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A NOTE ON EXISTENCE AND NON-EXISTENCE OF HORIZONS IN SOME ASYMPTOTICALLY FLAT 3-MANIFOLDS

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ABSTRACT. We consider asymptotically flat manifolds of the form $(S^3 \setminus \{P\}, G^4 g)$, where G is the Green's function of the conformal Laplacian of (S^3, g) at a point P. We show if $Ric(g) \geq 2g$ and the volume of (S^3, g) is no less than one half of the volume of the standard unit sphere, then there are no closed minimal surfaces in $(S^3 \setminus \{P\}, G^4 g)$. We also give an example of (S^3, g) where $Ric(g) > 0$ but $(S^3 \setminus \{P\}, G^4 g)$ does have closed minimal surfaces.

1. Introduction

Let (N^3, g, p) be an initial data set satisfying the dominant energy constraint condition in general relativity. It is a fascinating question to ask under what conditions an *apparent horizon* (of a back hole) exists in (N^3, g, p). Here an apparent horizon is a 2-surface $\Sigma^2 \subset N^3$ satisfying

$$H_\Sigma = \mathrm{Tr}_\Sigma p, \tag{1}$$

where H_Σ is the mean curvature of Σ in N and $\mathrm{Tr}_\Sigma p$ is the trace of the restriction of p to Σ.

A fundamental result of Schoen and Yau states that *matter condensation* causes apparent horizons to be formed [11]. Their result is remarkable not only because it provides a general criteria to the existence question, but also because it leads to a refined problem – besides matter fields, what is the *pure effect of gravity* on the formation of apparent horizons?

To analyze this refined problem, one considers an asymptotically flat initial data set (N^3, g, p) in a *vacuum* spacetime. As the first step, one assumes (N^3, g, p) is time-symmetric (i.e. $p \equiv 0$). In this context, an apparent horizon is simply a *minimal surface*, and the relevant topological assumption is that N^3 is diffeomorphic to $\mathbb{R}^3$. (If N^3 has nontrivial topology, a closed minimal surface always exists by [8].)

There is a geometric construction of such an initial data set. Let $[g]$ be a conformal class of metrics on the three-sphere S^3. Recall the Yamabe constant of $(S^3, [g])$ is defined by

$$Y(S^3, [g]) = \inf_{v \in W^{1,2}(S^3)} \frac{\int_M [8|\nabla v|_g^2 + R(g)v^2] dV_g}{\left(\int_M v^6 dV_g\right)^{\frac{1}{3}}}, \tag{2}$$

where $R(g)$ is the scalar curvature of g. If $Y(S^3, [g]) > 0$, there exists a positive Green's function G of the conformal Laplacian $8\triangle_g - R(g)$ at any fixed point $P \in S^3$. Consider the new metric $G^4 g$ on $S^3 \setminus \{P\}$, it is easily checked that $(S^3 \setminus \{P\}, G^4 g)$ is

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asymptotically flat with zero scalar curvature. One basic fact about this construction is that the blowing-up manifold $(S^3 \setminus \{P\}, G^4 g)$, up to a constant scaling, depends only on the conformal class $[g]$. Precisely, if one replaces g by another metric $\bar{g} \in [g]$ and let $\bar{G}$ be the Green's function associated to $\bar{g}$, then the metric $\bar{G}^4 \bar{g}$ differs from $G^4 g$ only by a constant multiple. Therefore, it is of interest to seek conditions on $[g]$ that determine whether $(S^3 \setminus \{P\}, G^4 g)$ has a horizon.

So far, no such a conformal invariant condition has been found. However, there are results where conditions in terms of a single metric are given. In [1], Beig and Ó Murchadha studied the behavior of a *critical sequence*, i.e. a sequence of metrics $\{g_n\}$ on S^3 converging to a metric g_0 with zero scalar curvature. They showed the blowing-up manifold $(S^3 \setminus \{P\}, G_n^4 g_n)$ has a horizon for sufficiently large n. Their idea was further explored by Yan [12]. Given a metric g on S^3, assuming the diameter of $(S^3, g) \leq D$, the volume of $(S^3, g) \geq V$ and the Ricci curvature of g satisfies $Ric(g) \geq \mu g$, Yan showed that, for any $r > \frac{3}{2}$, there exists a small positive number $\delta = \delta(\mu, V, D, r) \leq 1$ such that, if $R(g) > 0$ and $||R(g)||_{L^r(S^3, g)} < \delta$, then the blowing-up manifold $(S^3 \setminus \{P\}, G^4 g)$ has a horizon.

One question arising from Yan's theorem is whether a *positive* Ricci curvature metric on S^3 can produce a blowing-up manifold with a horizon, as it is unclear whether Yan's theorem could be applied when $\mu > 0$. Another motivation to this question is, as a positive Ricci curvature metric can be deformed to the standard metric on S^3 through metrics of positive Ricci curvature, it is of potential interest to study how the horizon disappears in the corresponding deformation of the blowing-up manifold if it exists initially.

In this paper, we focus on conformal classes of metrics with a positive Ricci curvature metric. Our main result is the observation of a volume condition which guarantees non-existence of horizons in the blowing-up manifold. Throughout the paper, $\mathbb{S}^3$ denotes S^3 with the standard metric of constant curvature $+1$.

Theorem *Let $[g]$ be a conformal class of metrics on S^3 which has a metric of positive Ricci curvature. Consider*

$$V_{max}(S^3, [g]) = \sup_{\bar{g} \in [g]} \{Vol(S^3, \bar{g}) \mid Ric(\bar{g}) \geq 2\bar{g}\},$$

where $Vol(\cdot)$ is the volume functional. If

$$V_{max}(S^3, [g]) \geq \frac{1}{2} Vol(\mathbb{S}^3),$$

then the blowing-up manifold $(S^3 \setminus \{P\}, G^4 g)$ has no horizon.

We also give an example of (S^3, g) where $Ric(g) > 0$ and $(S^3 \setminus \{P\}, G^4 g)$ does have horizons.

2. Positive Ricci curvature and maximum volume

We first explain the volume assumption in the Theorem. Let M^n be a smooth, connected, closed manifold of dimension $n \geq 3$. Assume $[g]$ is a conformal class of metrics on M^n which has a metric of positive Ricci curvature. One can define

$$V_{max}(M^n, [g]) = \sup_{\bar{g} \in [g]} \{Vol(M^n, \bar{g}) \mid Ric(\bar{g}) \geq (n-1)\bar{g}\}. \tag{3}$$

The following result relating $V_{max}(M^n,[g])$ and the Yamabe constant of $(M^n,[g])$ was observed in [5].

Proposition 1. *Let $[g]$ be a conformal class of metrics on M^n which has a metric of positive Ricci curvature. Then the Yamabe constant of $(M^n,[g])$ satisfies*

$$Y(M^n,[g]) \geq n(n-1)V_{max}(M^n,[g])^{\frac{2}{n}}. \tag{4}$$

Proof. By definition,

$$Y(M^n,[g]) = \inf_{v\in W^{1,2}(M)} \frac{\int_M [c_n|\nabla v|^2_{\bar g} + R(\bar g)v^2]dV_{\bar g}}{\left(\int_M v^{\frac{2n}{n-2}}dV_{\bar g}\right)^{\frac{n-2}{n}}} \tag{5}$$

for any $\bar g \in [g]$, where $c_n = \frac{4(n-1)}{n-2}$.

Assume $Ric(\bar g) \geq (n-1)\bar g$, by a result of Ilias [7], which is based on the isoperimetric inequality of Gromov [9], we have

$$\int_M [c_n|\nabla v|^2_{\bar g} + n(n-1)v^2]dV_{\bar g} \geq \left(\int_M v^{\frac{2n}{n-2}}dV_{\bar g}\right)^{\frac{n-2}{n}} n(n-1)Vol(M^n,\bar g)^{\frac{2}{n}} \tag{6}$$

for any $v \in W^{1,2}(M)$. Note that $R(\bar g) \geq n(n-1)$, hence

$$\begin{aligned} Y(M^n,[g]) &\geq \inf_{v\in W^{1,2}(M)} \frac{\int_M [c_n|\nabla v|^2_{\bar g} + n(n-1)v^2]dV_{\bar g}}{\left(\int_M v^{\frac{2n}{n-2}}dV_{\bar g}\right)^{\frac{n-2}{n}}} \\ &\geq n(n-1)Vol(M^n,\bar g)^{\frac{2}{n}}. \qquad (7) \end{aligned}$$

Taking the supremum over $\bar g \in [g]$ satisfying $Ric(\bar g) \geq (n-1)\bar g$, we have

$$Y(M^n,[g]) \geq n(n-1)V_{max}(M^n,[g])^{\frac{2}{n}}. \tag{8}$$

□

As an immediate corollary, we see the assumption

$$V_{max}(S^3,[g]) \geq \frac{1}{2}Vol(\mathbb{S}^3)$$

in the Theorem implies

$$\begin{aligned} Y(S^3,[g]) &\geq 6\left(\frac{1}{2}\right)^{\frac{2}{3}} Vol(\mathbb{S}^3)^{\frac{2}{3}} \\ &= Y(RP^3,[g_0]), \qquad (9) \end{aligned}$$

where RP^3 is the three dimensional projective space and g_0 is the standard metric on RP^3 which has constant sectional curvature $+1$.

3. An upper bound of the Sobolev constant when a horizon is present

One basic fact relating the conformal class $[g]$ on S^3 and the blowing-up metric $h = G^4 g$ on $\mathbb{R}^3 = S^3 \setminus \{P\}$ is

$$Y(S^3, [g]) = 8S(h), \tag{10}$$

where $S(h)$ is the Sobolev constant of the asymptotically flat manifold $(\mathbb{R}^3, h)$ [3]. Recall $S(h)$ is defined by

$$S(h) = \inf_{u \in W^{1,2}(\mathbb{R}^3, h)} \left\{ \frac{\int_{\mathbb{R}^3} |\nabla u|_h^2 \, dV_h}{(\int_{\mathbb{R}^3} u^6 \, dV_h)^{\frac{1}{3}}} \right\}. \tag{11}$$

The next proposition, which plays a key role in the derivation of the Theorem, was essentially established by Bray and Neves in [3] using the inverse mean curvature flow technique [6]. As the statement of Bray and Neves is different from what we need, we include the proof here.

Proposition 2. *Let h be a complete metric on $\mathbb{R}^3$ such that $(\mathbb{R}^3, h)$ is asymptotically flat. If $(\mathbb{R}^3, h)$ has nonnegative scalar curvature and has a closed minimal surface, then*

$$S(h) < \frac{1}{8} Y(RP^3, [g_0]). \tag{12}$$

Proof. Since $(\mathbb{R}^3, h)$ has a closed minimal surface, the *outermost* minimal surface $\mathcal{S}$ in $(\mathbb{R}^3, h)$, i.e. the closed minimal surface that is not enclosed by any other minimal surface [2], exists and consists of a finite union of disjoint, embedded minimal two-spheres and projective planes. As our background manifold is $\mathbb{R}^3$, $\mathcal{S}$ must consist of embedded minimal two-spheres alone, furthermore each component of $\mathcal{S}$ necessarily bounds a three-ball.

We fix a component Σ of $\mathcal{S}$ and denote by Ω the three-ball that Σ bounds in $\mathbb{R}^3$. Let ϕ be the weak solution to the inverse mean curvature flow in $(\mathbb{R}^3 \setminus \bar{\Omega},\ h)$ with initial condition Σ [6]. ϕ satisfies

$$\phi \geq 0, \ \phi|_\Sigma = 0, \quad \lim_{x \to \infty} \phi = \infty.$$

Let Σ_t be the set $\partial\{u < t\}$ for $t > 0$ and Σ_0 be the starting surface Σ, then the family of surfaces $\{\Sigma_t\}$ satisfies the following properties [6]:

(1) $\{\Sigma_t\}$ consists of $C^{1,\alpha}$ surfaces. For a.e. t, Σ_t has weak mean curvature H and $H = |\nabla u|_h$ for a.e. $x \in \Sigma_t$.
(2) $|\Sigma_t| = e^t |\Sigma_0|$, where $|\Sigma_t|$ denotes the area of Σ_t.
(3) Since $(\mathbb{R}^3, h)$ has nonnegative scalar curvature, Σ is connected and $\mathbb{R}^3 \setminus \bar{\Omega}$ is simply connected, the Hawking quasi-local mass of Σ_t,

$$m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma_t} H^2 d\mu \right),$$

is monotone increasing. Here $d\mu$ is the induced surface measure.

Now we restrict attention to functions $u \in W^{1,2}(\mathbb{R}^3, h)$ that have the form

$$u(x) = \begin{cases} f(0) & x \in \Omega \\ f(\phi(x)) & x \in \mathbb{R}^3 \setminus \Omega \end{cases} \tag{13}$$

for some C^1 functions $f(t)$ defined on $[0,\infty)$. By the coarea formula and Property 1 above, we have

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla u|_h^2 dV_h &= \int_0^\infty f'(t)^2\left(\int_{\Sigma_t} H d\mu\right)dt\\ &\le \int_0^\infty f'(t)^2\sqrt{16\pi|\Sigma|(e^t-e^{\frac{t}{2}})}dt, \qquad (14)\end{aligned}$$

where the inequality follows from Property 2, 3 and Hölder's inequality. Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^3}u^6 dV_h &\ge \int_0^\infty f(t)^6\left(\int_{\Sigma_t} H^{-1} d\mu\right)dt\\ &\ge \int_0^\infty f(t)^6 e^{2t}|\Sigma|^2[16\pi|\Sigma|(e^t-e^{\frac{t}{2}})]^{-\frac{1}{2}}dt. \qquad (15)\end{aligned}$$

Therefore,

$$\frac{\int_{\mathbb{R}^3}|\nabla u|_h^2 dV_h}{\left(\int_{\mathbb{R}^3}u^6 dV_h\right)^{\frac{1}{3}}} \le \frac{(16\pi)^{\frac{2}{3}}\int_0^\infty f'(t)^2(e^t-e^{\frac{t}{2}})^{\frac{1}{2}}dt}{\left(\int_0^\infty f(t)^6 e^{2t}(e^t-e^{\frac{t}{2}})^{-\frac{1}{2}}dt\right)^{\frac{1}{3}}}. \qquad (16)$$

To pick an optimal $f(t)$ that minimizes the right side of (16), we consider the half spatial Schwarzschild manifold

$$(M^3, g_S) = (\mathbb{R}^3\setminus B_1(0), (1+\frac{1}{|x|})^4\delta_{ij})$$

and the quotient manifold $(\tilde{M}^3, \tilde{g}_S)$ obtained from (M^3, g_S) by identifying the antipodal points of $\{|x|=1\}$. Up to scaling, $(\tilde{M}^3, \tilde{g}_S)$ is isometric to $(RP^3\setminus\{Q\}, G_0^4 g_0)$, the blowing-up manifold of (RP^3, g_0) by its Green function at a point Q. Hence, the Sobolev constant $S(\tilde{g}_S)$ of $(\tilde{M}^3, \tilde{g}_S)$ equals $\frac{1}{8}Y(RP^3,[g_0])$. On the other hand, $S(\tilde{g}_S)$ is achieved by a function u_0 that is a constant on each coordinate sphere $\{|x|=t\}$ in $\tilde{M}$, and the level set of the solution ϕ_0 to the inverse mean curvature flow starting at $\{|x|=1\}$ in (M, g_S) is also given by coordinate spheres. Therefore, lifted as a function on (M^3, g_S), u_0 has the form of

$$u_0 = f_0\circ\phi_0$$

for some explicitly determined function $f_0(t)$, and

$$S(\tilde{g}_S) = \frac{\int_M|\nabla u_0|_{g_S}^2 dV_{g_S}}{(\int_M u_0^6\, dV_{g_S})^{\frac{1}{3}}} = \frac{(16\pi)^{\frac{2}{3}}\int_0^\infty f_0'(t)^2(e^t-e^{\frac{t}{2}})^{\frac{1}{2}}dt}{\left(\int_0^\infty f_0(t)^6 e^{2t}(e^t-e^{\frac{t}{2}})^{-\frac{1}{2}}dt\right)^{\frac{1}{3}}}, \qquad (17)$$

where the second equality holds because the Hawking quasi-local mass remains unchanged along the level sets of ϕ_0. Now consider $u = f_0\circ\phi$ on $(\mathbb{R}^3, h)$. It was verified in [3] that $u\in W^{1,2}(\mathbb{R}^3, h)$. Therefore, we have

$$\begin{aligned}S(h)\le\frac{\int_{\mathbb{R}^3}|\nabla u|_h^2\, dV_h}{(\int_{\mathbb{R}^3}u^6\, dV_h)^{\frac{1}{3}}} &\le \frac{(16\pi)^{\frac{2}{3}}\int_0^\infty f_0'(t)^2(e^t-e^{\frac{t}{2}})^{\frac{1}{2}}dt}{\left(\int_0^\infty f_0(t)^6 e^{2t}(e^t-e^{\frac{t}{2}})^{-\frac{1}{2}}dt\right)^{\frac{1}{3}}}\\ &= S(\tilde{g}_S) = \frac{1}{8}Y(RP^3,[g_0]). \qquad (18)\end{aligned}$$

To show the strict inequality, we assume $S(h) = \frac{1}{8}Y(RP^3, [g_0])$. Then, $S(h)$ is achieved by $u = f_0 \circ \phi$. It follows from the Euler-Lagrange equation of the Sobolev functional (11) that u satisfies

$$\triangle_h u + Cu^5 = 0 \quad \text{on } \mathbb{R}^3, \tag{19}$$

where $C = S(h)||u||^{-4}_{L^6(\mathbb{R}^3,h)}$. However, $u \equiv f_0(0)$ on Ω and $f_0(0) \neq 0$ (Indeed, up to a constant multiple, $f_0(t) = (2e^t - e^{\frac{t}{2}})^{-\frac{1}{2}}$ [3]). Hence, $C = 0$, which contradicts to the fact that u is not a constant. Therefore, the strict inequality $S(h) < \frac{1}{8}Y(RP^3, [g_0])$ holds. □

Proof of the Theorem: Suppose $(S^3 \setminus \{P\}, G^4 g)$ has a horizon, then it follows from (10) and Proposition 2 that

$$Y(S^3, [g]) < Y(RP^3, [g_0]). \tag{20}$$

On the other hand, the assumption $V_{max}(S^3, [g]) \geq \frac{1}{2}Vol(\mathbb{S}^3)$ implies

$$Y(S^3, [g]) \geq Y(RP^3, [g_0]) \tag{21}$$

by (9), which is a contradiction. Hence, there are no horizons in $(S^3 \setminus \{P\}, G^4 g)$. □

4. An example with horizons

In this section, we provide an example to show that there exist metrics on S^3 with positive Ricci curvature such that the blowing-up manifolds do have horizons.

Our example comes from a 1-parameter family of left-invariant metrics $\{g_\epsilon\}$ on S^3, commonly known as the *Berger metrics*. Precisely, we think S^3 as the Lie Group

$$SU(2) = \left\{ \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix} \ : \ |z|^2 + |w|^2 = 1 \right\},$$

where the Lie algebra of $SU(2)$ is spanned by

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then $\{g_\epsilon\}$ is defined by declaring X_1, X_2, X_3 to be orthogonal, X_1 to have length ϵ and X_2, X_3 to be unit vectors. Note that scalar multiplication on $S^3 \subset \mathbb{C}^2$ corresponds to multiplication on the left by matrices $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ on $SU(2)$, hence X_1 is exactly tangent to the circle fiber of the *Hopf fibration*

$$\pi : S^3 \longrightarrow S^2 = S^3/S^1$$

and g_ϵ shrinks the circle fiber as $\epsilon \to 0$. One fact of g_ϵ for small ϵ is that all sectional curvature of (S^3, g_ϵ) lies in the interval $[\epsilon^2, 4 - 3\epsilon^2]$ (see [10]), in particular g_ϵ has positive Ricci curvature.

Proposition 3. *Let $P \in S^3$ be a fixed point and G_ϵ be the Green's function of the conformal Laplacian of g_ϵ at P. Then $(S^3 \setminus \{P\}, G_\epsilon^4 g_\epsilon)$ has a horizon for ϵ sufficiently small.*

Proof. For each $\epsilon \in (0,1]$, we consider the rescaled metric $\bar{g}_\epsilon = \epsilon^{-2} g_\epsilon$ and the Green's function $\bar{G}_\epsilon$ associated to $\bar{g}_\epsilon$ at P. Then, with respect to $\bar{g}_\epsilon$, X_1 becomes a unit vector and X_2, X_3 have large length ϵ^{-1} as $\epsilon \to 0$. Let $U \subset S^3$ be a fixed neighborhood of P such that $\pi|_U$ is a trivial fiberation. Let O be a fixed point in the product manifold $S^1 \times \mathbb{R}^2$. By a scaling argument, there exists a family of diffeomorphisms

$$\Psi_\epsilon : U \longrightarrow \Psi_\epsilon(U) \subset S^1 \times \mathbb{R}^2,$$

such that $\Psi_\epsilon(P) = O \in \Psi_\epsilon(U)$, $\{\Psi_\epsilon(U)\}_{1 \geq \epsilon > 0}$ forms an exhaustion family of $S^1 \times \mathbb{R}^2$ as $\epsilon \to 0$, and the push forward metrics $\hat{g}_\epsilon = \Psi_\epsilon^{-1*}(\bar{g}_\epsilon|_U)$ on $\Psi_\epsilon(U)$ converge in C^2 norm on compact sets to a flat metric $\hat{g}$ on $S^1 \times \mathbb{R}^2$. Now fix another point $Q \in \Psi_1(U)$ that is different from O and consider the normalized function

$$\hat{G}_\epsilon(x) = \frac{\bar{G}_\epsilon \circ \Psi_\epsilon^{-1}(x)}{\bar{G}_\epsilon \circ \Psi_\epsilon^{-1}(Q)} \tag{22}$$

for $x \in \Psi_\epsilon(U) \setminus \{O\}$. Then $\hat{G}_\epsilon$ satisfies

$$\begin{cases} 8\triangle_{\hat{g}_\epsilon} \hat{G}_\epsilon - R(\hat{g}_\epsilon)\hat{G}_\epsilon = 0 \text{ on } \Psi_\epsilon(U) \setminus \{O\} \\ \hat{G}_\epsilon = 1 \text{ at } Q \end{cases}. \tag{23}$$

Since $\hat{G}_\epsilon$ is positive and $\hat{g}_\epsilon$ converges to $\hat{g}$ as $\epsilon \to 0$, it follows from the Harnack inequality that $\hat{G}_\epsilon$ coverges to a positive function $\hat{G}$ on $(S^1 \times \mathbb{R}^2) \setminus \{O\}$ in C^2 norm on any compact set away from $\{O\}$. Furthermore, $\hat{G}$ satisfies

$$\begin{cases} \triangle_{\hat{g}} \hat{G} = 0 \text{ on } (S^1 \times \mathbb{R}^2) \setminus \{O\} \\ \hat{G} = 1 \text{ at } Q \end{cases}. \tag{24}$$

On the other hand, the fact that the geodesic ball in $(S^1 \times \mathbb{R}^2, \hat{g})$ only has quadratic volume growth implies $(S^1 \times \mathbb{R}^2, \hat{g})$ does not have a positive Green's function for the usual Laplacian $\triangle_{\hat{g}}$ [4]. Therefore, $\hat{G} \equiv 1$ on $(S^1 \times \mathbb{R}^2) \setminus \{O\}$. Hence, the metrics $\hat{G}_\epsilon^4 \hat{g}_\epsilon$ converge to $\hat{g}$ in C^2 norm on any compact set away from $\{O\}$. Now let $V \subset S^1 \times \mathbb{R}^2$ be a small open ball containing O such that ∂V is an embedded two sphere whose mean curvature vector computed with respect to $\hat{g}$ points towards O. Then, for sufficiently small ϵ, the mean curvature vector of ∂V computed with respect to $\hat{G}_\epsilon^4 \hat{g}_\epsilon$ still points towards O. As $(\Psi_\epsilon(U), \hat{G}_\epsilon^4 \hat{g}_\epsilon)$ is isometric to $(U, \bar{G}_\epsilon^4 \bar{g}_\epsilon)$, the mean curvature vector of the boundary of $\Psi_\epsilon^{-1}(V)$ in $(S^3 \setminus \{P\}, \bar{G}_\epsilon^4 \bar{g}_\epsilon)$ must point towards the blowing-up point P. On the other hand, as $(S^3 \setminus \{P\}, \bar{G}_\epsilon^4 \bar{g}_\epsilon)$ is asymptotically flat, its infinity is foliated by two spheres whose mean curvature vector points away from P. Therefore, it follows from standard geometric measure theory that there exists an embedded minimal two sphere in $\Psi_\epsilon(V)$, hence $(S^3 \setminus \{P\}, \bar{G}_\epsilon^4 \bar{g})$ has a horizon. □

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