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A GENERALIZATION OF THE CASSELS-TATE DUAL EXACT SEQUENCE

Cristian D. González-Avilés and Ki-Seng Tan

Abstract. We extend the first part of the well-known Cassels-Tate dual exact sequence for abelian varieties A over global fields K in two directions: we treat the p-primary component in the function field case, where p is the characteristic of K, and we dispense with the assumption that the Tate-Shafarevich group of A is finite.

1. Introduction

Let K be a global field and let m be a positive integer *which is prime to the characteristic of* K (in the function field case). Let A be an abelian variety over K. Then there exists an exact sequence of discrete groups

$$0 \to Ш(A)(m) \to H^1(K,A)(m) \to \bigoplus_{\text{all } v} H^1(K_v,A)(m) \to Б(A)(m) \to 0,$$

where K_v is the henselization of K at v, $M(m)$ denotes the m-primary component of a torsion abelian group M, and $Б(A)$ is defined to be the cokernel of the localization map $H^1(K,A) \to \bigoplus_{\text{all } v} H^1(K_v,A)$. The Pontrjagyn dual of the preceding exact sequence is an exact sequence of compact groups

$$0 \leftarrow Ш(A)(m)^* \leftarrow H^1(K,A)(m)^* \leftarrow \prod_{\text{all } v} H^0(K_v,A^t)^\frown \leftarrow Б(A)(m)^* \leftarrow 0,$$

where A^t is the abelian variety dual to A and, for any abelian group M, $M^\frown$ denotes the m-adic completion $\varprojlim_n M/m^n$ of M. Now, if $Ш(A)(m)$ is *finite* (or, more generally, if $Ш(A)(m)$ contains no nontrivial elements which are divisible by m^n for every $n \geq 1$), then $Ш(A)(m)^*$ and $Б(A)(m)^*$ are canonically isomorphic to $Ш(A^t)(m)$ and $A^t(K)^\frown$, respectively, and the preceding exact sequence induces an exact sequence

$$0 \leftarrow Ш(A^t)(m) \leftarrow H^1(K,A)(m)^* \leftarrow \prod_{\text{all } v} H^0(K_v,A^t)^\frown \leftarrow A^t(K)^\frown \leftarrow 0$$

which is known as *the Cassels-Tate dual exact sequence*. See [9, Theorem II.5.6(b), p.247]. The aim of this paper is to extend the first part of the above exact sequence

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to the case where m is divisible by the characteristic of K (in the function field case) and no hypotheses are made on $Ш(A)$. The following is the main result of the paper. Let m and n be *arbitrary* positive integers. Set

$$\mathrm{Sel}(A^t)_{m^n} = \mathrm{Ker}\left[H^1(K, A^t_{m^n}) \to \bigoplus_{\text{all } v} H^1(K_v, A^t)\right]$$

and

$$T_m \,\mathrm{Sel}(A^t) = \varprojlim_n \mathrm{Sel}(A^t)_{m^n}.$$

Then the following holds[1]:

Main Theorem. *For any positive integer m, there exists a natural exact sequence of compact groups*

$$0 \leftarrow Ш(A)(m)^* \leftarrow H^1(K, A)(m)^* \leftarrow \prod_{\text{all } v} H^0(K_v, A^t)^{\wedge} \leftarrow T_m \mathrm{Sel}(A^t) \leftarrow 0.$$

We should note that a similar statement holds true if above the henselizations of K are replaced by its completions. See [9, Remark I.3.10, p.59].

This paper grew out of questions posed to the authors by B.Poonen, in connection with the forthcoming paper [11]. We expect that the above theorem will be useful in [op.cit.].

2. Settings and notations

Let K be a global field and let A be an abelian variety over K. In the function field case, we let p denote the characteristic of K. All cohomology groups below are either Galois cohomology groups or flat cohomology groups. For any non-archimedean prime v of K, K_v will denote the field of fractions of the henselization of the ring of v-integers of K. If v is an archimedean prime, K_v will denote the completion of K at v, and we will write $H^0(K_v, A)$ for the quotient of $A(K_v)$ by its identity component. Note that, for any prime v of K, the group $H^1(K_v, A)$ is canonically isomorphic to $H^1(\widehat{K}_v, A)$, where $\widehat{K}_v$ denotes the completion of K at v. See [9, Remark I.3.10(ii), p.58]. Now let X denote either the spectrum of the ring of integers of K (in the number field case) or the unique smooth complete curve over the field of constants of K with function field K (in the function field case). In what follows, U denotes a nonempty open subset of X such that A has good reduction over U. When N is a quasi-finite flat group scheme on U, we endow $H^r(U, N)$ with the discrete topology. Now let m and n be arbitrary positive integers, and let M be an abelian topological group. We will write M/m^n for $M/m^nM = M \otimes_{\mathbb{Z}} \mathbb{Z}/m^n$ and $M^{\wedge}$ for the m-adic completion $\varprojlim_n M/m^n$ of M. Further, we set $\mathbb{Z}_m = \prod_{\ell\,|m} \mathbb{Z}_\ell$, $\mathbb{Q}_m = \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Q}$ and define $M^* = \mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}_m/\mathbb{Z}_m)$. Finally, the m-primary component of a torsion group M will be denoted by $M(m)$.

[1]To see why the exact sequence of the theorem extends the Cassels-Tate dual exact sequence recalled above, see exact sequence (6) below, note that $T_m\, Ш(A) = T_m(\,Ш(A)_{m\text{-div}})$ vanishes if $Ш(A)_{m\text{-div}} = 0$, and use [9, Corollary III.9.5, p.370].

3. Proof of the Main Theorem

Both A and its dual variety A^t extend to abelian schemes $\mathcal{A}$ and $\mathcal{A}^t$ over U (see [2, Ch.1, §1.4.3]). By [5, VIII.7.1(b)], the canonical Poincaré biextension of (A^t, A) by $\mathbb{G}_m$ extends to a biextension over U of $(\mathcal{A}^t, \mathcal{A})$ by $\mathbb{G}_m$. Further, by [op.cit., VII.3.6.5], (the isomorphism class of) this biextension corresponds to a map $\mathcal{A}^t \otimes^{\mathbf{L}} \mathcal{A} \to \mathbb{G}_m[1]$ in the derived category of the category of smooth sheaves on U. This map in turn induces (see [9, p.283]) a canonical pairing $H^1(U, \mathcal{A}^t) \times H^1_{\mathrm{c}}(U, \mathcal{A}) \to H^3_{\mathrm{c}}(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$, where the $H^r_{\mathrm{c}}(U, \mathcal{A})$ are the cohomology groups with compact support of the sheaf $\mathcal{A}$ defined in [9, p.271].

Remark 3.1. The smoothness of $\mathcal{A}$ implies that the groups $H^r(U, \mathcal{A})$ and $H^r_{\mathrm{c}}(U, \mathcal{A})$ agree with the analogous groups defined for the étale topology. See [9, Proposition III.0.4(d), p.272].

For any positive integer m and any $n \geq 1$, the above pairing induces a pairing

$$H^1(U, \mathcal{A}^t_{m^n}) \times H^1_{\mathrm{c}}(U, \mathcal{A})/m^n \to \mathbb{Q}/\mathbb{Z}. \tag{1}$$

On the other hand, the map $\mathcal{A}^t \otimes^{\mathbf{L}} \mathcal{A} \to \mathbb{G}_m[1]$ canonically defines a map $\mathcal{A}^t_{m^n} \times \mathcal{A}_{m^n} \to \mathbb{G}_m$, which induces a pairing

$$H^1(U, \mathcal{A}^t_{m^n}) \times H^2_{\mathrm{c}}(U, \mathcal{A}_{m^n}) \to \mathbb{Q}/\mathbb{Z}. \tag{2}$$

The preceding pairing induces an isomorphism

$$H^2_{\mathrm{c}}(U, \mathcal{A}_{m^n}) \xrightarrow{\sim} H^1(U, \mathcal{A}^t_{m^n})^*. \tag{3}$$

See [9, Corollary II.3.3, p.217] for the case where m prime to p, and [op.cit., Theorem III.8.2, p.361] for the case where m is divisible by p. The pairings (1) and (2) are compatible, in the sense that the following diagram commutes:

$$\begin{tikzcd} H^1(U, \mathcal{A}^t_{m^n}) \times H^1_{\mathrm{c}}(U, \mathcal{A})/m^n \arrow[r] \arrow[d, "\mathrm{id} \times \partial"] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ H^1(U, \mathcal{A}^t_{m^n}) \times H^2_{\mathrm{c}}(U, \mathcal{A}_{m^n}) \arrow[r] & \mathbb{Q}/\mathbb{Z}, \end{tikzcd} \tag{4}$$

where $\partial \colon H^1_{\mathrm{c}}(U, \mathcal{A})/m^n \hookrightarrow H^2_{\mathrm{c}}(U, \mathcal{A}_{m^n})$ is induced by the connecting homomorphism $H^1_{\mathrm{c}}(U, \mathcal{A}) \to H^2_{\mathrm{c}}(U, \mathcal{A}_{m^n})$ coming from the exact sequence

$$0 \to \mathcal{A}_{m^n} \to \mathcal{A} \stackrel{m^n}{\to} \mathcal{A} \to 0.$$

Now define

$$\mathrm{Sel}(A^t)_{m^n} = \mathrm{Ker}\left[H^1(K, A^t_{m^n}) \to \bigoplus_{\text{all } v} H^1(K_v, A^t) \right]$$

and

$$T_m \,\mathrm{Sel}(A^t) = \varprojlim_n \mathrm{Sel}(A^t)_{m^n}.$$

Now the kernel-cokernel sequence of the pair of maps $H^1(K, A^t_{m^n}) \to H^1(K, A^t)_{m^n} \to \bigoplus_{\text{all } v} H^1(K_v, A^t)$ [9, Proposition I.0.24, p.19] yields an exact sequence

$$0 \to A^t(K)/m^n \to \mathrm{Sel}(A^t)_{m^n} \to Ш(A^t)_{m^n} \to 0. \tag{5}$$

Taking inverse limits, we obtain an exact sequence

$$0 \to A^t(K)^{\widehat{}} \to T_m \operatorname{Sel}(A^t) \to T_m Ш(A^t) \to 0. \tag{6}$$

See [1, Proposition 10.2, p.104]. Now define[2]

$$D^1(U, \mathcal{A}^t_{m^n}) = \operatorname{Ker}\left[H^1(U, \mathcal{A}^t_{m^n}) \to \prod_{v \notin U} H^1(K_v, A^t)\right]$$

and

$$\begin{aligned} D^1(U, \mathcal{A}^t) &= \operatorname{Im}\left[H^1_{\mathrm{c}}(U, \mathcal{A}^t) \to H^1(U, \mathcal{A}^t)\right] \\ &= \operatorname{Ker}\left[H^1(U, \mathcal{A}^t) \to \prod_{v \notin U} H^1(K_v, A^t)\right]. \end{aligned}$$

Note that the pairing $H^1(U, \mathcal{A}^t) \times H^1_{\mathrm{c}}(U, \mathcal{A}) \to \mathbb{Q}/\mathbb{Z}$ induces a pairing $D^1(U, \mathcal{A}^t) \times D^1(U, \mathcal{A}) \to \mathbb{Q}/\mathbb{Z}$.

By [9, Proposition III.0.4(a), p.271] and the right-exactness of the tensor product functor, there exists a natural exact sequence

$$\bigoplus_{v \notin U} H^0(K_v, A)/m^n \to H^1_{\mathrm{c}}(U, \mathcal{A})/m^n \to D^1(U, \mathcal{A})/m^n \to 0. \tag{7}$$

Lemma 3.2. *The map $H^1(U, \mathcal{A}^t_{m^n}) \hookrightarrow H^1(K, A^t_{m^n})$ induces an isomorphism*

$$D^1(U, \mathcal{A}^t_{m^n}) \simeq \operatorname{Sel}(A^t)_{m^n}.$$

Proof. By [9, Lemma II.5.5, p.246] and Remark 3.1 above, the map $H^1(U, \mathcal{A}^t) \hookrightarrow H^1(K, A^t)$ induces an isomorphism

$$D^1(U, \mathcal{A}^t)_{m^n} \simeq Ш(A^t)_{m^n}.$$

Now $H^1(U, \mathcal{A}^t_{m^n}) \to \prod_{v \in U} H^1(K_v, A^t)$ factors through $H^1(U, \mathcal{A}^t) \to \prod_{v \in U} H^1(K_v, A^t)$, which is the zero map (see [9, (5.5.1), p.247] and Remark 3.1 above). Consequently, $H^1(U, \mathcal{A}^t_{m^n}) \hookrightarrow H^1(K, A^t_{m^n})$ maps $D^1(U, \mathcal{A}^t_{m^n})$ into $\operatorname{Sel}(A^t)_{m^n}$. To prove surjectivity, we consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}^t(U)/m^n\mathcal{A}^t(U) \arrow[r] \arrow[d] & H^1(U, \mathcal{A}^t_{m^n}) \arrow[r] \arrow[d] & H^1(U, \mathcal{A}^t)_{m^n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A^t(K)/m^nA^t(K) \arrow[r] & H^1(K, A^t_{m^n}) \arrow[r] & H^1(K, A^t)_{m^n} \arrow[r] & 0. \end{tikzcd}$$

Note that the properness of $\mathcal{A}^t$ over U implies that the left-hand vertical map in the above diagram is an isomorphism (see [op.cit., p.242]). Now let $c \in \operatorname{Sel}(A^t)_{m^n}$, write c' for its image in $Ш(A^t)_{m^n}$ under the map in (5) and let $\xi' \in D^1(U, \mathcal{A}^t)_{m^n} \subset H^1(U, \mathcal{A}^t)_{m^n}$ be the pullback of c' under the isomorphism $D^1(U, \mathcal{A}^t)_{m^n} \simeq Ш(A^t)_{m^n}$ recalled above. Then the fact that the left-hand vertical map in the above diagram is an isomorphism implies that ξ' can be pulled back to a class $\xi \in H^1(U, \mathcal{A}^t_{m^n})$ which maps down to c. Clearly $\xi \in D^1(U, \mathcal{A}^t_{m^n})$, and this completes the proof. □

The following proposition generalizes [9, Theorem II.5.2(c), p.244].

[2] In these definitions, the products extend over all primes of K, including the archimedean primes, not in U.

Proposition 3.3. *There exists a canonical isomorphism*

$$(\,T_m\operatorname{Sel}(A^t))^* \xrightarrow{\sim} H^2_{\mathrm{c}}(U,\mathcal{A})(m).$$

Proof. There exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{c}}(U,\mathcal{A})/m^n \arrow[r] \arrow[dr, "c"] & H^2_{\mathrm{c}}(U,\mathcal{A}_{m^n}) \arrow[r] \arrow[d, "\simeq"] & H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n} \arrow[r] & 0 \\ & & H^1(U,\mathcal{A}^t_{m^n})^*, & & \end{tikzcd}$$

where the vertical map is the isomorphism (3). Clearly, the above diagram induces an isomorphism $\operatorname{Coker} c \simeq H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n}$. On the other hand, there exists a natural exact commutative[3] diagram

$$\begin{tikzcd} \bigoplus_{v\notin U} H^0(K_v,A)/m^n \arrow[r] \arrow[d, "\simeq"] & H^1_{\mathrm{c}}(U,\mathcal{A})/m^n \arrow[r] \arrow[d, "c"] & D^1(U,\mathcal{A})/m^n \arrow[r] \arrow[d, "\psi"] & 0 \\ \bigoplus_{v\notin U} H^1(K_v,A^t)^*_{m^n} \arrow[r] & H^1(U,\mathcal{A}^t_{m^n})^* \arrow[r] & D^1(U,\mathcal{A}^t_{m^n})^* \arrow[r] & 0, \end{tikzcd}$$

where the top row is (7), the right-hand vertical map ψ is the composite of the natural map $D^1(U,\mathcal{A})/m^n \to D^1(U,\mathcal{A}^t)^*_{m^n}$ induced by the pairing $D^1(U,\mathcal{A}^t)\times D^1(U,\mathcal{A}) \to \mathbb{Q}/\mathbb{Z}$ and the natural map $D^1(U,\mathcal{A}^t)^*_{m^n} \to D^1(U,\mathcal{A}^t_{m^n})^*$, and the left-hand vertical map is induced by the canonical Poincaré biextensions of (A^t,A) by $\mathbb{G}_m$ over K_v for each $v\notin U$. That the latter map is an isomorphism follows from [9, Remarks I.3.5 and I.3.7, pp.53 and 56, and Theorem III.7.8, p.354] and the fact that the pairings defined in [loc.cit.] are compatible with the pairing induced by the canonical Poincaré biextension (see [4, Appendix]). The above diagram and the identification $\operatorname{Coker} c = H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n}$ yield an exact sequence

$$D^1(U,\mathcal{A})/m^n \to D^1(U,\mathcal{A}^t_{m^n})^* \to H^2_{\mathrm{c}}(U,\mathcal{A})_{m^n} \to 0$$

Taking direct limits, we obtain an exact sequence

$$D^1(U,\mathcal{A})\otimes\mathbb{Q}_m/\mathbb{Z}_m \to \Big(\varprojlim D^1(U,\mathcal{A}^t_{m^n})\Big)^* \to H^2_{\mathrm{c}}(U,\mathcal{A})(m) \to 0$$

But $D^1(U,\mathcal{A})\otimes\mathbb{Q}_m/\mathbb{Z}_m = 0$ since $D^1(U,\mathcal{A})$ is torsion and $\mathbb{Q}_m/\mathbb{Z}_m$ is divisible. Now lemma 3.2 completes the proof. □

By Remark 3.1 and [9, proof of Lemma II.5.5, p.247, and Proposition II.2.3, p. 203], there exist exact sequences[4]

$$H^1(U,\mathcal{A}) \xrightarrow{c_U} \bigoplus_{v\notin U} H^1(K_v,A) \to H^2_{\mathrm{c}}(U,\mathcal{A})$$

and

$$0 \to H^1(U,\mathcal{A}) \xrightarrow{i_U} H^1(K,A) \xrightarrow{\lambda_U} \bigoplus_{v\in U} H^1(K_v,A),$$

[3]The commutativity of this diagram follows from that of diagram (4).

[4]In the second exact sequence, "$v\in U$" is shorthand for "v is a closed point of U".

where c_U and λ_U are natural localization maps and i_U is induced by the inclusion $\mathrm{Spec}K \hookrightarrow U$. If $U \subset V$ is an inclusion of nonempty open subsets of X, then there exists a natural commutative diagram

$$\begin{tikzcd} H^1(V,\mathcal{A}) \arrow[r, "c_V"] \arrow[d] & \bigoplus_{v\notin V} H^1(K_v,A) \arrow[d] \\ H^1(U,\mathcal{A}) \arrow[r, "c_U"] & \bigoplus_{v\notin U} H^1(K_v,A). \end{tikzcd}$$

Define

$$Б(A)_U = \operatorname{coker}\left[c_U\colon H^1(U,\mathcal{A}) \to \bigoplus_{v\notin U} H^1(K_v,A)\right],$$

which we regard as a subgroup of $H^2_c(U,\mathcal{A})$. The preceding diagram shows that an inclusion $U \subset V$ of nonempty open subsets of X induces a map $Б(A)_V \to Б(A)_U$. Define

$$Б(A) = \varinjlim Б(A)_U = \operatorname{coker}\left[H^1(K,A) \to \bigoplus_{\text{all } v} H^1(K_v,A)\right],$$

where the limit is taken over the directed family of all nonempty open subsets U of X such that A has good reduction over U, ordered by $V \le U$ if and only if $U \subset V$. For each U as above and every $n \ge 1$, there exists an exact sequence

$$\bigoplus_{v\notin U} H^1(K_v,A)_{m^n} \to (Б(A)_U)_{m^n} \to (\operatorname{Im} c_U)/m^n.$$

Since $\operatorname{Im} c_U$ is torsion, we conclude that there exists a surjection

$$\bigoplus_{v\notin U} H^1(K_v,A)(m) \twoheadrightarrow Б(A)_U(m). \tag{8}$$

On the other hand, there exists a natural injection $T_m \operatorname{Sel}(A^t) \hookrightarrow \prod_{\text{all } v} H^0(K_v,A^t)\hat{}$ (when m is prime to p, this is contained in [9, Corollary I.6.23(b), p.111]. If m is divisible by p, the assertion follows from the $p-$analogue of Proposition I.6.19 of [op.cit.], which in turn follows from [10][5]. Consequently, there is a surjection

$$\bigoplus_{\text{all } v} (H^0(K_v,A^t)\hat{}\,)^* \to (T_m \operatorname{Sel}(A^t))^*$$

Further, as noted in the proof of Proposition 3.3, the canonical Poincaré biextensions induce an isomorphism

$$\bigoplus_{\text{all } v} (H^0(K_v,A^t)\hat{}\,)^* \simeq \bigoplus_{\text{all } v} H^1(K_v,A)(m),$$

[5] See, especially, Propositions 5 and 6 of [10], which extend the results on pp.773 and 734 of [12] to the case $p|m$.

whence there exists a surjection

$$\bigoplus_{\text{all } v} H^1(K_v, A)(m) \twoheadrightarrow (T_m \operatorname{Sel}(A^t))^*. \tag{9}$$

The maps (8) and (9) fit into a commutative diagram

$$\begin{tikzcd}
\bigoplus_{\text{all } v} H^1(K_v, A)(m) \arrow[r, two heads, "(9)"] & (T_m \operatorname{Sel}(A^t))^* \arrow[r, "\sim"] & H^2_{\mathrm{c}}(U, \mathcal{A})(m) \\
\bigoplus_{v \notin U} H^1(K_v, A)(m) \arrow[u, hook] \arrow[r, two heads, "(8)"] & Б(A)_U(m), \arrow[u, hook, dashed] \arrow[ur, hook] &
\end{tikzcd}$$

where the isomorphism on the top row exists by Proposition 3.3. Taking the direct limit over U in the above diagram, we conclude that there exists an isomorphism

$$Б(A)(m) \xrightarrow{\sim} (T_m \operatorname{Sel}(A^t))^*,$$

as desired.

Remark 3.4. Recently (see [7, Theorem 1.2]), the Cassels-Tate dual exact sequence has been extended to 1-motives M over number fields, under the assumption that the Tate-Shafarevich group of M is finite. Using [6, Remark 5.10], it should not be difficult to extend this result to global function fields, provided the p-primary components of the groups involved are ignored. In this paper we have managed to remove the latter restriction when M is an abelian variety, but the problem remains for general 1-motives M.

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[6] After this paper was completed, we learned that the existence of a natural duality between $Б(A)(m)$ and $T_m \operatorname{Sel}(A^t)$ had already been observed by Cassels in the case of elliptic curves over number fields. See [3, p.153]. Therefore, the Main Theorem of this paper may be regarded as a natural generalization of Cassels' result.

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Departamento de Matemáticas, Universidad Andrés Bello, Chile
E-mail address: `cristiangonzalez@unab.cl`

Department of Mathematics, National Taiwan University, Taipei 106, Taiwan
E-mail address: `tan@math.ntu.tw`