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SUPER-RIGID DONALDSON-THOMAS INVARIANTS

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Abstract. We solve the part of the Donaldson-Thomas theory of Calabi-Yau threefolds which comes from super-rigid rational curves. As an application, we prove a version of the conjectural Gromov-Witten/Donaldson-Thomas correspondence of [MNOP] for contributions from super-rigid rational curves. In particular, we prove the full GW/DT correspondence for the quintic threefold in degrees one and two.

1. Introduction

Let Y be a smooth complex projective Calabi-Yau threefold. Let $I_n(Y,\beta)$ be the moduli space of ideal sheaves $I_Z \subset \mathcal{O}_Y$, where the associated subscheme Z has maximal dimension equal to one, the holomorphic Euler characteristic $\chi(\mathcal{O}_Z)$ is equal to n, and the associated 1-cycle has class $\beta \in H_2(Y)$.

Recall that $I_n(Y,\beta)$ has a natural symmetric obstruction theory [Th00], [BF05]. Hence we have the (degree zero) virtual fundamental class of $I_n(Y,\beta)$, whose degree $N_n(Y,\beta) \in \mathbb{Z}$ is the associated Donaldson-Thomas invariant.

Let

$$C = \sum_{i=1}^{s} d_i C_i$$

be an effective cycle on Y, and assume that the C_i are pairwise disjoint, smoothly embedded rational curves with normal bundle $N_{C_i/Y} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Such curves are called *super-rigid rational curves* in Y [Pa99, BP01]. Assume that the class of C is β.

Let $J_n(Y,C) \subset I_n(Y,\beta)$ be the locus corresponding to subschemes $Z \subset Y$ whose associated cycle under the Hilbert-Chow morphism is equal to C (see Definition 2.1). Since $J_n(Y,C) \subset I_n(Y,\beta)$ is open and closed (see Remark 2.2), we get an induced virtual fundamental class on $J_n(X,C)$ by restriction. We call

$$N_n(Y,C) = \deg[J_n(Y,C)]^{\mathrm{vir}}$$

the *contribution of* C to the Donaldson-Thomas invariant $N_n(Y,\beta)$.

The goal of this paper is to compute the invariants $N_n(Y,C)$.

To formulate our results, we define a series $P_d(q) \in \mathbb{Z}[[q]]$, for all integers $d \geq 0$ by

$$\prod_{m=1}^{\infty} (1+q^m v)^m = \sum_{d=0}^{\infty} P_d(q) v^d \,. \tag{1}$$

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Moreover, recall the McMahon function

$$M(q) = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \,. \tag{2}$$

Then we prove (Theorem 2.14) that

$$\sum_{n=0}^{\infty} N_n(Y,C) q^n = M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q) \,.$$

Maulik, Nekrasov, Okounkov, and Pandharipande have conjectured a beautiful correspondence between Gromov-Witten theory and Donaldson-Thomas theory which we call the GW/DT correspondence.

As an application of the above formula we prove the GW/DT correspondence for the contributions from super-rigid rational curves (Theorem 3.1). In particular, we prove the full degree β GW/DT correspondence (Conjecture 3 of [MNOP]) for any β for which it is known that all cycle representatives are supported on super-rigid rational curves (Corollary 3.2). For example, our results yield the GW/DT correspondence for the quintic threefold in degrees one and two (Corollary 3.3). As far as we know, these are the first instances of the GW/DT conjecture to be proved for compact Calabi-Yau threefolds.

The *local* GW/DT correspondence for super-rigid rational curves follows from the results of [MNOP] as a special case of the correspondence for toric Calabi-Yau threefolds. In contrast to Gromov-Witten theory, passing from the local invariants of super-rigid curves to *global* invariants is non-trivial in Donaldson-Thomas theory, and can be regarded as the main contribution of this paper.

1.1. Weighted Euler characteristics. Our main tool will be the weighted Euler characteristics introduced in [Be05]. Every scheme X has a canonical $\mathbb{Z}$-valued constructible function ν_X on it. The weighted Euler characteristic $\widetilde{\chi}(X)$ of X is defined as

$$\widetilde{\chi}(X) = \chi(X, \nu_X) = \sum_{n\in\mathbb{Z}} n\, \chi\left(\nu_X^{-1}(n)\right) \,.$$

More generally, we use relative weighted Euler characteristics $\widetilde{\chi}(Z\,, X)$ defined as

$$\widetilde{\chi}(Z\,, X) = \chi(Z, f^*\nu_X)\,,$$

for any morphism $f : Z \to X$. Three fundamental properties are

(i) if $X \to Y$ is étale, then $\widetilde{\chi}(Z\,, X) = \widetilde{\chi}(Z\,, Y)$,
(ii) if $Z = Z_1 \sqcup Z_2$ is a disjoint union, $\widetilde{\chi}(Z\,, X) = \widetilde{\chi}(Z_1\,, X) + \widetilde{\chi}(Z_2\,, X)$,
(iii) $\widetilde{\chi}(Z_1\,, X_1)\,\widetilde{\chi}(Z_2\,, X_2) = \widetilde{\chi}\big(Z_1 \times Z_2\,, X_1 \times X_2\big)$.

The main result of [Be05], Theorem 4.18, asserts that if X is a projective scheme with a symmetric obstruction theory on it, then

$$\deg[X]^{\mathrm{vir}} = \widetilde{\chi}(X)\,.$$

Thus we can calculate $N_n(Y,C)$ as $\widetilde{\chi}\left(J_n(Y,C)\right)$.

We will also need the following fact. If X is an affine scheme with an action of an algebraic torus T and an isolated fixed point $p \in X$, and X admits a symmetric obstruction theory compatible with the T-action, then

$$\nu_X(p) = (-1)^{\dim T_p X}\,,$$

where T_pX is the Zariski tangent space of X at p. This is the main technical result of [BF05], Theorem 3.4.

Finally, we will use the following result from [BF05]. If X is a smooth threefold (not necessarily proper), then

$$\sum_{m=0}^{\infty} \widetilde{\chi}(\operatorname{Hilb}^m X) q^m = M(-q)^{\chi(X)} .$$

In the case where X is projective and Calabi-Yau, the above proves Conjecture 1 of [MNOP]. Note that Conjecture 1 of [MNOP] has also been proved in [Li06] and [LP06].

2. The Calculation

2.1. The open subscheme $J_n(Y,C)$.

Definition 2.1. Let $C_1,\ldots,C_s$ be pairwise distinct, super-rigid rational curves on Y and let $(d_1,\ldots,d_s)$ be an s-tuple of non-negative integers. Let $C=\sum_i d_iC_i$ be the associated 1-cycle on Y and let β be the class of C in homology. Define

$$J_n(Y,C) \subset I_n(Y,\beta)$$

to be the open and closed subscheme consisting of subschemes $Z \subset Y$ whose associated 1-cycle is equal to C.

Remark 2.2. To see that $J_n(Y,C)$ is, indeed, open and closed, consider the Hilbert-Chow morphism, see [Ko96], Chapter I, Theorem 6.3, which is a morphism

$$f : I_n(Y,\beta)^{sn} \longrightarrow \operatorname{Chow}(Y,d) ,$$

where $\operatorname{Chow}(Y,d)$ is the Chow scheme of 1-dimensional cycles of degree $d=\deg\beta$ on Y. It is a projective scheme. Moreover, $I_n(Y,\beta)^{sn}$ is the semi-normalization of $I_n(Y,\beta)$. The structure morphism $I_n(Y,\beta)^{sn} \to I_n(Y,\beta)$ is a homeomorphism of underlying Zariski topological spaces. Therefore the Hilbert-Chow morphism descends to a continuous map of Zariski topological spaces

$$|f| : |I_n(Y,\beta)| \longrightarrow |\operatorname{Chow}(Y,d)| .$$

Because the C_i are super-rigid, the cycle C corresponds to an isolated point of $|\operatorname{Chow}(Y,d)|$. So the preimage of this point under $|f|$ is open and closed in $|I_n(Y,\beta)|$. The open subscheme of $I_n(Y,\beta)$ defined by this open subset is $J_n(Y,C)$.

Definition 2.3. As $J_n(Y,C)$ is open in $I_n(Y,\beta)$, it has an induced (symmetric) obstruction theory and hence a virtual fundamental class of degree zero. Since $J_n(Y,C)$ is closed in $I_n(Y,\beta)$ it is projective, and so we can consider the degree of the virtual fundamental class

$$N_n(Y,C) = \deg[J_n(Y,C)]^{\mathrm{vir}} ,$$

and call it the *contribution of* C to the Donaldson-Thomas invariant $N_n(Y,\beta)$.

2.2. The closed subset $\widetilde{J}_n(Y,C)$.

Definition 2.4. Let $C=\sum_i d_iC_i$ be as above and denote by $\operatorname{supp} C$ the reduced closed subscheme of Y underlying C. Let

$$\widetilde{J}_n(Y,C)\subset J_n(Y,C)\subset I_n(Y,\beta)$$

be the closed subset consisting of subschemes $Z\subset Y$ whose underlying closed subset $Z^{\mathrm{red}}\subset Y$ is contained in $\operatorname{supp} C$.

Remark 2.5. To see that $\widetilde{J}_n(Y,C)$ is closed in $I_n(Y,\beta)$, let $W_m\subset Y$ be the m-th infinitesimal neighborhood of $\operatorname{supp} C\subset Y$. For fixed numerical invariants n and β there exists a sufficiently large m, such that for any subscheme $Z\subset Y$, with invariants n and β, and such that $Z^{\mathrm{red}}\subset \operatorname{supp} C$, we have $Z\subset W_m$. For such an m, consider the Hilbert scheme $I_n(W_m,\beta)$, which is a closed subscheme of $I_n(Y,\beta)$, as W_m is a closed subscheme of Y. The underlying closed subset of $I_n(Y,\beta)$ is equal to $\widetilde{J}_n(Y,C)$.

Remark 2.6. Informally speaking, $J_n(Y,C)$ parameterizes subschemes whose one dimensional components are confined to C, but may have embedded points anywhere in Y, whereas $\widetilde{J}_n(Y,C)$ parameterizes subschemes where both the one dimensional components and the embedded points are supported on C.

2.3. The open Calabi Yau $N=\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. We consider the open Calabi-Yau N, which is the total space of the vector bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ on $\mathbb{P}^1$. We denote by $C_0\subset N$ the zero section. We consider the Hilbert scheme $I_n(N,[dC_0])$.

Let $\overline{N}$ denote $\mathbb{P}\big(\mathcal{O}\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-1)\big)$ and let $D_\infty=\overline{N}\backslash N$. Since $3D_\infty$ is an anti-canonical divisor of $\overline{N}$, the corresponding section defines a trivialization of K_N. $\overline{N}$ is naturally a toric variety, D_∞ is an invariant divisor, and we let T_0 be the subtorus whose elements act trivially on K_N. Then T_0 induces a T_0-equivariant symmetric obstruction theory on $I_n(N,[dC_0])$, by Proposition 2.4 of [BF05]. Moreover, the T_0 fixed points in $I_n(N,[dC_0])$ are isolated points whose Zariski tangent spaces have no trivial factors as T_0 representations (the proof of Lemma 4.1, Part (a) and (b) in [BF05] is easily adapted to prove this).

As in [MNOP], the T_0 fixed points in $I_n(N,[dC_0])$ correspond to subschemes which are given by monomial ideals on the restriction to the two affine charts of N. The number of such fixed points is given by $p(n,d)$ described below.

Let $p(n,d)$ be the number of triples $(\pi_0,\lambda,\pi_\infty)$, where π_0 and π_∞ are 3-dimensional partitions and λ a 2-dimensional partition. The 3-dimensional partitions each have one infinite leg with asymptotics λ, and no other infinite legs. Moreover, $d=|\lambda|$ and n is given by ([MNOP] Lemma 5)

$$n=|\pi_0|+|\pi_\infty|+\sum_{(i,j)\in\lambda}(i+j+1)\,,$$

where the size of a three dimensional partition with an infinite leg of shape λ along the z axis is defined by

$$|\pi|=\#\{(i,j,k)\in\mathbb{Z}^3_{\geq 0}:\quad (i,j,k)\in\pi,(i,j)\notin\lambda\}.$$

Proposition 2.7. *We have*

$$\widetilde{\chi}\big(I_n(N,[dC_0])\big)=(-1)^{n-d}p(n,d)\,.$$

Proof. By Corollary 3.5 of [BF05], we have

$$\widetilde{\chi}\big(I_n(N,[dC_0])\big)=\sum_p(-1)^{\dim T_p}\,,$$

where the sum is over all T_0-fixed points on $I_n(N,[dC_0])$ and T_p is the Zariski tangent space of $I_n(N,[dC_0])$ at p. The parity of $\dim T_p$ can be easily deduced from Theorem 2 of [MNOP] (just as in the proof of Lemma 4.1 (c) in [BF05]). The result is $n-d$. So all we have to notice is that $p(n,d)$ is the number of fixed points of T_0 on $I_n(N,[dC_0])$. □

Corollary 2.8. *We have*

$$\widetilde{\chi}\Big(\widetilde{J}_n(N,dC_0)\,,I_n(N,[dC_0])\Big)=(-1)^{n-d}p(n,d)\,.$$

Proof. We just have to notice that all T_0-fixed points are contained in $\widetilde{J}_n(N,dC_0)$. □

2.4. The box counting function $p(n,d)$**.** Counting three dimensional partitions with given asymptotics has been shown by Okounkov, Reshetikhin, and Vafa [ORV] to be equivalent to the topological vertex formalism which occurs in Gromov-Witten theory. They give general formulas for the associated generating functions in terms of q values of Schur functions which we will use to prove the following Lemma.

Lemma 2.9. *The generating function for* $p(n,d)$ *is given by*

$$\sum_{n=0}^{\infty}p(n,d)\,q^n=M(q)^2P_d(q)\,,$$

where the power series $P_d(q)$ *and* $M(q)$ *are defined in Equations (1) and (2).*

Proof. The generating function for the number of 3-dimensional partitions with one infinite leg of shape λ is given by equation 3.21 in [ORV]:

$$\sum_{\substack{\text{3d partitions }\pi\\ \text{asymptotic to }\lambda}}q^{|\pi|}=M(q)q^{-\binom{\lambda}{2}-\frac{|\lambda|}{2}}s_{\lambda^t}(q^{1/2},q^{3/2},q^{5/2},\dots)$$

where λ^t is the transpose partition, $\binom{\lambda}{2}=\sum_i\binom{\lambda_i}{2}$, $|\lambda|=\sum_i\lambda_i$, and

$$s_{\lambda^t}(q^{1/2},q^{3/2},q^{5/2},\dots)$$

is the Schur function associated to λ^t evaluated at $x_i=q^{(2i-1)/2}$. Using the homogeneity of Schur functions and writing

$$s_{\lambda^t}(q)=s_{\lambda^t}(1,q,q^2,\dots)$$

we can rewrite the right hand side of the above equation as

$$M(q)q^{-\binom{\lambda}{2}}s_{\lambda^t}(q).$$

Observing that

$$\sum_{(i,j)\in\lambda}(i+j+1)=|\lambda|+\binom{\lambda}{2}+\binom{\lambda^t}{2},$$

we get

$$\sum_{n,d=0}^{\infty}p(n,d)q^nv^d=M(q)^2\sum_{\lambda}s_{\lambda^t}(q)^2q^{|\lambda|+\binom{\lambda^t}{2}-\binom{\lambda}{2}}v^{|\lambda|}.$$

The hook polynomial formula for $s_{\lambda^t}(q)$ (I.3 ex 2 pg 45,[Mac95]) is

$$s_{\lambda^t}(q) = q^{\binom{\lambda}{2}} \prod_{x\in\lambda^t} (1-q^{h(x)})^{-1} \tag{3}$$

from which one easily sees that

$$s_{\lambda^t}(q) = q^{\binom{\lambda}{2}-\binom{\lambda^t}{2}} s_\lambda(q).$$

Therefore

$$\begin{aligned}
\sum_{n,d=0}^{\infty} p(n,d)q^n v^d &= M(q)^2 \sum_\lambda s_\lambda(q)s_{\lambda^t}(q)q^{|\lambda|}v^{|\lambda|} \\
&= M(q)^2 \sum_\lambda s_\lambda(q,q^2,q^3,\dots)s_{\lambda^t}(v,vq,vq^2,\dots) \\
&= M(q)^2 \prod_{i,j=1}^{\infty} (1+q^{i+j-1}v)
\end{aligned}$$

where the last equality comes from the orthogonality of Schur functions (I.4 equation $(4.3)'$ of [Mac95]). By rearranging this last sum and taking the v^d term, the lemma is proved. □

Remark 2.10. From the proof of the lemma we see that

$$P_d(q) = q^d \sum_{\lambda\vdash d} s_\lambda(q)s_{\lambda^t}(q).$$

From Equation (3), it is immediate that $P_d(q)$ is a rational function in q. Moreover, using the formula for total hooklength (pg 11, I.1 ex 2, [Mac95]), it is easy to check that $P_d(q)$ is invariant under $q \mapsto 1/q$.

2.5. General Y.

Lemma 2.11. *Let C be a super-rigid rational curve on the Calabi-Yau threefold Y. Then*

$$\widetilde{\chi}\big(\widetilde{J}_n(Y,dC)\,, J_n(Y,dC)\big) = (-1)^{n-d}p(n,d)\,,$$

for all n, d.

Proof. First of all, by Theorem 3.2 of [La81], an analytic neighborhood of C in Y is isomorphic to an analytic neighborhood of C_0 in N. Therefore, by the analytic theory of Hilbert schemes (or Douady spaces), see [Do66], we obtain an analytic isomorphism of $\widetilde{J}_n(Y,dC)$ with $\widetilde{J}_n(N,dC_0)$ which extends to an isomorphism of a tubular neighborhood of $\widetilde{J}_n(Y,dC)$ in $I_n(Y,[dC])$ with a tubular neighborhood of $\widetilde{J}_n(N,dC_0)$ in $I_n(N,[dC_0])$.

The formula for $\nu_X(P)$ in terms of a linking number, Proposition 4.22 of [Be05], shows that $\nu_X(P)$ is an invariant of the underlying analytic structure of a scheme X. Thus, we have

$$\widetilde{\chi}\big(\widetilde{J}_n(Y,dC)\,, I_n(Y,[dC])\big) = \widetilde{\chi}\big(\widetilde{J}_n(N,dC_0)\,, I_n(N,[dC_0])\big)\,.$$

Finally, apply Corollary 2.8. □

Lemma 2.12. *Let $f : X \to Y$ be an étale morphism of separated schemes of finite type over $\mathbb{C}$. Let $Z \subset X$ be a constructible subset. Assume that the restriction of f to the closed points of Z, $f : Z(\mathbb{C}) \to Y(\mathbb{C})$, is injective. Then we have*

$$\widetilde{\chi}\big(f(X)\,,Y\big) = \widetilde{\chi}(Z\,,X)\,.$$

We remark that by Chevalley's theorem (EGA IV, Cor. 1.8.5), $f(Z)$ is constructible, so that $\widetilde{\chi}(f(X)\,,Y)$ is defined.

Proof. Without loss of generality, $Z \subset X$ is a closed subscheme and so $Z \to Y$ is unramified.

We claim that there exists a decomposition $Y = Y_1 \sqcup \ldots \sqcup Y_n$ into locally closed subsets, such that, putting the reduced structure on Y_i, the induced morphism $Z_i = Z \times_Y Y_i \to Y_i$ is either an isomorphism, or Z_i is empty.

In fact, by generic flatness (EGA IV, Cor. 6.9.3), we may assume without loss of generality that $Z \to Y$ is flat, hence étale. By Zariski's Main Theorem (EGA IV, Cor. 18.12.13), we may assume that $Z \to Y$ is finite, hence finite étale. Then, by our injectivity assumption, the degree of $Z \to Y$ is 1 and so $Z \to Y$ is an isomorphism.

Once we have this decomposition of Y, the lemma follows from additivity of the Euler characteristic over such decompositions and the étale invariance of the canonical constructible function ν. □

Now we consider the case of a curve with several components. Let

$$C_{\vec{d}} = \sum_{i=1}^{s} d_i C_i$$

be an effective cycle, where the C_i are pairwise disjoint super-rigid rational curves in Y. We assume $d_i > 0$, for all $i = 1, \ldots, s$.

For an $(s+1)$-tuple of non-negative integers $\vec{m} = (m_0, m_1, \ldots, m_s)$, we let $|\vec{m}| = \sum_{i=0}^{s} m_i$. Consider, for $|\vec{m}| = n$ the open subscheme

$$U_{\vec{m}} \subset \mathrm{Hilb}^{m_0}(Y) \times \prod_{i=1}^{s} J_{m_i}(Y, d_i C_i)\,,$$

consisting of subschemes $(Z_0, (Z_i))$ with pairwise disjoint support.

Lemma 2.13. *Mapping $(Z_0, (Z_i))$ to $Z_0 \cup \bigcup_i Z_i$ defines an étale morphism*

$$f : U_{\vec{m}} \longrightarrow J_n(Y, C_{\vec{d}})\,.$$

Proof. This is straightforward. See also Lemma 4.4 in [BF97]. □

Let us write $\mathring{Y}$ for $Y \setminus \operatorname{supp} C$ and remark that

$$Z_{\vec{m}} = \mathrm{Hilb}^{m_0}(\mathring{Y}) \times \prod_{i>0} \widetilde{J}_{m_i}(Y, d_i C_i)$$

is contained in $U_{\vec{m}}$. Moreover, the restriction $f : Z_{\vec{m}} \to J_n(Y, C_{\vec{d}})$ is injective on closed points. Finally, every closed point of $J_n(Y, C_{\vec{d}})$ is contained in $f(Z_{\vec{m}})$, for a unique $\vec{m}$, such that $|\vec{m}| = n$.

We will apply Lemma 2.12 to the diagram

$$\begin{tikzcd}
Z_{\vec{m}} = \operatorname{Hilb}^{m_0}(\mathring{Y}) \times \prod_{i>0} \widetilde{J}_{m_i}(Y, d_iC_i) \arrow[d, "\text{closed subset}"'] \arrow[dr, "\text{injective}"] & \\
U_{\vec{m}} \arrow[r, "f", "\text{étale}"'] \arrow[d, "\text{open embedding}"'] & J_n(Y, C_{\vec{d}}) \\
\operatorname{Hilb}^{m_0}(Y) \times \prod_{i>0} J_{m_i}(Y, d_iC_i) &
\end{tikzcd}$$

Thus, we may calculate as follows:

$$\begin{aligned}
&\widetilde{\chi}\big(J_n(Y, C_{\vec{d}})\big) \\
&\quad = \sum_{|\vec{m}|=n} \widetilde{\chi}\big(f(Z_{\vec{m}})\,, J_n(Y, C_{\vec{d}})\big) \\
&\quad = \sum_{|\vec{m}|=n} \widetilde{\chi}\Big(Z_{\vec{m}}\,, U_{\vec{m}}\Big) \\
&\quad = \sum_{|\vec{m}|=n} \widetilde{\chi}\Big(\operatorname{Hilb}^{m_0}(\mathring{Y}) \times \prod_{i>0} \widetilde{J}_{m_i}(Y, d_iC_i)\,, \operatorname{Hilb}^{m_0}(Y) \times \prod_{i>0} J_{m_i}(Y, d_iC_i)\Big) \\
&\quad = \sum_{|\vec{m}|=n} \widetilde{\chi}\Big(\operatorname{Hilb}^{m_0}(\mathring{Y})\,, \operatorname{Hilb}^{m_0}(Y)\Big) \prod_{i>0} \widetilde{\chi}\Big(\widetilde{J}_{m_i}(Y, d_iC_i)\,, J_{m_i}(Y, d_iC_i)\Big) \\
&\quad = \sum_{|\vec{m}|=n} \widetilde{\chi}\big(\operatorname{Hilb}^{m_0}(\mathring{Y})\big) \prod_{i>0} (-1)^{m_i - d_i} p(m_i, d_i)\,.
\end{aligned}$$

Now we perform the summation:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widetilde{\chi}\big(J_n(Y, C_{\vec{d}})\big) q^n \\
&\qquad = \sum_{n=0}^{\infty} \left(\sum_{|\vec{m}|=n} \widetilde{\chi}\big(\operatorname{Hilb}^{m_0}(\mathring{Y})\big) \prod_{i=1}^{s} (-1)^{m_i - d_i} p(m_i, d_i) \right) q^n \\
&\qquad = \sum_{n=0}^{\infty} \sum_{|\vec{m}|=n} \widetilde{\chi}\big(\operatorname{Hilb}^{m_0}(\mathring{Y})\big) q^{m_0} \prod_{i=1}^{s} (-1)^{d_i} p(m_i, d_i) (-q)^{m_i} \\
&\qquad = \left(\sum_{m_0=0}^{\infty} \widetilde{\chi}\big(\operatorname{Hilb}^{m_0}(\mathring{Y})\big) q^{m_0} \right) \sum_{\vec{m}'} \prod_{i=1}^{s} (-1)^{d_i} p(m_i, d_i) (-q)^{m_i} \\
&\qquad = M(-q)^{\chi(\mathring{Y})} \prod_{i=1}^{s} (-1)^{d_i} \sum_{m_i=0}^{\infty} p(m_i, d_i)(-q)^{m_i} \\
&\qquad = M(-q)^{\chi(\mathring{Y})} \prod_{i=1}^{s} M(-q)^2 (-1)^{d_i} P_{d_i}(-q) \\
&\qquad = M(-q)^{\chi(\mathring{Y})} M(-q)^{2s} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q)
\end{aligned}$$

$$= M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q)$$

By the main result of [Be05], Theorem 4.18, we have

$$N_n(Y, C_{\vec{d}}) = \widetilde{\chi}\big(J_n(Y, C_{\vec{d}})\big)\,.$$

This finishes the proof of:

Theorem 2.14. *The partition function for the contribution of $C_{\vec{d}}$ to the Donaldson-Thomas invariants of Y is given by*

$$Z(Y, C_{\vec{d}}) = \sum_{n=0}^{\infty} N_n(Y, C_{\vec{d}})\, q^n = M(-q)^{\chi(Y)} \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q)\,.$$

Corollary 2.15. *Define the reduced partition function*

$$Z'(Y, C_{\vec{d}}) = \frac{Z(Y, C_{\vec{d}})}{Z(Y, 0)}\,.$$

Then we have

$$Z'(Y, C_{\vec{d}}) = \prod_{i=1}^{s} (-1)^{d_i} P_{d_i}(-q)\,,$$

a rational function in q, invariant under $q \mapsto 1/q$.

Proof. Behrend and Fantechi prove [BF05] that

$$Z(Y, 0) = \sum_{n=0}^{\infty} \widetilde{\chi}(\mathrm{Hilb}^n Y) q^n = M(-q)^{\chi(Y)};$$

the formula for $Z'(Y, C_{\vec{d}})$ then follows immediately from Theorem 2.14. For the proof that $Z'(Y, C_{\vec{d}})$ is a rational function invariant under $q \mapsto 1/q$, see Remark 2.10. □

3. The super-rigid GW/DT correspondence.

3.1. The usual GW/DT correspondence. We can formulate the The Gromov-Witten/Donaldson-Thomas correspondence of [MNOP] as follows.

Let Y be a Calabi-Yau threefold and let

$$Z_{DT}(Y, \beta) = \sum_{n \in \mathbb{Z}} N_n(Y, \beta) q^n$$

be the partition function for the degree β Donaldson-Thomas invariants. Let

$$Z'_{DT}(Y, \beta) = \frac{Z_{DT}(Y, \beta)}{Z_{DT}(Y, 0)}$$

be the reduced partition function.

In Gromov-Witten theory, the reduced partition function for the degree β Gromov-Witten invariants, $Z'_{GW}(Y, \beta)$, is given by the coefficients of the exponential of the $\beta \neq 0$ part of the potential function:

$$1 + \sum_{\beta \neq 0} Z'_{GW}(Y, \beta) v^{\beta} = \exp\left(\sum_{\beta \neq 0} N_g^{GW}(Y, \beta) u^{2g-2} v^{\beta} \right). \tag{4}$$

Here

$$N_g^{GW}(Y,\beta) = \deg[\overline{M}_g(Y,\beta)]^{\mathrm{vir}}$$

is the genus g, degree β Gromov-Witten invariant of Y.

The conjectural GW/DT correspondence states that

(i) The degree 0 partition function in Donaldson-Thomas theory is given by

$$Z_{DT}(Y,0) = M(-q)^{\chi(Y)},$$

(ii) $Z'_{DT}(Y,\beta)$ is a rational function in q, invariant under $q \mapsto 1/q$, and

(iii) the equality

$$Z'_{GW}(Y,\beta) = Z'_{DT}(Y,\beta)$$

holds under the change of variables $q = -e^{iu}$.

Part (i) is proved for all Y in [BF05].

3.2. The super-rigid GW/DT correspondence. In an entirely parallel manner, we can formulate the GW/DT correspondence for $N_n(Y, C_{\vec{d}})$, the contribution from a collection of super-rigid rational curves $C_{\vec{d}} = \sum_i d_i C_i$.

Just as in Donaldson-Thomas theory, there is an open component of the moduli space of stable maps

$$\overline{M}_g(Y, C_{\vec{d}}) \subset \overline{M}_g(Y,\beta)$$

parameterizing maps whose image lies in the support of $C_{\vec{d}}$. There are corresponding invariants given by the degree of the virtual class:

$$N_g^{GW}(Y, C_{\vec{d}}) = \deg[\overline{M}_g(Y, C_{\vec{d}})]^{\mathrm{vir}}$$

We define $Z'_{GW}(Y, C_{\vec{d}})$ by replacing $N_g^{GW}(Y,\beta)$ on the right side of formula (4) by $N_g^{GW}(Y, C_{\vec{d}})$.

Then we can formulate our results as follows.

Theorem 3.1. *The GW/DT correspondence holds for the contributions from super-rigid rational curves. Namely, let $C_{\vec{d}} = d_1C_1 + \cdots + d_sC_s$ be a cycle supported on pairwise disjoint super-rigid rational curves C_i in a Calabi-Yau threefold Y, and let $Z'_{DT}(Y, C_{\vec{d}})$ and $Z'_{GW}(Y, C_{\vec{d}})$ be defined as above. Then*

(ii) *$Z'_{DT}(Y, C_{\vec{d}})$ is a rational function of q, invariant under $q \mapsto 1/q$, and*

(iii) *the equality*

$$Z'_{DT}(Y, C_{\vec{d}}) = Z'_{GW}(Y, C_{\vec{d}})$$

holds under the change of variables $q = -e^{iu}$.

Proof. For (ii), see Corollary 2.15. To prove (iii), we reproduce a calculation well known to the experts (e.g. [Ka06]).

By the famous multiple cover formula of Faber-Pandharipande [FP00] (see also [Pa99]),

$$N_g^{GW}(Y, C_{\vec{d}}) = \sum_{i=1}^{s} c(g, d_i)\,,$$

where $c(g,d)$ is given by

$$\sum_{g\geq 0} c(g,d)u^{2g-2} = \frac{1}{d}\left(\sin\left(\frac{du}{2}\right)\right)^{-2}.$$

We compute $Z'_{GW}(Y, C_{\vec{d}})$ and make the substitution $q = -e^{iu}$:

$$\begin{aligned}
1 + \sum_{(d_1,\dots,d_s)\neq 0}^{\infty} & Z'_{GW}(Y, C_{\vec{d}}) v_1^{d_1} \cdots v_s^{d_s} \\
&= \exp\left(\sum_{j=1}^{s}\sum_{d_j=1}^{\infty}\sum_{g=0}^{\infty} c(g, d_j) u^{2g-2} v_j^{d_j}\right) \\
&= \prod_{j=1}^{s} \exp\left(\sum_{d_j=1}^{\infty} \frac{v_j^{d_j}}{d_j}\left(2\sin\frac{d_j u}{2}\right)^{-2}\right) \\
&= \prod_{j=1}^{s} \exp\left(\sum_{d_j=1}^{\infty} \frac{v_j^{d_j}}{d_j} \frac{-e^{id_j u}}{(1-e^{id_j u})^2}\right) \\
&= \prod_{j=1}^{s} \exp\left(\sum_{d_j=1}^{\infty}\sum_{m_j=1}^{\infty} \frac{-m_j}{d_j} e^{id_j m_j u} v_j^{d_j}\right) \\
&= \prod_{j=1}^{s} \exp\left(\sum_{m_j=1}^{\infty} m_j \log\left(1 - v_j e^{im_j u}\right)\right) \\
&= \prod_{j=1}^{s}\prod_{m_j=1}^{\infty} (1 - (-q)^{m_j} v_j)^{m_j} \\
&= \prod_{j=1}^{s}\sum_{d_j=0}^{\infty} P_{d_j}(-q)(-v_j)^{d_j} \\
&= \sum_{(d_1,\dots,d_s)}\prod_{j=1}^{s} (-1)^{d_j} P_{d_j}(-q) v_j^{d_j}.
\end{aligned}$$

Therefore,

$$Z'_{GW}(Y, C_{\vec{d}}) = \prod_{j=1}^{s} (-1)^{d_j} P_{d_j}(-q)$$

and so by comparing with Corollary 2.15 the theorem is proved. □

The following corollary is immediate.

Corollary 3.2. *Let Y be a Calabi-Yau threefold and let $\beta \in H_2(Y, \mathbb{Z})$ be a curve class such that all cycle representatives of β are supported on a collection of pairwise disjoint, super-rigid rational curves. Then the degree β GW/DT correspondence holds:*

$$Z'_{DT}(Y, \beta) = Z'_{GW}(Y, \beta).$$

For example, we have:

Corollary 3.3. *Let $Y \subset \mathbb{P}^4$ be a quintic threefold, and let L be the class of the line. Then for β equal to L or $2L$, the GW/DT correspondence holds.*

Proof. By deformation invariance of both Donaldson-Thomas and Gromov-Witten invariants, it suffices to let Y be a generic quintic threefold. It is well known that there are exactly 2875 pairwise disjoint lines on Y and they are all super-rigid. The conics on Y are all planar and hence rational, and it is known that there are exactly 609250 pairwise disjoint conics and they are super-rigid as well. For these facts and more, see [Ka86]. □

Note that we cannot prove the GW/DT conjecture by this method for the quintic in degree three (and higher) due to the presence of elliptic curves in degree three.

Explicit formulas for the reduced Donaldson-Thomas partition function of a generic quintic threefold Y in degrees one and two are given below:

$$\begin{aligned} Z'_{DT}(Y,L) &= 2875\frac{q}{(1-q)^2} \\ Z'_{DT}(Y,2L) &= 609250\frac{q}{(1-q)^2}\cdot 2875\cdot\frac{-2q^3}{(1+q)^4(1-q)^2} \\ &= -3503187500\frac{1}{(q-q^{-1})^4} \end{aligned}$$

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