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DEGREE AND HOLOMORPHIC EXTENSIONS

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Abstract. Let D be a bounded convex domain in $\mathbb{C}^N$, $N \geq 2$. We prove that a continuous map $\Phi\colon\ bD \to \mathbb{C}^N$ extends holomorphically through D if and only if for every polynomial map $P\colon\ \mathbb{C}^N \to \mathbb{C}^N$ such that $\Phi + P \neq 0$ on bD, the degree of $\Phi + P|bD$ is nonnegative. We also prove another such theorem for more general domains.

1. Introduction

Let $D \subset \mathbb{R}^n$ be a bounded open set and let $\Psi\colon\ bD \to R^n \setminus \{0\}$ be a continuous map. Let $\tilde{\Psi}$ be a continuous extension of Ψ to $\overline{D}$. Approximate $\tilde{\Psi}$ on $\overline{D}$ uniformly by a map G smooth in a neighbourhood of $\overline{D}$ such that $G(bD) \subset \mathbb{R}^n \setminus \{0\}$. Perturbing G slightly we may assume that the origin 0 is a regular value of G so $G^{-1}(0) \cap D$ is a finite subset of D and each point in $G^{-1}(0) \cap D$ is a regular point of G. Let ν be the number of points in $G^{-1}(0) \cap D$ at which the derivative DG preserves orientation minus the number of points in $G^{-1}(0) \cap D$ at which DG reverses orientation. The number ν depends neither on the choice of the extension $\tilde{\Psi}$ of Ψ nor on the choice of G provided that G approximates $\tilde{\Psi}$ on $\overline{D}$ well enough [D]. It is called the degree of Ψ, $\nu = \deg \Psi$. It is known that if $\{\Psi_t,\ 0 \leq t \leq 1\}$, is a continuous family of continuous maps from bD to $\mathbb{R}^n \setminus \{0\}$ then $\deg \Psi_1 = \deg \Psi_0$. In the special case when $D \subset \mathbb{C}$ is a bounded domain with smooth boundary and $\Psi\colon\ bD \to \mathbb{C} \setminus \{0\}$ is a continuous function then $2\pi \deg \Psi$ equals the change of argument of Ψ along bD. [D]

Let $D \subset \mathbb{C}^N$ be a bounded domain and suppose that $\Phi\colon\ bD \to \mathbb{C}^N \setminus \{0\}$ is a continuous map which extends holomorphically through D. Then $\deg \Phi \geq 0$. To see this, observe first that perturbing Φ slightly does not change the degree and implies that all the zeros of the holomorphic extension $\tilde{\Phi}$ of Φ are regular points of $\tilde{\Phi}$. Since $\tilde{\Phi}$ is holomorphic, at each regular point a of $\tilde{\Phi}$ the derivative $(D\tilde{\Phi})(a)$, a $\mathbb{C}$-linear map looked upon as a linear map from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2N}$, preserves orientation. In particular, $\deg \Phi$ is equal to the number of points $a \in D$ such that $\tilde{\Phi}(a) = 0$ hence $\deg \Phi \geq 0$.

Assume that $\Psi\colon bD \to \mathbb{C}^N$ is a continuous map. If Ψ extends holomorphically through D then by the preceding discussion $\deg(\Psi + F) \geq 0$ for every continuous map $F\colon\ bD \to \mathbb{C}^N$ that extends holomorphically through D and is such that $\Psi + F \neq 0$ on bD. It is known that the converse is true if D is a smoothly bounded domain in $\mathbb{C}$. In the present paper we prove the converse for a large class of domains in $\mathbb{C}^N$, $N \geq 2$.

2. The main results

Our main results are the following two theorems.

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THEOREM 2.1 *Let D be a bounded convex domain in $\mathbb{C}^N,\ N \geq 2$. A continuous map $\Phi\colon\ bD \to \mathbb{C}^N$ extends holomorphically through D if and only if for every polynomial map $P\colon\ \mathbb{C}^N \to \mathbb{C}^N$ such that $\Phi + P \neq 0$ on bD, the degree of $\Phi + P|bD$ is nonnegative.*

THEOREM 2.2 *Let $N \geq 2$ and let $D \subset \mathbb{C}^N$ be a bounded domain with $\mathcal{C}^2$ boundary such that $\overline{D}$ has a Stein neighbourhood basis. A continuous map $\Phi\colon\ bD \to \mathbb{C}^N$ extends holomorphically through D if and only if for each holomorphic map G from a neighbourhood of $\overline{D}$ (that may depend on G) to $\mathbb{C}^N$ such that $\Phi + G \neq 0$ on bD, the degree of $\Phi + G|bD$ is nonnegative.*

The theorems are known in the case when $N = 1$: Given a bounded domain D in $\mathbb{C}$ let $A(D)$ be the algebra of all continuous functions on $\overline{D}$ which are holomorphic on D. If bD consists of finitely many pairwise disjoint simple closed curves then a continuous function Φ on bD extends holomorphically through D if and only if for each $G \in A(D)$ such that $\Phi + G \neq 0$ on bD, the change of argument of $\Phi + G$ along bD is nonnegative [G2][G2]. In fact, $A(D)$ here may be replaced by any dense subset of $A(D)$, for instance, by the set of functions holomorphic in a neighbourhood of $\overline{D}$ (which may depend on the function) [S2]. In particular, if D is simply connected, it suffices to take for G the polynomials.

3. The degree of a special map on the intersection of bD with a complex line

Denote by L the z_1-axis in $\mathbb{C}^N,\ N \geq 2$,

$$L = \{(\zeta, 0, \cdots, 0)\colon\ \zeta \in \mathbb{C}\}.$$

PROPOSITION 3.1 *Let $D \subset \mathbb{C}^N$ be a bounded domain. Suppose that L meets D and that $L \cap bD$ is the boundary of $L \cap D$ in L. Let $\Omega = \{\zeta \in \mathbb{C}\colon\ (\zeta, 0, \cdots, 0) \in D\}$ so $b\Omega = \{\zeta \in \mathbb{C}\colon\ (\zeta, 0, \cdots, 0) \in bD\}$. Let $\varphi\colon\ bD \to \mathbb{C}$ be a continuous function such that $\varphi(\zeta, 0, \cdots 0) \neq 0$ $(\zeta \in b\Omega)$. Define a continuous map $\Phi\colon\ bD \to \mathbb{C}^N \setminus \{0\}$ by*

$$\Phi(z_1, \cdots, z_N) = \big(\varphi(z_1, \cdots, z_N), z_2, \cdots, z_N\big) \quad \big((z_1, \cdots, z_N) \in bD\big).$$

Then deg Φ equals the degree of the map $\zeta \mapsto \varphi(\zeta, 0, \cdots, 0)$ $(\zeta \in b\Omega)$.

Proof. We first show the following;
There is an $\epsilon > 0$ such that whenever φ_1 is a continuous function on bD such that $|\varphi_1(z) - \varphi(z)| < \epsilon$ $(z \in L \cap bD)$ then the degrees of the maps $\zeta \mapsto \varphi(\zeta, 0, \cdots, 0)$ $(\zeta \in b\Omega)$ and $\zeta \mapsto \varphi_1(\zeta, 0, \cdots, 0)$ $(\zeta \in b\Omega)$ are the same and, moreover, if $\Phi_1(z) = (\varphi_1(z), z_2, \cdots, z_N)$ $(z \in bD)$ then deg $\Phi_1 =$ deg Φ.
To see this, recall first that by our assumption, $\varphi(z) \neq 0$ $(z \in L \cap bD)$ so there is an $\epsilon > 0$ such that if $\varphi_1\colon\ bD \to \mathbb{C}$ is a continuous function such that $|\varphi_1 - \varphi| < \epsilon$ on $L \cap bD$ then $(1-\lambda)\varphi + \lambda\varphi_1 \neq 0$ on $L \cap bD$ for each $\lambda,\ 0 \leq \lambda \leq 1$. In particular, $(1-\lambda)\varphi(\zeta, 0, \cdots, 0) + \lambda\varphi_1(\zeta, 0, \cdots, 0) \neq 0$ $(\zeta \in b\Omega,\ 0 \leq \lambda \leq 1)$ which implies that the degrees of the maps $\zeta \mapsto \varphi(\zeta, 0, \cdots, 0)$ $(\zeta \in b\Omega)$ and $\zeta \mapsto \varphi_1(\zeta, 0, \cdots, 0)$ $(\zeta \in b\Omega)$ are the same. Fix such φ_1 and let $\Phi_1(z) = (\varphi_1(z), z_2, \cdots, z_N)$. Consider $(1-\lambda)\Phi(z) + \lambda\Phi_1(z) = ((1-\lambda)\varphi(z) + \lambda\varphi_1(z), z_2, \cdots, z_N)$. If $z \in L \cap bD$ then $(1-\lambda)\varphi(z) + \lambda\varphi_1(z) \neq 0$ so $(1-\lambda)\Phi(z) + \lambda\Phi_1(z) \neq 0$ $(0 \leq \lambda \leq 1)$. If $z \in bD \setminus L$ then $(z_2, \cdots z_N) \neq 0$ so again $(1-\lambda)\Phi(z) + \lambda\Phi_1(z) \neq 0$ $(0 \leq \lambda \leq 1)$. Thus, $\Psi_\lambda = (1-\lambda)\Phi + \lambda\Phi_1,\ 0 \leq \lambda \leq 1$, is

a continuous family of continuous maps from bD to $\mathbb{C}^N \setminus \{0\}$ so $\deg \Phi = \deg \Psi_0 = \deg \Psi_1 = \deg \Phi_1$. The statement is proved.

Choose a smooth complex valued function ω on $\mathbb{C}$ which satisfies

$$|\omega(\zeta) - \varphi(\zeta, 0, \cdots, 0)| < \varepsilon \quad (\zeta \in b\Omega)$$

and is such that 0 is its regular value. Define a smooth function φ_1 on $\mathbb{C}^N$ by

$$\varphi_1(z_1, z_2, \cdots, z_N) = \omega(z_1)$$

and define $\Phi_1(z) = (\varphi_1(z), z_2, \cdots z_N)$. By the preceding paragraph the proof of Proposition 3.1 will be complete once we have shown that $\deg \Phi_1$ is the same as the degree of the map $\zeta \mapsto \omega(\zeta)$ $(\zeta \in b\Omega)$.

By the assumption, $\omega(x+iy) = u(x,y) + iw(x,y) = (u(x,y), w(x,y))$ has finitely many zeros $a_j = p_j + iq_j$, $1 \le j \le m$, in Ω and each of these zeros is a regular point of ω. Moreover, by the construction, the map Φ_1 has precisely the zeros $(a_j, 0, \cdots, 0)$, $1 \le j \le m$, in D.

Suppose that $a = p + iq = (p,q)$ is one of the zeros of ω so $(a, 0, \cdots 0)$ is a zero of Φ_1. Since a is a regular point of ω the derivative

$$(D\omega)(a) = \begin{bmatrix} \frac{\partial u}{\partial x}(p,q), & \frac{\partial u}{\partial y}(p,q), \\ \frac{\partial w}{\partial x}(p,q), & \frac{\partial w}{\partial y}(p,q) \end{bmatrix}$$

looked upon as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, is nonsingular.
Since $\Phi_1(x_1, y_1, \cdots, x_N, y_N) = (u(x_1,y_1), w(x_1,y_1), x_2, y_2, \cdots, x_N, y_N)$ the derivative of Φ_1 at $(a, 0, \cdots, 0) = (p, q, 0, \cdots, 0)$, looked upon as a linear map from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2N}$ is

$$(D\Phi_1)(a, 0, \cdots, 0) = \begin{bmatrix} \frac{\partial u}{\partial x_1}(p,q), & \frac{\partial u}{\partial y_1}(p,q), & O \\ \frac{\partial w}{\partial x_1}(p,q), & \frac{\partial w}{\partial y_1}(p,q), & O \\ O, & O, & I \end{bmatrix}$$

where I is the identity matrix of order $2N-2$. Thus, $(D\Phi_1)(a, 0, \cdots, 0)$ is also nonsingular and $\det(D\omega)(a) = \det(D\Phi_1)(a, 0, \cdots, 0)$. It follows that the maps $(D\omega)(a)$ and $(D\Phi_1)(a, 0, \cdots, 0)$ either both preserve orientation or both reverse orientation. Since $\Phi_1(z) = 0$ for $z \in D$ if and only if $z = (a_j, 0, \cdots, 0)$ for some j, $1 \le j \le m$, it follows that the degree of $\zeta \mapsto \omega(\zeta)$ $(\zeta \in b\Omega)$ equals $\deg \Phi_1$. This completes the proof.

We shall also need

PROPOSITION 3.2 *Let $\Phi = (\Phi_1, \cdots, \Phi_N)$ be a continuous map from bD to $\mathbb{C}^N \setminus \{0\}$. Let $t_j > 0$ $(1 \le j \le N)$ and let $\Psi = (t_1\Phi_1, \cdots, t_N\Phi_N)$. Then $\deg \Psi = \deg \Phi$.*

Proof. $\Theta_\lambda = (1-\lambda)\Phi + \lambda\Psi$, $0 \le \lambda \le 1$ is a continuous family of maps from bD to $\mathbb{C}^N \setminus \{0\}$ such that $\Theta_0 = \Phi$, $\Theta_1 = \Psi$. It follows that $\deg \Psi = \deg \Phi$. This completes the proof.

4. Proofs of Theorems 2.1 and 2.2

Lemma 4.1 *Let $D \subset \mathbb{C}$ be a bounded open set with $\mathcal{C}^1$ boundary. A continuous function Φ on bD extends holomorphically through D if and only if for each function G, holomorphic in a neighbourhood of $\overline{D}$, such that $\Phi + G \ne 0$ on bD, the degree of $\Phi + G|bD$ is nonnegative.*

Proof. Observe first that $D = D_1 \cup \cdots \cup D_m$ where $D_j,\ 1 \le j \le m$, are domains with pairwise disjoint closures and each $bD_j,\ 1 \le j \le m$, consists of finitely many pairwise disjoint simple closed curves. The only if part follows from the argument principle. To prove the if part, assume that $\Phi\colon\ bD \to \mathbb{C}$ is a continous function that does not extend holomorphically through D. So for some j, the function $\Phi|bD_j$ does not extend holomorphically through D_j which implies [G2] that there is a function $H \in A(D_j)$ such that $\Phi + H \ne 0$ on bD_j and that $\deg\ (\Phi|bD_j + H|bD_j)$ is negative. Since H can be approximated on $\overline{D_j}$ arbitrarily well by rational functions with poles outside $\overline{D_j}$ [S2] we may assume that H is holomorphic on a neighbourhood U of $\overline{D_j}$ whose closure misses $bD_k,\ 1 \le k \le m,\ k \ne j$. Adding a sufficiently large constant T_k to $H|bD_k,\ k \ne j, 1 \le k \le N$, will make the degree of $H|bD_k + T_k$ equal zero. So putting $H \equiv T_k$ on $bD_k,\ 1 \le k \le m,\ k \ne j$, we get a function H, holomorphic on a neighbourhood of $\overline{D}$ such that $\Phi + H \ne 0$ on bD and such that $\deg\ (\Phi + H|bD)$ is negative. This proves the if part and completes the proof.

Proof of Theorem 2.2. The only if part was proved at the end of Section 2. To prove the if part assume that $\Phi = (\Phi_1, \cdots, \Phi_N)$ does not extend holomorphically through D. Then one of the components, say Φ_1, does not extend holomorphically through D which implies that there is a complex line L meeting D and meeting bD transversely such that $\Phi_1|(L \cap bD)$ does not extend holomorphically through $L \cap D$ [GS]. After a translation and rotation we may assume with no loss of generality that L is the z_1-axis. Let $\Omega = \{\zeta \in \mathbb{C}\colon\ (\zeta, 0, \cdots, 0\} \in D\}$, so $b\Omega = \{\zeta \in \mathbb{C}\colon\ (\zeta, 0, \cdots, 0) \in bD\}$. The function $\zeta \mapsto \Phi_1(\zeta, 0, \cdots, 0)$ is continuous on $b\Omega$ and does not extend holomorphically through Ω which, by Lemma 4.1 implies that there is a holomorphic function g on an open neighbourhood U of $\overline{\Omega}$ in $\mathbb{C}$ such that $\Phi_1(\zeta, 0, \cdots, 0) + g(\zeta) \ne 0$ $(\zeta \in b\Omega)$ and such that the map $\zeta \mapsto \Phi_1(\zeta, 0, \cdots, 0) + g(\zeta)$ $(\zeta \in b\Omega)$ has negative degree. Since $\overline{D}$ has a Stein neighbourhood basis it has arbitrarily small pseudoconvex neighbourhoods so there is a pseudoconvex domain Σ containing $\overline{D}$ such that $\{(\zeta, 0, \cdots, 0)\colon\ \zeta \in U\} \cap \Sigma$ is a closed subset of Σ and hence a closed one dimensional submanifold of the pseudoconvex domain Σ. It follows [GR, p. 245] that there is a holomorphic function H_1 on Σ such that

$$H_1(\zeta, 0, \cdots, 0) = g(\zeta) \quad ((\zeta, 0, \cdots 0) \in \Sigma).$$

Thus, H_1 is holomorphic in a neighbourhood of $\overline{D}$.

Write $z = (z_1, \cdots, z_N)$. By Proposition 3.1 the map from bD to $\mathbb{C}^N \setminus \{0\}$ given by

$$z \mapsto (\Phi_1(z) + H_1(z), z_2, \cdots, z_N) \quad (z \in bD) \tag{4.1}$$

has the same degree as the map from $b\Omega$ to $\mathbb{C} \setminus \{0\}$ given by

$$\zeta \mapsto \Phi_1(\zeta, 0, \cdots, 0) + g(\zeta) \quad (\zeta \in b\Omega)$$

which implies that the degree of the map (4.1) is negative.

Perturbing the map (4.1) slightly will not change the degree so one can choose $T > 0$ so large that the map

$$z \mapsto \big(\Phi_1(z) + H_1(z), z_2 + \Phi_2(z)/T, \cdots, z_N + \Phi_N(z)/T\big) \tag{4.2}$$

maps bD into $\mathbb{C}^N \setminus \{0\}$ and has the same degree as the map (4.1). By Proposition 3.2 the degree of the map

$$z \mapsto (\Phi_1(z) + H_1(z), \Phi_2(z) + Tz_2, \cdots, \Phi_N(z) + Tz_N) \quad (z \in bD)$$

is the same as the degree of the map (4.2). So, setting $H_j(z) \equiv Tz_j$ $(z \in U,\ 2 \le j \le N)$ we have constructed a holomorphic map $H\colon\ U \to \mathbb{C}^N$ such that $\Phi + H \ne 0$ on bD and such that $\deg\,(\Phi + H|bD)$ is negative. This completes the proof of Theorem 2.2.

Proof of Theorem 2.1 If D is convex then Ω in the proof, being convex, is a simply connected domain so g can be chosen to be a polynomial and for H_1 one can take a polynomial on $\mathbb{C}^N$ defined by $H_1(z_1, \cdots, z_N) = g(z_1)$ to have $H_1(\zeta, 0, \cdots, 0) = g(\zeta)$ $((\zeta, 0, \cdots, 0) \in \overline{D})$. One finishes the proof as the proof of Theorem 1.2. Theorem 2.1 is proved.

5. Consequences and remarks

A continuous function $\Phi\colon\ bD \to \mathbb{C}$ extends holomorphically through D if and only if the map $z \mapsto (\Phi(z), 0, \cdots, 0)$ extends holomorphically through D which gives

COROLLARY 5.1 *Let $N \ge 2$ and let $D \subset \mathbb{C}^N$ be a bounded domain with $\mathcal{C}^2$ boundary such that $\overline{D}$ has a Stein neighbourhood basis. A continuous function $\Phi\colon\ bD \to C$ extends holomorphically through D if and only if for any N-tuple of functions $G_1, \cdots, G_N$ holomorphic in a neighbourhood of $\overline{D}$ and such that $(\Phi + G_1, G_2, \cdots, G_N) \ne 0$ on bD, the degree of the map $z \mapsto (\Phi(z) + G_1(z), G_2(z), \cdots, G_N(z))$ $(z \in bD)$ is nonnegative.*

This strenghtens the main result of [S1] where it was shown that Φ extends holomorphically through D if and only if the degree of $z \mapsto (H_1(z, \Phi(z), \cdots, H_N(z, \Phi(z)))$ $(z \in bD)$ is nonnegative whenever $H_1, \cdots, H_N$ are holomorphic functions on a neighbourhood of $\overline{D} \times \mathbb{C}$ in $\mathbb{C}^{N+1}$ such that $(H_1(z, \Phi(z), \cdots, H_N(z, \Phi(z))) \ne 0$ $(z \in bD)$.

COROLLARY 5.2 *Let $D \subset \mathbb{C}^N$ be a convex domain. A continuous function $\Phi\colon\ bD \to \mathbb{C}$ extends holomorphically through D if and only if for every N-tuple of polynomials $P_1, \cdots, P_N\colon\ \mathbb{C}^N \to \mathbb{C}$ such that $(\Phi + P_1, P_2, \cdots, P_N) \ne 0$ on bD, the degree of the map $z \mapsto (\Phi(z) + P_1(z), P_2(z), \cdots, P_N(z))$ $(z \in bD)$ is nonnegative.*

Theorem 2.1 and Corollary 5.2 hold for more general domains:
Suppose that $\mathcal{U}$ is an open subset of the set of all complex lines in $\mathbb{C}^N$, $N \ge 2$, passing through the origin, and let $D \subset \mathbb{C}^N$ be a bounded domain with $\mathcal{C}^2$ boundary such that $L \cap bD$ is connected for every complex line L of the form $L = z + \Sigma$, $z \in \mathbb{C}^n$, $\Sigma \in \mathcal{U}$, which intersects bD transversely. Then the statement of Theorem 1.1 holds. To see this, notice first that the assumptions imply that bD is connected. Further, if Φ is a continuous function on bD such that for each L as above, $\Phi|L \cap bD$ extends holomorphically through $L \cap D$ then Φ satisfies weak tangential Cauchy Riemann equations on bD [GS] and hence, since bD is connected, Φ extends holomorphically through D [K]. Thus, if Φ is a continuous function on bD that does not extend holomorphically through D then there is a complex line L as above such that $\Phi|L \cap bD$ does not extend holomorphically through $L \cap D$. Proceeding as in the proof of Theorem 2.2 we notice that the connectedness of $L \cap bD$ implies that Ω is a simply connected domain and we can finish the proof as the proof of Theorem 2.1.

If $A\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is an invertible $\mathbb{R}$-linear map and D is a bounded domain that contains the origin then $\deg\,(A|bD)$ equals $+1$ if A preserves orientation on $\mathbb{R}^{2N} = \mathbb{C}^N$ and -1 if A reverses orientation on $\mathbb{R}^{2N} = \mathbb{C}^N$. Recall that an invertible

$\mathbb{C}$-linear map preserves orientation. So, if $A\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is a $\mathbb{C}$-linear map then $A+B$ preserves orientation whenever $B\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is a $\mathbb{C}$-linear map such that $A+B$ is invertible. This property characterizes $\mathbb{C}$-linearity:

PROPOSITION 5.3 *Let $A\colon\ \mathbb{C}^N \to \mathbb{C}^N$ be an $\mathbb{R}$-linear map such that $A+B$ preserves orientation whenever $B\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is a $\mathbb{C}$-linear map such that $A+B$ is invertible. Then A is $\mathbb{C}$-linear.*

Proof. Assume that $A\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is an $\mathbb{R}$-linear map which is not $\mathbb{C}$-linear. Write $A = (A_1, \cdots, A_N)$ where $A_j,\ 1 \le j \le N$, are $\mathbb{R}$-linear functionals at least one of which, say A_1, is not $\mathbb{C}$-linear. It follows that there is a complex line L through the origin such that $A_1|L$ is not $\mathbb{C}$-linear. With no loss of generality assume that L is the z_1-axis. Thus, $A_1((\zeta, 0, \cdots, 0)) = \alpha\zeta + \beta\overline{\zeta}$ $(\zeta \in \mathbb{C})$ where $\beta \neq 0$. There is $T > 0$ so large that the map from $\mathbb{C}^N$ to $\mathbb{C}^N$ given by

$$z \mapsto \big(\beta\overline{z_1}, Tz_2 + tA_2(z), \cdots, Tz_N + tA_N(z)\big) \tag{5.1}$$

is invertible for each $t,\ 0 \le t \le 1$. Since the map (5.1), looked upon as a linear map from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2N}$ is invertible for each $t,\ 0 \le t \le 1$, and reverses orientation for $t = 0$ it follows that it reverses orientation for each $t,\ 0 \le t \le 1$. In particular, if we define

$$H(z) = (-\alpha z_1, Tz_2, \cdots, Tz_N)q\ (z \in \mathbb{C}^N)$$

it follows that

$$z \mapsto (A + H)(z) = (\beta\overline{z_1}, A_2(z) + Tz_2, \cdots, A_N(z) + Tz_N)$$

is an invertible map which reverses orientation. This completes the proof.

Proposition 5.3 can be viewed as the simplest case of Theorem 2.1. It shows that for a small class of maps Φ - $\mathbb{R}$-linear maps - a small class of holomorphic maps P - $\mathbb{C}$-linear maps - is needed to check the holomorphic extendibility. It is an obvious question whether one can go further and ask whether for the set of all polynomial maps in $z_1, \cdots, z_N, \overline{z_1}, \cdots \overline{z_N}$ of degree $\le m$, to check the holomorphic extendibility through D it is enough to take for P the holomorphic polynomials of degree $\le m$. We prove that this is the case when D is a ball:

PROPOSITION 5.4 *Let $D \subset \mathbb{C}^N$ be an open ball, let $m \in \mathbb{N}$ and let $\Phi\colon\ \mathbb{C}^N \to \mathbb{C}^N$ be a polynomial map in $z_1, \cdots, z_N, \overline{z_1}, \cdots, \overline{z_N}$ of degree $\le m$. If deg $(\Phi + P)|bD$ is nonnegative whenever $P\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is a holomorphic polynomial of degree $\le m$ such that $\Phi + P \neq 0$ on bD, then $\Phi|bD$ extends holomorphically through D (as a holomorphic polynomial of degree $\le m$).*

Proof. Denote $\Delta(a, r) = \{\zeta \in \mathbb{C} :\ |\zeta - a| < r\}$. Note first that the fact that $\Phi\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial map in $z_1, \cdots, z_N, \overline{z_1}, \cdots, \overline{z_N}$ of degree $\le m$ is invariant with respect to affine $\mathbb{C}$-linear change of coordinates. Note also that if $p\colon\ \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $\le m$ in ζ and $\overline{\zeta}$ then given $a \in \mathbb{C}$ and $r > 0$ there are polynomials q and s of degree $\le m$ such that

$$p(\zeta, \overline{\zeta}) = q(\zeta - a) + \overline{s(\zeta - a)}\ \ (\zeta \in b\Delta(a, r)).$$

Assume that $\Phi\colon\ \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial of degree $\le m$ in $z_1, \cdots, z_N, \overline{z_1}, \cdots, \overline{z_N}$. Then for every $Z, W \in \mathbb{C}^N$, $\zeta \mapsto \Phi(Z + \zeta W)\colon\ \mathbb{C} \to \mathbb{C}^N$ is a polynomial of degree $\le m$ in $\zeta, \overline{\zeta}$.

Suppose that $\Phi|bD$ does not extend holomorphically through D. Then one of the components $\Phi_1, \cdots, \Phi_N$, say Φ_1, does not extend from $L \cap bD$ holomorphically through $L \cap D$. After a translation and a unitary change of coordinates we may assume that L is the z_1-axis. Then $L \cap D$ is an open disc so $\{\zeta \in \mathbb{C}\colon\ (\zeta, 0, \cdots, 0) \in D\}$ is an open disc, $\Omega = \Delta(a, r)$ and $b\Omega = b\Delta(a, r)$ is a circle. Now, $\zeta \mapsto \Phi_1(\zeta, 0, \cdots, 0)$ is a complex valued polynomial in ζ and $\overline{\zeta}$ of degree $\leq m$ so

$$\Phi_1(\zeta, 0, \cdots, 0) = q(\zeta - a) + \overline{s(\zeta - a)} \quad (\zeta \in b\Omega)$$

where q and s are polynomials of degree $\leq m$. Since $\Phi_1|L \cap bD$ does not extend holomorphically through $L \cap D$ it follows that the polynomial s is nonconstant so there is a $b \in \mathbb{C}$ such that the change of argument of $\zeta \mapsto \overline{s(\zeta - a)} - b$ along $b\Omega$ is negative. It follows that the degree of

$$\zeta \mapsto \Phi_1(\zeta, 0, \cdots, 0) - q(\zeta - a) - b \quad (\zeta \in b\Omega)$$

is negative. Define

$$P_1(z_1, \cdots, z_N) = -q(z_1 - a) - b.$$

Then $P_1\colon\ \mathbb{C}^N \to \mathbb{C}$ is a holomorphic polynomial of degree $\leq m$ and the degree of the map

$$\zeta \mapsto \Phi_1(\zeta, 0, \cdots, 0) + P_1(\zeta, 0, \cdots, 0) \quad (\zeta \in b\Omega) \tag{5.2}$$

is negative. As in the proof of Theorem 2.2 we see that there is $T > 0$ so large that if $P_j(z) = Tz_j$ $(2 \leq j \leq N)$ and if $P = (P_1, \cdots, P_N)$ then $\Phi + P \neq 0$ on bD and the degree of the map $z \mapsto \Phi(z) + P(z)$ $(z \in bD)$ coincides with the degree of the map (5.2). Thus we have constructed a holomorphic polynomial map $P\colon\ \mathbb{C}^N \to \mathbb{C}^N$ of degree $\leq m$ such that $\Phi + P \neq 0$ on bD and such that the degree of $(\Phi + P)|bD$ is negative. This completes the proof.

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