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FORMAL HODGE THEORY

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Abstract. We introduce *formal* (mixed) Hodge structures (of level ≤ 1) in such a way that the Hodge realization of Deligne's 1-motives extends to a realization from Laumon's 1-motives to formal Hodge structures (of level ≤ 1) providing an equivalence of categories.

Let $\mathrm{MHS}_1^{\mathrm{fr}}$ denote the category of torsion free graded polarizable mixed Hodge structures of level ≤ 1. We have a nice algebraic description of this category *via* $\mathcal{M}_1^{\mathrm{fr}}$ the category of Deligne's 1-motives [5] (*cf.* also [3], including torsion, one obtains 1-motives with torsion describing MHS_1). Actually, Deligne's Hodge realization provide an equivalence

$$T_{Hodge} : \mathcal{M}_1^{\mathrm{fr}} \xrightarrow{\simeq} \mathrm{MHS}_1^{\mathrm{fr}}$$

such that Cartier duality on $\mathcal{M}_1^{\mathrm{fr}}$ is transformed in $\underline{\mathrm{Hom}}(-, \mathbb{Z}(1))$ on $\mathrm{MHS}_1^{\mathrm{fr}}$. Moreover, we have a natural generalization of Deligne's 1-motives due to Laumon [6]. A Laumon 1-motive $M := [F \stackrel{u}{\to} G]$ is a commutative formal group $F = F^0 \times F_{\text{ét}}$, with torsion free étale part $F_{\text{ét}}$, a commutative connected algebraic group G and a map of abelian fppf-sheaves $u : F \to G$. Let $\mathcal{M}_1^{a,\mathrm{fr}}$ denote the category of Laumon's 1-motives and refer to its objects as 1-motives for short. Note that Cartier duality on $\mathcal{M}_1^{\mathrm{fr}}$ canonically extends to $\mathcal{M}_1^{a,\mathrm{fr}}$ (see [6]).

The purpose of this note is to introduce the abelian category FHS_1 of *formal* mixed Hodge structures (of level ≤ 1) in order to extend the Hodge realization T_{Hodge} of Deligne's 1-motives $\mathcal{M}_1^{\mathrm{fr}}$ to a realization $T_{\oint}$ from Laumon's 1-motives $\mathcal{M}_1^{a,\mathrm{fr}}$ to $\mathrm{FHS}_1^{\mathrm{fr}} \subset \mathrm{FHS}_1$. We have that $\mathrm{MHS}_1^{\mathrm{fr}} \subset \mathrm{FHS}_1^{\mathrm{fr}}$ in a canonical way, *i.e.*, there is a fully faithful embedding such that the natural involution (Cartier duality) on $\mathrm{MHS}_1^{\mathrm{fr}}$ extends to an involution on $\mathrm{FHS}_1^{\mathrm{fr}}$.

For the sake of exposition we here confine our study to level ≤ 1 mixed Hodge structures. However, it is conceivable and suitable to consider formal mixed Hodge structures with arbitrary Hodge numbers: generalizing our definition below it's not that difficult (we will treat such a matter nextly, *cf.* [1, 2.12] for the general setting). For example, enriched Hodge structures [4] (of level ≤ 1) can easily be recovered as "special" formal Hodge structures (see also [2] for details). In [2] we are also providing a "sharp" De Rham realization generalizing Deligne's construction of De Rham realization in [5]. The main result of this paper can be summarized in the following way.

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Theorem. *There is an equivalence of categories with involution*

$$T_{\oint} : \mathcal{M}_1^{a,\mathrm{fr}} \xrightarrow{\simeq} \mathrm{FHS}_1^{\mathrm{fr}}$$

between Laumon's 1-motives and torsion free formal Hodge structures (of level ≤ 1) providing a diagram

$$\begin{array}{ccc} \mathcal{M}_1^{\mathrm{fr}} & \stackrel{\simeq}{\to} & \mathrm{MHS}_1^{\mathrm{fr}} \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathcal{M}_1^{a,\mathrm{fr}} & \stackrel{\simeq}{\to} & \mathrm{FHS}_1^{\mathrm{fr}} \end{array}$$

where

- $\mathcal{M}_1^{\mathrm{fr}} \hookrightarrow \mathcal{M}_1^{a,\mathrm{fr}}$ *and* $\mathrm{MHS}_1^{\mathrm{fr}} \hookrightarrow \mathrm{FHS}_1^{\mathrm{fr}}$ *are canonical inclusions,*
- $\mathcal{M}_1^{a,\mathrm{fr}} \to \mathcal{M}_1^{\mathrm{fr}}$ *and* $\mathrm{FHS}_1^{\mathrm{fr}} \to \mathrm{MHS}_1^{\mathrm{fr}}$ *are "forgetful functors" denoted* $(\ \ ) \rightsquigarrow (\ \ )_{\text{ét}}$*, which are left inverses of the inclusions,*
- $T_{\oint}(M)$ *coincide with* $T_{Hodge}(M)$ *if* $M = M_{\text{ét}}$ *and, in general, we have a formula*

$$T_{\oint}(M)_{\text{ét}} = T_{Hodge}(M_{\text{ét}}).$$

The plan of the paper is the following. In Section 1 we introduce the category FHS_1. In Section 2 we construct $T_{\oint}$ proving the theorem.

1. Formal Hodge Structures

1.1. Paradigma. Consider a commutative formal group $H = H^0 \times H_{\mathbb{Z}}$ over $\mathbb{C}$ along with a mixed Hodge structure on the étale part $H_{\mathbb{Z}}$, *i.e.*, say $H_{\text{ét}} := (H_{\mathbb{Z}}, W_*, F^*_{Hodge}) \in \mathrm{MHS}_1$ for short. For the mixed Hodge structure $H_{\text{ét}} \in \mathrm{MHS}_1$ we here denote $H_{\mathbb{Z}}$ the finitely generated abelian underlying group, along with the weight filtration $W_{-2} \subseteq W_{-1}$ of $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes \mathbb{Q}$ and $F^0_{Hodge} \subseteq H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$ the Hodge filtration. We say that H is *free* if the étale part of the formal group is free, so that: $H_{\mathbb{Z}} = \mathbb{Z}^r$ and $H^0 = \widehat{\mathbb{C}}^s$ non-canonically. (Note that here $\widehat{\mathbb{C}}$ denotes the connected formal additive group). For H free we also denote by $W_* H_{\text{ét}}$ and $\mathrm{gr}_*^W H_{\text{ét}}$ the corresponding objects of MHS_1.

1.1.1. Definition. Define a *formal Hodge structure* (of level ≤ 1) as follows: *(i)* a formal group H such that $H_{\text{ét}} \in \mathrm{MHS}_1$, *(ii)* a finite dimensional $\mathbb{C}$-vector space V with a two steps filtration $V^0 \subseteq V^1 \subseteq V$ by sub-spaces, *(iii)* a group homomorphism $v : H \to V$ and *(iv)* a $\mathbb{C}$-isomorphism $\sigma : H_{\mathbb{C}}/F^0_{Hodge} \stackrel{\simeq}{\to} V/V^0$ restricting to an isomorphism $W_{-2}H_{\mathbb{C}} \cong V^1/V^0$. We further assume that the following condition holds: if $v_{\mathbb{Z}} : H_{\mathbb{Z}} \to V$ is the induced map, $c : H_{\mathbb{Z}} \to H_{\mathbb{C}}/F^0_{Hodge}$ is the canonical map and $\mathrm{pr} : V \twoheadrightarrow V/V^0$ is the projection then the following

$$\begin{array}{ccc} H_{\mathbb{Z}} & \xrightarrow{v_{\mathbb{Z}}} & V \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle pr} \\ H_{\mathbb{C}}/F^0_{Hodge} & \xrightarrow{\sigma} & V/V^0 \end{array} \qquad (1)$$

commutes. Denote (H, V) for short such a structure.

Define a morphism ϕ between (H,V) and (H',V') as follows. We let $\phi := (f,g)$ be a pair of maps in the following commutative square

$$\begin{array}{ccc} H & \xrightarrow{v} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H' & \xrightarrow{v'} & V' \end{array} \tag{2}$$

where $f : H \to H'$ is a homomorphism of formal groups such that $f_{\text{ét}} : H_{\text{ét}} \to H'_{\text{ét}}$ is a map in MHS_1 and $g : V \to V'$ is a $\mathbb{C}$-homomorphism compatible with the filtrations, *i.e.*, $g(V^i) \subseteq V'^i$ for $i = 0, 1$. We further assume that the following diagram commutes

$$\begin{array}{ccc} H_{\mathbb{C}}/F^0_{Hodge} & \xrightarrow{\sigma} & V/V^0 \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ H'_{\mathbb{C}}/F^0_{Hodge} & \xrightarrow{\sigma'} & V'/V'^0 \end{array} \tag{3}$$

where $\overline{f}$ and $\overline{g}$ are the canonically induced maps.

1.1.2. Definition. Let FHS_1 denote the *category* whose objects are (H,V), the morphisms are $\phi = (f,g)$ as above and the composition is given by gluing the squares (2) (the condition (3) is preserved by gluing). Let $\mathrm{FHS}_1^{\mathrm{fr}} \subset \mathrm{FHS}_1$ denote the full subcategory given by (H,V) such that H is free.

1.1.3. Proposition. *The category* FHS_1 *is abelian. A short exact sequence*

$$0 \to (H,V) \to (H',V') \to (H'',V'') \to 0$$

is given by an exact sequence on each component (formal groups and filtered vector spaces) so that

$$0 \to H_{\text{ét}} \to H'_{\text{ét}} \to H''_{\text{ét}} \to 0$$

is exact in MHS_1.

Proof. Straightforward. □

1.2. Étale structures. We can recover mixed Hodge structures as follows.

1.2.1. Definition. Define $(H,V)_{\text{ét}} := (H_{\mathbb{Z}}, V/V^0)$ where $(H_{\mathbb{Z}})_{\text{ét}} := H_{\text{ét}}$, $v_{\text{ét}} : H_{\mathbb{Z}} \to V/V^0$ is the composition of pr and $v_{\mathbb{Z}}$ (*cf.* 1.1.1) and

$$(V/V^0)^0 := 0 \subseteq (V/V^0)^1 := V^1/V^0 \subseteq V/V^0.$$

Say that a formal Hodge structure is *étale* if $(H,V) = (H,V)_{\text{ét}}$, *i.e.*, if $H^0 = V^0 = 0$.

Given $(H_{\mathbb{Z}}, W_*, F^*_{Hodge}) \in \mathrm{MHS}_1$ there is a natural way to provide an étale one as follows. Set $H := H_{\mathbb{Z}}$, $H_{\text{ét}} := (H_{\mathbb{Z}}, W_*, F^*_{Hodge})$, $H^0 = 0$, $V := H_{\mathbb{C}}/F^0_{Hodge}$, $V^0 := 0$, $V^1 := W_{-2}H_{\mathbb{C}}$, σ is the identity and the map $v := c$ is induced by the canonical map $t : H_{\mathbb{Z}} \to H_{\mathbb{C}}$. Denote

$$c(H_{\mathbb{Z}}, W_*, F^*_{Hodge}) := (H_{\mathbb{Z}}, H_{\mathbb{C}}/F^0_{Hodge})$$

the *canonical* étale formal Hodge structure associated to a mixed Hodge structure, providing a functor $c : \mathrm{MHS}_1 \to \mathrm{FHS}_1$.

1.2.2. Lemma. *The full subcategory* $\mathrm{FHS}_1^{\text{ét}}$ *of étale structures is equivalent to* MHS_1 *via* c *and the forgetful functor* $(H,V)\mapsto H_{\text{ét}}$. *The functor* $e:(H,V)\mapsto (H,V)_{\text{ét}}$ *is a left inverse of the inclusion* $\mathrm{FHS}_1^{\text{ét}}\subset \mathrm{FHS}_1$ *and, for* $(H',V')\in \mathrm{FHS}_1^{\text{ét}}$, *we have*

$$\mathrm{Hom}((H,V),(H',V'))\subseteq \mathrm{Hom}((H,V)_{\text{ét}},(H',V'))$$

where the equality holds if $v(H^0)\subseteq V^0$ *(cf. 1.3.1 below).*

Proof. Actually, for the equivalence, we are easily left to show that if (H,V) is étale then $c(H_{\text{ét}}):=(H_{\mathbb{Z}},H_{\mathbb{C}}/F^0_{Hodge})\cong (H,V)$. The claimed isomorphism is $(1,\sigma)$ granted by (1) since $V^0=H^0=0$.

For the other claims, let $(H,V)\in \mathrm{FHS}_1$ and $(H',V')\in \mathrm{FHS}_1^{\text{ét}}$ and consider a map $\phi=(f,g):(H,V)\to(H',V')$ whence induced maps $\overline{f}$ and $\overline{g}$ and a diagram

$$\begin{array}{ccc} H & \xrightarrow{v} & V \\ \uparrow\downarrow & & \downarrow \\ H_{\mathbb{Z}} & \xrightarrow{v_{\text{ét}}} & V/V^0 \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ H' & \xrightarrow{v'} & V' \end{array}$$

In fact $H'=H'_{\text{ét}}$ is étale thus $f(H^0)=0$ and f factors through $H_{\mathbb{Z}}$ yielding $\overline{f}$ and, similarly, we get a filtered map $\overline{g}:V/V^0\to V'$ since $V'^0=0$ and $g(V^0)=0$. Now $\overline{\phi}:=(\overline{f},\overline{g})$ yields a map by diagram chase. Note that if $v(H^0)\subseteq V^0$ then $(H,V)\to(H,V)_{\text{ét}}$ (*cf.* (5)) and we can lift back, by composition, any morphism $\phi':(H,V)_{\text{ét}}\to(H',V')$ as the condition (3) is tautological. □

1.2.3. Remark. Note that under the equivalence we then get a canonical inclusion $c:\mathrm{MHS}_1^{\mathrm{fr}}\hookrightarrow \mathrm{FHS}_1^{\mathrm{fr}}$ such that $e:\mathrm{FHS}_1^{\mathrm{fr}}\to \mathrm{MHS}_1^{\mathrm{fr}}$ is a left inverse and for $H'\in \mathrm{MHS}_1^{\mathrm{fr}}$

$$\mathrm{Hom}_{\mathrm{FHS}_1^{\mathrm{fr}}}((H,V),(H'_{\mathbb{Z}},H'_{\mathbb{C}}/F^0_{Hodge}))\subseteq \mathrm{Hom}_{\mathrm{MHS}_1^{\mathrm{fr}}}(H_{\text{ét}},H')$$

1.3. Connnected structures. A $\mathbb{C}$-vector space V will be regarded as an object $(0,V)$ of FHS_1 filtered as $V=V^1=V^0$. Similarly, a formal group H is regarded as an object $(H,0)$ of FHS_1 so that $H=H^0\times H_{\mathbb{Z}}$ and $H_{\mathbb{Z}}$ is pure of weight zero.

For $(H,V)\in\mathrm{FHS}_1$ we have that V^0 is a substructure of (H,V) and we can consider the quotient $(H,V)/V^0=(H,V/V^0)$ in FHS_1. We can also regard $(H,V)_{\text{ét}}$ as a substructure of $(H,V)/V^0$ and we obtain a canonical exact sequence

$$0\to(H,V)_{\text{ét}}\to(H,V)/V^0\to H^0\to 0 \tag{4}$$

1.3.1. Definition. Say that $(H,V)\in\mathrm{FHS}_1$ is *connected* if $H=H^0$ is connected, *i.e.*, if $(H,V)_{\text{ét}}=0$. Denote $\pi(H,V):=(H^0,V)$ the connected structure given by $V=V^1=V^0$ and the restriction of v to $H^0\subseteq H$. Let FHS_1^0 denote the full subcategory of FHS_1 determined by connected structures.

Say that $(H,V)\in\mathrm{FHS}_1$ is *special* if $v(H^0)\subseteq V^0$, *i.e.*, if $v:H\to V$ restricts to $v^0:H^0\to V^0$. Denote FHS_1^s the full subcategory of special structures and $(H,V)^0:=(H^0,V^0)\in\mathrm{FHS}_1^0$ the connected structure determined by $(H,V)\in\mathrm{FHS}_1^s$.

1.3.2. Lemma. *The functor* $(H,V)\mapsto\pi(H,V)$ *is a left inverse of the inclusion* $\iota:\mathrm{FHS}_1^0\subset\mathrm{FHS}_1$. *The category* FHS_1^0 *is equivalent to the category of linear mappings between finite dimensional* $\mathbb{C}$*-vector spaces. For* $(H',V')\in\mathrm{FHS}_1^0$ *and* $(H,V)\in\mathrm{FHS}_1^s$

$$\mathrm{Hom}((H',V'),(H,V))\cong\mathrm{Hom}((H',V'),(H,V)^0)$$

Proof. The first claim is clear. Moreover, the equivalence is provided by $(H,V)\mapsto \mathrm{Lie}\,(H)\to V$. Finally, a map from (H',V') connected to (H,V) special is given by a commutative square

$$\begin{array}{ccc} H' & \longrightarrow & V' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H & \longrightarrow & V \end{array}$$

such that $f(H')\subseteq H^0$ and $g(V')\subseteq V^0$. □

1.3.3. Remark. Note that (H,V) with $H_{\text{ét}}$ pure of weight zero exists if and only if $V=V^1=V^0$. Thus if (H,V) is special then $(H,V)^0$ is the largest connected formal substructure of (H,V) and we have a *non canonical* extension

$$0\to (H^0,V^0)\to (H,V)\to (H,V)_{\text{ét}}\to 0 \tag{5}$$

From lemmas 1.2.2 and 1.3.2 it follows that the functors $(H,V)\mapsto (H,V)^0$ and $(H,V)\mapsto (H,V)_{\text{ét}}$ are, respectively, a right adjoint of $\mathrm{FHS}_1^0\subset \mathrm{FHS}_1^s$ and a left adjoint of $\mathrm{FHS}_1^{\text{ét}}\subset \mathrm{FHS}_1^s$. However, special structures do have disadvantages, see 2.2.5 and 2.3.2.

1.3.4. Proposition. *The category* FHS_1^0 *forms a Serre abelian subcategory of* FHS_1 *yielding the extension*

$$0\to \mathrm{FHS}_1^0 \xrightarrow{\iota} \mathrm{FHS}_1 \xrightarrow{e} \mathrm{MHS}_1\to 0$$

where $\pi\iota=1$ *and* $ec=1$.

Proof. It follows from the lemmas 1.2.2, 1.3.2 and (4). In fact, it is clear (*cf.* 1.1.3) that FHS_1^0 forms a Serre subcategory. Since $e(\mathrm{FHS}_1^0)=0$ we have a factorisation $\overline{e}:\mathrm{FHS}_1/\mathrm{FHS}_1^0\to \mathrm{MHS}_1$ *via* the canonical projection $t:\mathrm{FHS}_1\to \mathrm{FHS}_1/\mathrm{FHS}_1^0$ and the equivalence $\mathrm{FHS}_1^{\text{ét}}\cong \mathrm{MHS}_1$. Since $e=\overline{e}t$ and $ec=1$ then $\overline{e}tc=1$. We also have $tc\overline{e}\cong 1$ since applying t to (4) for $(H,V)\in \mathrm{FHS}_1$ we get a natural isomorphism

$$tc(H_{\text{ét}})\cong t(H,V)_{\text{ét}}\cong t(H,V)$$

□

1.4. Construction. We provide a Laumon 1-motive out of a *free* formal mixed Hodge structure (of level ≤ 1). The construction is similar to [5, p. 55-56].

For $(H,V)\in \mathrm{FHS}_1^{\text{fr}}$ the Laumon 1-motive $\overrightarrow{(H,V)}:=[F\xrightarrow{u} G]$ functorially associated to (H,V) is given as follows. Set $F:=H^0\times \mathrm{gr}_0^W(H_{\mathbb{Z}})$. Since (1) holds true $W_{-1}(H_{\mathbb{Z}})$ injects in V *via* $v_{\mathbb{Z}}:H_{\mathbb{Z}}\to V$ in such a way that $W_{-1}(H_{\mathbb{Z}})\cap V^0=0$. Set $G(\mathbb{C}):=V/W_{-1}(H_{\mathbb{Z}})$ obtaining a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{-1}(H_{\mathbb{Z}}) & \longrightarrow & H & \longrightarrow & F & \longrightarrow & 0 \\ & & \| \downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle u}\downarrow & & \\ 0 & \longrightarrow & W_{-1}(H_{\mathbb{Z}}) & \longrightarrow & V & \longrightarrow & G(\mathbb{C}) & \longrightarrow & 0 \end{array} \tag{6}$$

where u is just induced by v. Regarding the complex group $G(\mathbb{C})$ we then have it in a diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V^0 & \xrightarrow{=} & V^0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_{-1}(H_{\mathbb{Z}}) & \xrightarrow{v_{\mathbb{Z}}} & V & \longrightarrow & G(\mathbb{C}) & \longrightarrow & 0 \\
 & & \| \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_{-1}(H_{\mathbb{Z}}) & \xrightarrow{c} & H_{\mathbb{C}}/F^0_{Hodge} & \longrightarrow & J(W_{-1}(H_{\text{ét}})) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

obtained *via* σ and (1). This is showing that $G(\mathbb{C})$ is an extension of the complex torus $J(W_{-1}(H_{\text{ét}}))$ by a $\mathbb{C}$-vector group. Thus, by G.A.G.A., we get the algebraic group G.

2. Formal Hodge realization

2.1. Paradigma. For a Laumon 1-motive $M = [F \xrightarrow{u} G] \in \mathcal{M}_1^{a,\text{fr}}$ over a field k (algebraically closed of characteristic zero) we here denote $F = F^0 \times F_{\text{ét}}$ the formal group where $F_{\text{ét}}$ is further assumed torsion free. Denote $V(G) := \mathbb{G}_a^n \subseteq G$ the additive factor and display the connected algebraic group G as an extension

$$0 \to V(G) \to G \to G_\times \to 0 \tag{7}$$

where $G_\times$ is the semi-abelian quotient. The algebraic group $G_\times$ is an extension of an abelian variety A by a torus T.

2.1.1. Definition. For $M = [F \xrightarrow{u} G] \in \mathcal{M}_1^{a,\text{fr}}$ set $M_{\text{ét}} := [F_{\text{ét}} \xrightarrow{u_{\text{ét}}} G_\times] \in \mathcal{M}_1^{\text{fr}}$. Say that M is *étale* if $M = M_{\text{ét}}$, *i.e.*, it is a Deligne 1-motive. Say that M is *connected* if $M_{\text{ét}} = 0$, *i.e.*, $F = F^0$ is connected and $G = V(G)$ is a vector group. Say that M is *special* if $u(F^0) \subseteq V(G)$ and set $M^0 := [F^0 \to V(G)]$.

2.1.2. Lemma. *The functor $M \mapsto M_{\text{ét}}$ is a left inverse of the inclusion $\mathcal{M}_1^{\text{fr}} \subset \mathcal{M}_1^{a,\text{fr}}$ of Deligne's 1-motives and for $M' \in \mathcal{M}_1^{\text{fr}}$ we have*

$$\operatorname{Hom}(M, M') \subseteq \operatorname{Hom}(M_{\text{ét}}, M')$$

If M is special we then get an extension

$$0 \to M^0 \to M \to M_{\text{ét}} \to 0 \tag{8}$$

such that if M' is étale then $\operatorname{Hom}(M_{\text{ét}}, M') \cong \operatorname{Hom}(M, M')$ *and if M' is connected then* $\operatorname{Hom}(M', M^0) \cong \operatorname{Hom}(M', M)$.

Proof. Let $M = [F \xrightarrow{u} G] \in \mathcal{M}_1^{a,\mathrm{fr}}$, $M' = [F' \xrightarrow{u'} G'] \in \mathcal{M}_1^{\mathrm{fr}}$. Let $(f, g) : M \to M'$ be a map. Then get a diagram (*cf.* the proof of 1.2.2)

$$\begin{array}{ccc} F & \xrightarrow{u} & G \\ \uparrow\downarrow & & \downarrow \\ F_{\text{ét}} & \xrightarrow{u_{\text{ét}}} & G_\times \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ F' & \xrightarrow{u'} & G' \end{array}$$

where $\overline{f}$ and $\overline{g}$ are the induced maps since M' is étale, yielding a map $(\overline{f}, \overline{g}) : M_{\text{ét}} \to M'$. In fact, $\mathrm{Hom}(F, F') = \mathrm{Hom}(F_{\text{ét}}, F')$ because F' is étale and F^0 is mapped to zero and $\mathrm{Hom}(G, G') = \mathrm{Hom}(G_\times, G')$ because $\mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_a, A) = 0$ and G' is semi-abelian. Moreover, $M \to M_{\text{ét}}$ if M is special, yielding (8). For the isomorphisms then note that $\mathrm{Hom}(M^0, M') = 0$ if M' is étale and, equivalently, $\mathrm{Hom}(M', M_{\text{ét}}) = 0$ if M' is connected. □

In general, we can regard $M_{\text{ét}}$ as a sub-1-motive of $M/V(G)$ and we obtain (*cf.* (4)) a canonical exact sequence

$$0 \to M_{\text{ét}} \to M/V(G) \to F^0[1] \to 0 \tag{9}$$

Denote $M_{\text{ét}}^\natural = [F_{\text{ét}} \xrightarrow{u^\natural} G^\natural] \in \mathcal{M}_1^{a,\mathrm{fr}}$ (*cf.* [5]) the universal $\mathbb{G}_a$-extension of $M_{\text{ét}}$. The algebraic group $G^\natural$ can be represented by an extension

$$0 \to \mathrm{Ext}(M_{\text{ét}}, \mathbb{G}_a)^\vee \to G^\natural \to G_\times \to 0 \tag{10}$$

where $\mathrm{Ext}(M_{\text{ét}}, \mathbb{G}_a)^\vee$ is given by the dual vector space of $\mathbb{G}_a$-extensions of $M_{\text{ét}}$. The map $u^\natural : F_{\text{ét}} \to G^\natural$ is a canonical lifting of $u_{\text{ét}} : F_{\text{ét}} \to G_\times$.

Set $k = \mathbb{C}$. Recall that Deligne's Hodge realization (see [5])

$$T_{Hodge}(M_{\text{ét}}) := (H_\mathbb{Z}, W_*, F^0_{Hodge})$$

of $M_{\text{ét}}$ is given by the pull-back

$$\begin{array}{ccc} H_\mathbb{Z} & \xrightarrow{\overline{v}_\mathbb{Z}} & \mathrm{Lie}\,(G_\times) \\ \downarrow & & \downarrow{\scriptstyle \exp} \\ F_{\text{ét}} & \xrightarrow{u_{\text{ét}}} & G_\times \end{array}$$

Here $W_{-1} := H_1(G_\times)$, $W_{-2} := H_1(T)$ and

$$F^0_{Hodge} := \mathrm{Ker}(H_\mathbb{C} \to \mathrm{Lie}\,(G_\times))$$

2.1.3. Lemma. ([5, 10.1]) *For $k = \mathbb{C}$ we have an isomorphism*

$$M_{\text{ét}}^\natural \cong [H_\mathbb{Z}/W_{-1} \xrightarrow{\overline{t}} H_\mathbb{C}/W_{-1}]$$

here $\overline{t}$ is the induced map $t : H_\mathbb{Z} \to H_\mathbb{C} \mod W_{-1}(H_\mathbb{Z})$.

Actually (see [5, 10.1.8]) we have a bifiltered isomorphism (*i.e.*, "periods")

$$\tau : \mathrm{Lie}\,(G^\natural) \xrightarrow{\cong} H_\mathbb{C}$$

such that

$$
(11)\qquad
\begin{array}{ccccc}
H_{\mathbb{Z}} & \xrightarrow{v^\natural} & \mathrm{Lie}\,(G^\natural) & \xrightarrow{\tau} & H_{\mathbb{C}}\\
\| & & \downarrow & & \downarrow\\
H_{\mathbb{Z}} & \xrightarrow{\overline{v}_{\mathbb{Z}}} & \mathrm{Lie}\,(G_\times) & \xrightarrow{\overline{\tau}} & H_{\mathbb{C}}/F^0_{Hodge}
\end{array}
$$

commutes. Here $t = \tau v^\natural$ where $v^\natural$ is the canonical map induced by $u^\natural$, $\mathrm{Lie}\,(G^\natural)$ is the pullback of (10) along exp, $H_1(G^\natural) \cong H_1(G_\times) = W_{-1}(H_{\mathbb{Z}})$ and $\mathrm{Ext}(M_{\text{ét}}, \mathbb{G}_a)^\vee \cong F^0_{Hodge}$.

2.1.4. Example. (*cf.* [1, 1.1 & 3.3]) For X proper over a field k, char $(k) = 0$, set $G := \mathrm{Pic}^0_{X/k}$ and let $M = [0 \to G]$ be the corresponding 1-motive. Here $G_\times \cong \mathbb{P}\mathrm{ic}^0_{X\cdot/k}$ and $G^\natural \cong \mathbb{P}\mathrm{ic}^{\natural,0}_{X\cdot/k}$ are given by simplicial Pic and $\natural -$ Pic functors of a smooth proper hypercovering $X\cdot$ of X. Thus $H_{\mathbb{Z}} = H^1(X_{\mathrm{an}}, \mathbb{Z})$, $\mathrm{Lie}\,(G^\natural) = H^1_{DR}(X)$ and $\tau : H^1_{DR}(X) \cong H^1(X_{\mathrm{an}}, \mathbb{C})$ by cohomological descent over $k = \mathbb{C}$.

2.2. Construction. Extending Deligne's Hodge realization for a given Laumon 1-motive $M = [F \xrightarrow{u} G]$ over $\mathbb{C}$ consider the pull-back $T_\oint(F)$ of $u : F \to G$ along $\exp : \mathrm{Lie}\,(G) \to G$, *i.e.*, it fits in the following diagram

$$
\begin{array}{ccccc}
0 \to H_1(G) & \longrightarrow & T_\oint(F) & \longrightarrow & F \to 0\\
\| & & \downarrow & & \downarrow u\\
0 \to H_1(G) & \longrightarrow & \mathrm{Lie}\,(G) & \xrightarrow{\exp} & G \to 0
\end{array}
$$

Here $T_\oint(F)$ is a formal group and we get a natural group homomorphism $v : T_\oint(F) \to \mathrm{Lie}\,(G)$. We are going to show that

$$T_\oint(M) := (T_\oint(F), \mathrm{Lie}\,(G)) \in \mathrm{FHS}_1^{\mathrm{fr}}$$

is a formal Hodge structure. Note that if M is connected then $T_\oint(M) = M$.

2.2.1. Remark. The additional data coming from Lie is really needed if we allow additive factors! For example, let $W \to V$ be a linear map between $\mathbb{C}$-vector spaces, and let $M = [\widehat{W} \xrightarrow{u} V]$ be the induced 1-motive where $\widehat{W}$ is the formal completion at the origin (*cf.* [6, 5.2.5]). Note that all connected 1-motives are obtained in this way (see 1.3.2). For any embedding $V \subsetneq V'$ of vector spaces, we obtain another 1-motive $M' = [\widehat{W} \xrightarrow{u'} V']$ such that $M \subsetneq M'$. For both M and M' then $T_\oint(\widehat{W})$ is the infinitesimal group $\widehat{W}$, $\mathrm{Ker}(u) = \mathrm{Ker}(u')$ and we cannot distinguish M by M' out of the formal group only.

2.2.2. Lemma. *We have that $T_\oint(F)$ is the formal group $F^0 \times H_{\mathbb{Z}}$ such that $H_{\mathbb{Z}}$ is the above extension of $F_{\text{ét}}$ by $H_1(G_\times)$.*

Proof. Since formal groups are closed under extensions (*cf.* [6, 4.3.1]) $T_\oint(F)$ is a formal group, *i.e.*, it is, by construction, an extension of F by $H_1(G)$. Observe that (7) yields $\mathrm{Lie}\,(G)$ as the pullback of $\mathrm{Lie}\,(G_\times)$ along exp and $H_1(G) \cong H_1(G_\times)$. We then get a natural identification of $H_{\mathbb{Z}}$ with the étale part of $T_\oint(F)$, *i.e.*, with the pullback of $F_{\text{ét}} \hookrightarrow F$ along $T_\oint(F) \to F$. □

2.2.3. Lemma. *If* $\sigma := \overline{\tau}^{-1} : H_{\mathbb{C}}/F^0_{Hodge} \xrightarrow{\simeq} \mathrm{Lie}\,(G_\times)$ *is the isomorphism induced by* (11) *then* σ *restricts to* $W_{-2}(H_{\mathbb{C}}) \cong \mathrm{Lie}\,(T)$ *and the following*

$$\begin{array}{ccc} H_{\mathbb{Z}} & \xrightarrow{v_{\mathbb{Z}}} & \mathrm{Lie}\,(G) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle pr} \\ H_{\mathbb{C}}/F^0_{Hodge} & \xrightarrow{\sigma} & \mathrm{Lie}\,(G_\times) \end{array}$$

commutes (here $v_{\mathbb{Z}}$ *is the restriction of* v *and* c *is the canonical map cf.* (1)*).*

Proof. Note that $c = \overline{\tau}\circ\overline{v}_{\mathbb{Z}}$ in (11) and $\overline{v}_{\mathbb{Z}} = \mathrm{pr}\circ v_{\mathbb{Z}}$ by Lemma 2.2.2. □

2.2.4. Definition. Denote $T_\oint(M)$ the formal Hodge structure $(H,V) \in \mathrm{FHS}_1^{\mathrm{fr}}$ where
- *(i)* $H := T_\oint(F) = F^0 \times H_{\mathbb{Z}}$, $H_{\text{ét}} := T_{Hodge}(M_{\text{ét}})$, granted by Lemma 2.2.2,
- *(ii)* $V := \mathrm{Lie}\,(G)$, $V^1 := \mathrm{Lie}\,(T) + V(G)$ and $V^0 := V(G)$,
- *(iii)* the map $v : T_\oint(F) \to \mathrm{Lie}\,(G)$ defined above, and
- *(iv)* the isomorphism $\sigma := \overline{\tau}^{-1} : H_{\mathbb{C}}/F^0_{Hodge} \xrightarrow{\simeq} \mathrm{Lie}\,(G_\times)$ grants (1) by Lemma 2.2.3.

We then have $T_\oint(M)_{\text{ét}} = T_{Hodge}(M_{\text{ét}}) \in \mathrm{MHS}_1^{\mathrm{fr}}$ and the construction is clearly functorial (since the diagram (11) is natural) providing a functor

$$T_\oint : \mathcal{M}_1^{a,\mathrm{fr}} \longrightarrow \mathrm{FHS}_1^{\mathrm{fr}}$$

such that $T_\oint(M) = T_{Hodge}(M)$ if M is étale (*via* 1.2.2) and $T_\oint(M) = M$ if M is connected.

2.2.5. Remark. Note that by applying $T_\oint$ to (9) we get (4), the extension (8) yields (5) and M is special $\iff$ $T_\oint(M)$ is special.

2.3. Conclusion. Summarizing up, see also 1.2 and 1.4, the theorem is proven, *e.g.*, in order to show that $T_\oint$ yields an equivalence of categories we can argue as in [5, 10.1.3]. For $(H,V) \in \mathrm{FHS}_1^{\mathrm{fr}}$ we have constructed, in 1.4, a 1-motive

$$\overrightarrow{(H,V)} := [H^0 \times \mathrm{gr}_0^W(H_{\mathbb{Z}}) \to V/W_{-1}(H_{\mathbb{Z}})]$$

It is clear that $T_\oint(\overrightarrow{(H,V)}) \cong (H,V)$, see (6), which is natural in (H,V). Conversely, for $M = [F \to G]$ we have $T_\oint(M) := (T_\oint(F), \mathrm{Lie}\,(G))$ such that $\overrightarrow{T_\oint(M)} \cong M$ functorially in M by construction. One obtains a duality on $\mathrm{FHS}_1^{\mathrm{fr}}$ after Cartier duality on $\mathcal{M}_1^{a,\mathrm{fr}}$ by defining

$$T_\oint(M)^\vee := T_\oint(M^\vee)$$

The lemmas 1.2.2 and 2.1.2 further explain the diagram of the main theorem and the remaining claims.

2.3.1. Example. (*cf.* 2.1.4) For X proper over $\mathbb{C}$ and $M = [0 \to \mathrm{Pic}^0_{X/\mathbb{C}}]$ we have $T_\oint(M) = (H^1(X_{\mathrm{an}}, \mathbb{Z}(1)), H^1(X, \mathcal{O}_X))$. Here we have $M_{\text{ét}} = [0 \to \mathbb{P}\mathrm{ic}^0_{X_\cdot/\mathbb{C}}]$ and a projection

$$\begin{array}{ccc} \mathrm{Lie}\,\mathrm{Pic}^0_{X/\mathbb{C}} & \xrightarrow{\simeq} & H^1(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ \mathrm{Lie}\,\mathbb{P}\mathrm{ic}^0_{X_\cdot/\mathbb{C}} & \xrightarrow{\simeq} & \mathbb{H}^1(X_\cdot, \mathcal{O}_{X_\cdot}) \end{array}$$

with kernel the additive factor of $\mathrm{Pic}^0_{X/\mathbb{C}}$. Further considering $M^\natural_{\text{ét}} = [0 \to \mathbb{P}\mathrm{ic}^{\natural,0}_{X_\cdot/\mathbb{C}}]$ and $T_\oint(M^\natural_{\text{ét}}) = (H^1(X_{\mathrm{an}}, \mathbb{Z}(1)), H^1_{DR}(X))$ we get the extension

$$0 \to F^0_{Hodge} \to T_\oint(M^\natural_{\text{ét}}) \to T_{Hodge}(M_{\text{ét}}) \to 0$$

2.3.2. Remark. Note that in 2.3.1 M is special but the dual $M^\vee$ is not special! Another more striking example is given by taking an abelian variety X and looking at the special 1-motive $[0 \to \mathrm{Pic}^{\natural,0}_{X/\mathbb{C}}]$ which is the universal extension of the dual of X. The Cartier dual

$$[0 \to \mathrm{Pic}^{\natural,0}_{X/\mathbb{C}}]^\vee = [\widehat{X} \to X]$$

is not special. Actually, in general, the Cartier dual of a connected 1-motive is connected and the dual of étale is étale but the Cartier dual of M special just fits in an extension

$$0 \to M^\vee_{\text{ét}} \to M^\vee \to (M^0)^\vee \to 0$$

dual to (8).

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