leq\varepsilon}(n-\pi p|Z'|^2)e^{-\pi p|Z'|^2}f_{x_0}(Z')dv_X(Z')\right|_{L^2}\leq\frac{C}{p}\,|f|_{L^2}.\end{aligned}$$

Recall that $\eta \in \mathscr{C}^\infty(X)$ was defined in (5). Set

$$(31)\qquad \begin{aligned} K_{\eta,\omega,p}(x,y) &= \langle d\eta(x), d_x K_{\omega,p}(x,y)\rangle_{g^{T^*X}},\\ (K_{\eta,\omega,p}f)(x) &= \int_X K_{\eta,\omega,p}(x,y) f(y) dv_X(y). \end{aligned}$$

Then from (19), (20) and (27), we get

(32)
$$\left| p^{-n-1} K_{\eta,\omega,p} f - 2\pi p^n \int_{|Z'|\leq\varepsilon} \sum_{i=1}^{2n} (\tfrac{\partial}{\partial Z_i}\eta)(x_0,0) Z'_i e^{-\pi p|Z'|^2} f_{x_0}(Z') dv_X(Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2},$$

here C is taken large enough so that both (30) and (32) hold and is independent on p.

Let $e^{-u\Delta}(x,x')$ be the smooth kernel of the heat operator $e^{-u\Delta}$ with respect to $dv_X(x')$. By the heat kernel expansion in [1, Theorems 2.23, 2.26], there exist $\Phi_i(x,y)$ smooth functions on $X\times X$ such that when $u\to 0$, we have the following asymptotic expansion

(33)
$$\left| \frac{\partial^l}{\partial u^l}\Big(e^{-u\Delta}(x,y) - (4\pi u)^{-n} \sum_{i=0}^{k} u^i \Phi_i(x,y) e^{-\frac{1}{4u} d(x,y)^2}\Big)\right|_{\mathscr{C}^m(X\times X)} = \mathscr{O}(u^{k-n-l-\frac{m}{2}+1}),$$

and

$$(34)\qquad \Phi_0(x,y) = 1.$$

If we still use the normal coordinate, then by (33), there exist $\phi_{i,x_0}(Z') := \Phi_i(0,Z')$ such that uniformly for $x_0\in X$, $Z'\in T_{x_0}X$, $|Z'|\leq\varepsilon$, we have the following asymptotic expansion when $u\to 0$,

(35)
$$\left| \frac{\partial^l}{\partial u^l}\Big(e^{-u\Delta}(0,Z') - (4\pi u)^{-n}\Big(1+\sum_{i=1}^{k} u^i \phi_{i,x_0}(Z')\Big) e^{-\frac{1}{4u}|Z'|^2}\Big)\right|_{\mathscr{C}^0(X)} = \mathscr{O}(u^{k-n-l+1}),$$

and

$$(36)\qquad \begin{aligned} &\Big| \langle d\eta(x_0), d_{x_0} e^{-u\Delta}\rangle_{g^{T^*X}}(0,Z')\\ &\quad - (4\pi u)^{-n} \sum_{i=1}^{2n} (\frac{\partial}{\partial Z_i}\eta)(x_0,0) \frac{Z'_i}{2u}\Big(1+\sum_{i=1}^{k} u^i\phi_{i,x_0}(Z')\Big)\Big) e^{-\frac{1}{4u}|Z'|^2}\\ &\quad - (4\pi u)^{-n}\sum_{i=1}^{k} u^i \langle d\eta(x_0), (d_{x_0}\Phi_i)(0,Z')\rangle e^{-\frac{1}{4u}|Z'|^2}\Big|_{\mathscr{C}^0(X)} = \mathscr{O}(u^{k-n+\frac{1}{2}}). \end{aligned}$$

Observe that

$$(37)\qquad \frac{1}{p}\Delta \exp\Big(-\frac{\Delta}{4\pi p}\Big) = -\frac{1}{p}(\tfrac{\partial}{\partial u} e^{-u\Delta})\Big|_{u=\frac{1}{4\pi p}}.$$

Now from (27), (30)–(37) with $k = n + 1$, we get

$$\begin{aligned} &\left| \Big(p^{-n} K_{\omega,p} - \exp\Big(-\frac{\Delta}{4\pi p}\Big)\Big) f \right|_{L^2} \leq \frac{C}{p} \, |f|_{L^2} \,, \\ &\left| \frac{1}{p}\Big(p^{-n} \Delta K_{\omega,p} - \Delta \exp\Big(-\frac{\Delta}{4\pi p}\Big)\Big) f \right|_{L^2} \leq \frac{C}{p} \, |f|_{L^2} \,. \end{aligned} \tag{38}$$

and

$$\left| \frac{1}{p}\Big(p^{-n} K_{\eta,\omega,p} - \langle d\eta, d\exp(-\frac{\Delta}{4\pi p})\rangle\Big) f \right|_{L^2} \leq \frac{C}{p} \, |f|_{L^2} \,. \tag{39}$$

Note that

$$\begin{aligned} (\Delta(\eta K_{\omega,p}))(x,y) = (\Delta\eta)(x) K_{\omega,p}(x,y) + \eta(x) \Delta_x K_{\omega,p}(x,y) \\ - 2\langle d\eta(x), d_x K_{\omega,p}(x,y)\rangle_{g^{T^*X}} \,, \end{aligned} \tag{40}$$

and $R_p = \frac{\mathrm{Vol}(X, dv_X)}{\mathrm{Vol}(X,\nu)} p^n + \mathscr{O}(p^{n-1})$. From (15), (38)-(40), we get (9).

To get the last part of Theorem 0.1, as we noticed in [2, §4.5], the constants in (19) will be uniformly bounded under our condition, thus we can take C in (9), (38)and (39) independent of h^L, $d\nu$. □

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Center of Mathematical Science, Zhejiang University and Department of Mathematics, UCLA, CA 90095-1555, USA

E-mail address: liu@math.ucla.edu

Centre de Mathématiques Laurent Schwartz, UMR 7640 du CNRS, Ecole Polytechnique, 91128 Palaiseau Cedex, France

E-mail address: ma@math.polytechnique.fr