

SIMPLE HOPF ALGEBRAS AND DEFORMATIONS OF FINITE GROUPS

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ABSTRACT. We show that certain twisting deformations of a family of supersolvable groups are simple as Hopf algebras. These groups are direct products of two generalized dihedral groups. Examples of this construction arise in dimensions 60 and p^2q^2 , for prime numbers p, q with $q|p-1$. We also show that certain twisting deformation of the symmetric group is simple as a Hopf algebra. On the other hand, we prove that every twisting deformation of a nilpotent group is semisolvable. We conclude that the notions of simplicity and (semi)solvability of a semisimple Hopf algebra are not determined by its tensor category of representations.

1. Introduction and Main Results

Let G be a finite group. The character table of G provides substantial information about the group G itself like, for instance, normal subgroups and their orders, the center $Z(G)$, the nilpotence or solvability, etc. In particular, whether G is simple (respectively, solvable) or not can be established by inspection of its character table [10]. The character table provides the same information about G as does the Grothendieck ring of the tensor category $\text{Rep } G$ [3]. The above information is thus *a fortiori* determined by $\text{Rep } G$.

Finite dimensional Hopf algebras with tensor equivalent categories of representations are obtained from one another by a twisting deformation. Properties of H invariant under twisting are of special interest because they depend only on the tensor category $\text{Rep } H$.

In this paper we give a series of examples showing that the notions of simplicity and (semi)solvability of a (semisimple) Hopf algebra are *not* twist invariants; that is, they are not categorical notions.

In a first part we show that certain twists of the symmetric group \mathbb{S}_n on n letters are simple as Hopf algebras, for $n \geq 5$ (Theorem 3.5). To prove this we give a necessary condition for a group-like element in the dual of a twisting of a finite group to be central, in the case where the twisting is 'lifted' from an abelian subgroup.

Let p, r and q be prime numbers such that q divides $p-1$ and $r-1$. In a second part we show that a family of supersolvable groups G of order prq^2 can be deformed through a twist into nontrivial simple Hopf algebras (Theorem 4.5). These twists are also lifted from an abelian subgroup of G . The proof relies on the comparison of the (co)representation theory of the given twistings [7] with that of an extension [12]. We

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also make use of the classification of semisimple Hopf algebras in dimension p and pq [26, 14, 6, 9].

It is known that a semisimple Hopf algebra of dimension p^n is always semisolvable [13, 12]. On the other hand, we prove that if the group G is nilpotent then any twisting of kG is semisolvable.

Our results imply:

(a) *There exists a simple semisimple Hopf algebra which is neither twist equivalent to a simple group nor to the dual of a simple group.*

This answers negatively Question 2.3 in [1].

(b) *There exists a semisimple Hopf algebra of dimension p^2q^2 which is simple as Hopf algebra.*

Therefore the analogue of Burnside's p^aq^b -Theorem for finite groups does not hold for semisimple Hopf algebras. This concerns an open question raised by S. Montgomery; see [1, Question 4.17].

(c) *There exists a nontrivial semisimple Hopf algebra which is simple in dimension 36.*

This example gives a negative answer to [11, Question, pp. 269]. In dimension < 60 this is the only possible such Hopf algebra, by [16].

(d) *There exists a semisimple Hopf algebra which is a bosonization but not an extension.*

This answers Question 2.13 of [1]. Indeed, the Hopf algebras in Theorem 4.5 can be built up as a Majid-Radford biproduct or bosonization.

We show that there are exactly two twistings of groups of order 60 that can be simple as Hopf algebras: the twisting of A_5 constructed in [19], and the (self-dual) twisting of $D_3 \times D_5$ discussed in Subsection 4.2. This contributes to the problem in [1, Question 2.4].

The paper is organized as follows. In Section 2 we recall known facts on normal Hopf subalgebras and the twisting construction. In Section 3 we prove some results on dual central group-like elements in twisting of finite groups, and present the construction for the symmetric groups. Section 4 contains the construction for a family of supersolvable groups; at the end of this section we state our results in dimensions 60 and 36. Finally, in Section 5 we discuss twisting of nilpotent groups.

Along this paper we shall work over an algebraically closed base field k of characteristic zero. The notation for Hopf algebras is standard: Δ , ϵ , S , denote the comultiplication, counit and antipode, respectively.

2. Preliminaries

We discuss in this section basic facts on normal Hopf subalgebras and twisting deformations.

2.1. Normal Hopf subalgebras. Let A be a finite dimensional Hopf algebra over k . The (left) adjoint action of A on itself is defined by ${}^hx = \sum h_1x(Sh_2)$, $x, h \in A$. A Hopf subalgebra $K \subseteq A$ is called *normal* if it is stable under the adjoint action; A is called *simple* if it contains no proper normal Hopf subalgebras. Dualizing the

notion of normal Hopf subalgebra, we obtain the notion of conormal quotient Hopf algebra. The notion of simplicity is self-dual; that is, A is simple if and only if A^* is.

Let $K \subseteq A$ be a normal Hopf subalgebra. Then $B = A/AK^+$ is a conormal quotient Hopf algebra and the sequence of Hopf algebra maps $k \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow k$ is an exact sequence of Hopf algebras. In this case we shall say that A is an *extension* of B by K . As a Hopf algebra, A is isomorphic to a bicrossed product $A \cong K^\tau \#_\sigma B$.

We recall the following results for future use.

Proposition 2.1. [16, Corollary 1.4.3]. *Let $K \subseteq A$ be a normal Hopf subalgebra. Suppose that $\dim K$ is the least prime number dividing $\dim A$. Then K is central in A .* \square

Corollary 2.2. *Let $\pi : A \rightarrow B$ be a conormal quotient Hopf algebra. Suppose that $\dim B$ is the least prime number dividing $\dim A$. Then $G(B^*) \subset Z(A^*) \cap G(A^*)$.* \square

Example 2.3. Let G be a finite group. The normal Hopf subalgebras of kG are of the form kH where H is a normal subgroup of G . In particular, kG is simple as a Hopf algebra if and only if G is a simple group. In addition, if kG is simple, then it possesses no nontrivial quotient Hopf algebra. Therefore k^G contains no proper Hopf subalgebra.

2.2. Twisting. Let A be a finite dimensional Hopf algebra. The category $\text{Rep } A$ of its finite dimensional representations is a finite *tensor category* with tensor product given by the diagonal action of A and unit object k .

Finite tensor categories of the form $\text{Rep } A$ are characterized, using tannakian reconstruction arguments, as those possessing a fiber functor with values in the category of vector spaces over k . The forgetful functor $\text{Rep } A \rightarrow \text{Vec}_k$ is a fiber functor and other fiber functors correspond to *twisting* the comultiplication of A in the following sense.

Definition 2.4. [4]. A *twist* in A is an invertible $J \in A \otimes A$ satisfying:

$$(2.1) \quad (\Delta \otimes id)(J)(J \otimes 1) = (id \otimes \Delta)(J)(1 \otimes J),$$

$$(2.2) \quad (\varepsilon \otimes id)(J) = (id \otimes \varepsilon)(J) = 1.$$

If $J \in A \otimes A$ is a twist, $(A^J, m, \Delta^J, \varepsilon, S^J)$ is a Hopf algebra with $A^J = A$, $\Delta^J(h) = J^{-1} \Delta(h) J$, and $S^J(h) = v^{-1} S(h) v$, $h \in A$, $v = m \circ (S \otimes id)(J)$.

The Hopf algebras A and A' are called *twist equivalent* if $A' \cong A^J$. It is known that A and A' are twist equivalent if and only if $\text{Rep } A \cong \text{Rep } A'$ as tensor categories [23]. Therefore, properties like (quasi)triangularity, semisimplicity or the structure of the Grothendieck ring are preserved under twisting deformation.

Remark 2.5. Let $\pi : A \rightarrow B$ be a Hopf algebra map and let $J \in A \otimes A$ be a twist. Then $(\pi \otimes \pi)(J)$ is a twist for B and $\pi : A^J \rightarrow B^{(\pi \otimes \pi)(J)}$ is a Hopf algebra map.

Note that if $J \in K \otimes K$ is a twist for the Hopf subalgebra $K \subset A$, then $J \in A \otimes A$ is also a twist for A . We shall say that such J is *lifted* from the Hopf subalgebra K [5, 25].

If $A = kG$ is a group algebra, with G a finite abelian group, the (gauge) equivalence classes of twists for A are in bijective correspondence with the group $H^2(G, k^*)$ [15, Proposition 3]. The twist J corresponding to the cocycle $\omega : G \times G \rightarrow k^*$ is given by

$$(2.3) \quad J = \sum_{\alpha, \beta \in \widehat{G}} \omega(\alpha, \beta) e_\alpha \otimes e_\beta,$$

where $e_\chi = \frac{1}{|G|} \sum_{h \in G} \chi(h^{-1})h$, $\chi \in \widehat{G}$, is a basis of orthogonal central idempotents of kG .

Twists in finite groups have been completely classified [15, 8]. Every twist in kG is lifted from a *minimal* subgroup $H \subseteq G$ for J ; that is, the components of $J_{21}^{-1}J$ span kH . Gauge equivalence classes of twists are classified by classes of pairs (H, ω) , where H is some solvable subgroup whose order is a square, and $\omega \in H^2(H, k^*)$ is a non-degenerate 2-cocycle on H .

When H is abelian, the twist corresponding to (H, ω) is given by (2.3).

Recall that an element $g \in H$ is called ω -regular if $\omega(g, h) = \omega(h, g)$, for all $h \in Z_H(g)$ (this definition depends only on the class of g under conjugation). Then the cocycle $\omega \in H^2(H, k^*)$ is non-degenerate if and only if $\{1\}$ is the only ω -regular class in H .

In particular, if H is abelian, the cocycle ω is non-degenerate if and only if the skew-symmetric bilinear form $\omega_{21}^{-1}\omega : H \times H \rightarrow k^*$, $(g, h) \mapsto \omega(g, h)\omega^{-1}(h, g)$, is non-degenerate.

The following lemma follows from [7]. See also [19, Lemma 2.11].

Lemma 2.6. *Let $J \in kG \otimes kG$ be the twist associated to the pair (H, ω) , where H is the minimal subgroup of J . Then $(kG)^J$ is cocommutative if and only if $H \trianglelefteq G$, H is abelian and ω is $\text{ad } G$ -invariant in $H^2(H, k^*)$. \square*

In particular, every nonsymmetric twist lifted from an abelian subgroup of a simple nonabelian group, or from an abelian subgroup not containing normal subgroups of G , gives rise to a noncocommutative Hopf algebra.

Example 2.7. [19]. Let \mathbb{A}_n be the alternating group in n elements, $n \geq 4$. Consider the subgroup $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $a = (12)(34)$ and $b = (13)(24)$. Let ω be a 2-cocycle whose cohomology class is nontrivial. Since ω is not symmetric, by Lemma 2.6, $A = (k\mathbb{A}_n)^J$ is a simple noncommutative and noncocommutative Hopf algebra, for all $n \geq 5$.

As an algebra, $(k\mathbb{A}_5)^J \cong k \times M_3(k)^{(2)} \times M_4(k) \times M_5(k)$.

Recall that there is a one to one correspondence between quotient Hopf algebras of H and hereditary subrings of the Grothendieck ring $K_0(H)$ [17]. The subring corresponding to the quotient $H \rightarrow \overline{H}$ is $K_0(\overline{H}) \subseteq K_0(H)$.

Proposition 2.8. [19]. *Let G be a finite simple group and let $J \in kG \otimes kG$ be a twist. Then $(kG)^J$ is simple as a Hopf algebra. \square*

Indeed, the result is true under the weaker assumption that J is a pseudo-twist in kG . We shall see that the converse of Proposition 2.8 is not true.

3. Simple deformations of the symmetric groups

Let (A, R) be a finite dimensional quasitriangular Hopf algebra. The map $f : G(A^*) \rightarrow G(A)$, given by $f(\eta) = R^{(1)}\eta(R^{(2)})$, where $R = R^{(1)} \otimes R^{(2)}$, is an antihomomorphism of groups. In addition, $\eta \in G(A^*)$ is central in A^* if and only if $f(\eta)$ is central in A [20, Proposition 3].

In what follows we shall consider a finite group G and an abelian subgroup $H \subseteq G$. Let $A = (kG)^J$, where $J \in kG \otimes kG$ is a twist lifted from H , written in the form (2.3). Observe that $G(A^*) = \widehat{G}$.

Theorem 3.1. *Suppose that $Z(G) = 1$. Let $\eta \in \widehat{G}$. Then $\eta \in G(A^*) \cap Z(A^*)$ if and only if $\eta|_H$ is ω -regular.*

If ω is non-degenerate, $\eta \in G(A^) \cap Z(A^*)$ if and only if $\eta|_H = 1$.*

Here, $\eta|_H$ is the restriction of η to H .

Proof. The Hopf algebra A is triangular with R -matrix

$$R = J_{21}^{-1}J = \sum_{\alpha, \beta \in \widehat{H}} \omega(\alpha, \beta) \omega^{-1}(\beta, \alpha) e_\alpha \otimes e_\beta.$$

Since $Z(G) = 1$, we have $Z(A) \cap G(A) = 1$. That is, $\eta \in G(A^*) \cap Z(A^*)$ if and only if $f(\eta) = 1$. Let $\eta \in G(A^*)$. By the orthogonality relations, $\eta(e_\chi) = \delta_{\chi, \eta|_H}$, for all $\chi \in \widehat{H}$. Therefore,

$$f(\eta) = \sum_{\alpha, \beta \in \widehat{H}} \omega(\alpha, \beta) \omega^{-1}(\beta, \alpha) e_\alpha \eta(e_\beta) = \sum_{\alpha \in \widehat{H}} \omega(\alpha, \eta|_H) \omega^{-1}(\eta|_H, \alpha) e_\alpha.$$

Then $f(\eta) = 1$ if and only if $\omega(\chi, \eta|_H) \omega^{-1}(\eta|_H, \chi) = 1$, $\forall \chi \in \widehat{H}$. \square

Corollary 3.2. *Suppose $Z(G) = 1$ and ω is non-degenerate. Then the order of H divides $[A^* : G(A^*) \cap Z(A^*)]$.*

Proof. In this case, $\eta|_H = 1$, for all $\eta \in G(A^*) \cap Z(A^*)$, by Theorem 3.1. Then the projection $A \rightarrow k(G(A^*) \cap Z(A^*))^*$ restricts trivially to H . The corollary follows from [18]. \square

Let $\pi : A \rightarrow kF$ be a Hopf algebra quotient with F an abelian group. Then the group $\widehat{F} \cong F$ can be identified with a subgroup of $G(A^*)$.

Theorem 3.3. *Suppose $Z(G) = 1$. Let $\pi : A \rightarrow kF$ be a quotient Hopf algebra, where $F \cong \mathbb{Z}_p$ and p is the least prime dividing $|G|$. Then π is normal if and only if $\mu|_H$ is ω -regular, for all $\mu \in \widehat{F}$. Assume ω is non-degenerate. Then π is normal if and only if $\mu|_H = 1$, for all $\mu \in \widehat{F}$.*

Proof. By Theorem 3.1, condition $\omega(\chi, \mu|_H) = \omega(\mu|_H, \chi)$, for all $\chi \in \widehat{H}$, $\mu \in \widehat{F}$, is equivalent to $\widehat{F} \subset Z(A^*) \cap G(A^*)$. In view of Proposition 2.2 this is equivalent to π being normal. \square

Let $\pi : \mathbb{S}_n \rightarrow \mathbb{Z}_2$ be the only nontrivial epimorphism. So that $\ker \pi = \mathbb{A}_n$. Let $H \subseteq \mathbb{S}_n$ be an abelian subgroup, and let $A = (k\mathbb{S}_n)^J$ be a twisting with $J \in kH \otimes kH$ given by (2.3). As noted in Remark 2.5, $\pi : A \rightarrow k\mathbb{Z}_2$ is a Hopf algebra map.

Consider the sign representation $\sigma : \mathbb{S}_n \rightarrow k^*$.

Corollary 3.4. $\pi : A \rightarrow k\mathbb{Z}_2$ is normal if and only if $\sigma|_H$ is ω -regular.

Assume ω is non-degenerate. Then π is normal if and only if $H \subseteq \mathbb{A}_n$. \square

Let $n \geq 4$. Consider the abelian subgroup $H = \langle t_1, t_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ of \mathbb{S}_n , generated by the transpositions $t_1 = (12)$, $t_2 = (34)$.

We have $\widehat{H} = \langle a_1, a_2 \rangle$, where $a_i(t_j) = 1$ if $i \neq j$, and $a_i(t_i) = -1$.

Let ω be the unique nontrivial cocycle on \widehat{H} (up to coboundaries). Then ω is non-degenerate. Let $J \in kH^{\otimes 2}$ be the corresponding twist.

Theorem 3.5. Suppose $n \geq 5$. Then $(k\mathbb{S}_n)^J$ is simple.

Proof. In this case $\pi : \mathbb{S}_n \rightarrow \mathbb{Z}_2$ is the only nontrivial quotient of \mathbb{S}_n . Since twisting preserves Grothendieck rings, $A = (k\mathbb{S}_n)^J$ has also a unique nontrivial quotient $\pi : A \rightarrow k\mathbb{Z}_2$. Since $H \not\subseteq \mathbb{A}_n$, by Corollary 3.4, π is not normal. Therefore $(k\mathbb{S}_n)^J$ is simple, as claimed. \square

Remark 3.6. Let $A = (k\mathbb{S}_n)^J$ as in Theorem 3.5. Then A^* is simple and it is not a twisting of any group. Otherwise A^* and A would be both quasitriangular and simple, hence $G(A) \cong G(A^*)$ [21, Proposition 4]. But this is not the case, since $H \subseteq G(A)$ and $|G(A^*)| = 2$.

Another family of examples arises from the construction in [2]. Let t_i denote the transposition $(2i-1, 2i) \in \mathbb{S}_{2n}$, $1 \leq i \leq n$. Consider the abelian subgroup $H = \langle t_i, 1 \leq i \leq n \rangle \cong (\mathbb{Z}_2)^n$ of \mathbb{S}_{2n} .

We have $\widehat{H} = \langle a_i : 1 \leq i \leq n \rangle$, where $a_i(t_j) = 1$ if $i \neq j$, and $a_i(t_i) = -1$. Note that $\sigma|_H = a_1 a_2 \cdots a_n$.

Consider the bicharacter $\omega : \widehat{H} \times \widehat{H} \rightarrow k^*$, $\omega(a_i, a_j) = -1$, $i < j$, $\omega(a_i, a_j) = 1$, $i \geq j$. This example does not fulfill the condition in Proposition 3.4 for $n \geq 2$ even and $a = a_1$, since we have $\omega(\sigma|_H, a_1) = (-1)^{n-1} = -1$, while $\omega(a_1, \sigma|_H) = 1$. Let J be the corresponding twist.

Theorem 3.7. Suppose $n \geq 3$, n even. Then $(k\mathbb{S}_{2n})^J$ is simple.

Proof. Again in this case $\pi : \mathbb{S}_{2n} \rightarrow \mathbb{Z}_2$ is the only nontrivial quotient of \mathbb{S}_{2n} . So that $A = (k\mathbb{S}_{2n})^J$ has also a unique nontrivial quotient $\pi : A \rightarrow k\mathbb{Z}_2$. By Theorem 3.5 π is not normal. Thus $(k\mathbb{S}_{2n})^J$ is simple. \square

Remark 3.8. Let us see that for any twist $J \in k\mathbb{S}_4 \otimes k\mathbb{S}_4$, $(k\mathbb{S}_4)^J$ is not simple; see [16, Chapter 6]. We know from [15, 8] that the minimal subgroup H for J is some solvable subgroup whose order is a square admitting a non-degenerate 2-cocycle. Then for $k\mathbb{S}_4$ a twist must be lifted from a subgroup H of order 4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Claim 3.9. Let $A = (k\mathbb{S}_4)^J$, where J is the twist lifted from a subgroup $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which is not normal and $\omega \neq 1$. Then $G(A) \cong D_4$.

Proof. Let $D \cong D_4$ be the Sylow 2-subgroup of \mathbb{S}_4 containing H . Then there are Hopf algebra inclusions $kH \cong (kH)^J \hookrightarrow (kD)^J \hookrightarrow (k\mathbb{S}_4)^J$.

Since $(kD)^J$ is not commutative and of dimension 8, $(kD)^J \cong kD$ as Hopf algebras [24]. Thus 8 divides $|G((k\mathbb{S}_4)^J)|$. As H contains no subgroup which is normal in \mathbb{S}_4 , by Lemma 2.6, $(k\mathbb{S}_4)^J$ is not cocommutative. Then $|G((k\mathbb{S}_4)^J)| = 8$ and $G(A) \cong D \cong D_4$. \square

Let the quotients $B = \mathbb{S}_4/K \cong \mathbb{S}_3$, and $\zeta : (k\mathbb{S}_4)^J \rightarrow (k\mathbb{S}_3)^{\zeta(J)} \cong k\mathbb{S}_3$. We claim that ζ is normal.

Indeed, we have $\dim A^{coB} = 4$. Then $\dim A^{coB} \cap kG(A) = \dim kG(A)^{coB} = 1, 2, 4$, since $|kG(A)| = 8$. Also, $\dim kG(A)^{coB} \dim \zeta(kG(A)) = 8$. If $\dim kG(A)^{coB} = 1, 2$, then $\dim \zeta(kG(A)) = 8, 4$, which is impossible. Then $\dim A^{coB} \cap kG(A) = 4$; that is, $A^{coB} \subset kG(A)$. Hence A^{coB} is a normal Hopf subalgebra.

4. Deformations of a family of supersolvable groups

Let p, q and r be prime numbers such that q divides $p-1$ and $r-1$. Let $G_1 = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ and $G_2 = \mathbb{Z}_r \rtimes \mathbb{Z}_q$ be the only nonabelian groups of orders pq and rq , respectively. Let $G = G_1 \times G_2$ and let $\mathbb{Z}_q \times \mathbb{Z}_q \cong H \subseteq G$ a subgroup of order q^2 . In particular, G is supersolvable and $Z(G) = 1$.

Let $1 \neq \omega \in H^2(\widehat{H}, k^*)$, $J \in kG \otimes kG$ the twist lifted from H corresponding to ω . Let also $A = (kG)^J$. Note that the cocycle ω is nondegenerate. Also, A is a nontrivial Hopf algebra of dimension prq^2 .

Lemma 4.1. $A \cong k^{(q^2)} \times M_q(k) \times \cdots \times M_q(k) \times M_{q^2}(k) \times \cdots \times M_{q^2}(k)$ as an algebra.
 $p+r-2$ copies
 $\frac{(p-1)(r-1)}{q^2}$ copies

Proof. As an algebra, $A = kG \simeq kG_1 \otimes kG_2$. □

The coalgebra structure of A follows from the result in [7] on representations of cotriangular semisimple Hopf algebras. In particular:

Lemma 4.2. $G(A) \cong H$ is of order q^2 . □

Proof. Let, for all $g \in G$, $H_g = H \cap gHg^{-1}$, and let $\tilde{\omega}$ be the 2-cocycle on H_g given by $\tilde{\omega}(x, y) = \omega^{-1}(g^{-1}xg, g^{-1}yg)\omega(x, y)$. By [7] the irreducible representations of A^* are classified by pairs (\bar{g}, X) , where $\bar{g} \in H \backslash G/H$ is a double coset modulo H and X is an irreducible representation of the twisted group algebra $k_{\tilde{\omega}}H_g$. The dimension of the representation $W_{\bar{g}, X}$ corresponding to (\bar{g}, X) is $\dim W_{\bar{g}, X} = [H : H_g] \dim X$.

Note that $\tilde{\omega}$ is trivial on $H_1 = H$. Thus $\dim W_{\bar{1}, X} = 1$, for all possible choices of X , giving $|H|$ distinct one-dimensional representations.

Also, $\dim W_{\bar{g}, X} = 1$ if and only if $H_g = H$ and $\dim X = 1$. That is, if and only if $g \in N_G(H)$ and $\dim X = 1$. In our example $N_G(H) = H$. So if $\dim W_{\bar{g}, X} = 1$, then $\bar{g} = \bar{1}$ and there are no more one-dimensional representations. □

Lemma 4.3. Suppose A is not simple and let $K \subseteq A$ be a proper normal Hopf subalgebra. Then $\dim K = pq$ or rq .

Moreover, K is necessarily commutative and not cocommutative and there is an exact sequence of one of the forms

$$k \rightarrow k^{G_1} \rightarrow A \rightarrow kG_2 \rightarrow k, \quad k \rightarrow k^{G_2} \rightarrow A \rightarrow kG_1 \rightarrow k.$$

Proof. Let $B = A/AK^+$. There is a Hopf algebra inclusion $B^* \subseteq A^*$ and B^* is normal in A^* . Since $Z(G) = 1$, we have $G(A) \cap Z(A) = 1$. In particular, $\dim K \neq q$ [26]. Suppose $\dim B^* = q$. Then $B^* = kG(B^*)$ and $G(B^*) \subseteq G(A^*) \cap Z(A^*)$. Since ω is non-degenerate, Corollary 3.2 implies that $q^2|[A^* : B^*] = prq$, which is impossible. Hence $\dim B \neq q$.

If $\dim B = q^2 p, q^2 r, q^2$, then $\dim K = r, p, pr$ and K is a group algebra or a dual group algebra [26, 14, 6, 9]. Thus A has group-like elements of order p or r , contradicting $|G(A)| = q^2$. Then $\dim K = pq, rq$. Since $|G(A)| = q^2$, $K \cong k^{G_1}$ or k^{G_2} . Similarly, these are the only possibilities for B^* . \square

Remark 4.4. The results in [7] actually imply that $A^* \cong A$ as algebras. Once we have established that A is simple, then we will see in Subsection 4.1 that $A^{*\text{cop}} \cong A$ as Hopf algebras.

Theorem 4.5. *A is simple as a Hopf algebra.*

Proof. Suppose not. We shall compute the dimensions of the irreducible A -modules to get a contradiction. By Lemma 4.3, without loss of generality, we may assume that A is a bicrossed product $A \cong k^{G_1 \tau} \#_{\sigma} kG_2$. By [12] the irreducible A -modules are classified by pairs (x, U) , where x is a representative of an orbit of the action of G_2 in G_1 and U is an irreducible projective representation of the stabilizer $(G_2)_x$ of x . The irreducible module $W_{(x,U)}$ corresponding to (x, U) is the induced module $W_{(x,U)} = \text{Ind}_B^A(kx \otimes U)$, where $B = k^{G_1} \#_{\sigma} (kG_2)_x$. Therefore $\dim W_{(x,U)} = [G_2 : (G_2)_x] \dim U$.

Hence the dimension of an irreducible module cannot be q^2 . This contradicts Lemma 4.1. The theorem is now established. \square

Remark 4.6. We point out the following consequence of Theorem 4.5. By Lemmas 4.1 and 4.2, A is a biproduct $A = R \# kH$ in the sense of Majid–Radford, with $\dim R = pr$. This gives a nontrivial braided Hopf algebra structure on R over H , in such a way that the corresponding biproduct is *simple*. This answers [1, Question 2.13].

Letting $p = r$ in Theorem 4.5 we obtain:

Theorem 4.7. *Let p, q , be prime numbers such that $q|p-1$. Then there exists a semisimple Hopf algebra of dimension $p^2 q^2$ which is simple as a Hopf algebra.* \square

The theorem disproves a conjectured ‘quantum version’ of Burnside’s $p^a q^b$ -Theorem in the context of semisimple Hopf algebras. It also gives a negative answer to [1, Question 2.3].

4.1. Self-duality. If G is finite group and J is a minimal twist in G , then the twisted group algebra $A = (kG)^J$ satisfies $A \cong A^{*\text{cop}}$ [8]. We show next that this is also true under other restrictions on G and J .

Proposition 4.8. *Let G be a finite solvable group and let $J \in kG \otimes kG$ be any twist. Assume $A = (kG)^J$ is simple. Then $A \cong A^{*\text{cop}}$.*

By comparing the descriptions of the representation theories of G and $(kG^J)^*$, we see that this proposition imposes severe restrictions on the possible groups G satisfying the assumptions.

Proof. Since A is simple, then $Z(G) = 1$. Suppose $F \leq G$. Then G acts on k^F by the adjoint action and the smash product $k^F \# kG$ is a Hopf algebra quotient of $D(G)$ (actually, $D(G)$ corresponds to $F = G$). For the Hopf algebra $D_F(G) = k^F \# kG$ we have $G(D_F(G)) \cap Z(D_F(G)) = \widehat{F} \times Z(G) = \widehat{F}$ and

$$D_F(G)/D_F(G)(k\widehat{F})^+ \cong k^{[F,F]} \# kG = D_{[F,F]}(G).$$

Denote $G = G^{(0)}$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, $i \geq 0$. Since G is solvable, there exists $m \geq 1$ such that $G^{(m)} = 1$. Iterating the construction above, we get a sequence

$$D(G) \xrightarrow{\pi_1} D_{G^{(1)}}(G) \xrightarrow{\pi_2} D_{G^{(2)}}(G) \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_m} D_{G^{(m)}}(G) = kG,$$

where every map π_i has central kernel. Since A is twist equivalent to kG then $D(A)$ is twist equivalent to $D(G)$. In view of Lemma 5.2, we get another sequence of Hopf algebra maps with central kernels

$$D(A) \xrightarrow{\pi_1} K_1 \xrightarrow{\pi_2} K_2 \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_m} K_m = (kG)^{J'}.$$

Since the maps π_i are normal and A and $A^{*\text{cop}}$ are simple and nontrivial, we may assume that the composition $\pi_1 \circ \cdots \circ \pi_m : D(A) \rightarrow (kG)^{J'}$ is injective when restricted to A and to $A^{*\text{cop}}$. By dimension, $A \cong (kG)^{J'} \cong A^{*\text{cop}}$. \square

4.2. Twisting deformations of groups of order 60. As we saw in Example 2.7, there is a nontrivial simple Hopf algebra of dimension 60, obtained as a twisting of the alternating group \mathbb{A}_5 . Another example arises as a twisting deformation of the group $D_3 \times D_5$, by Theorem 4.5. For this example, $A \cong A^{*\text{cop}} \cong k^{(4)} \oplus M_2(k)^{(6)} \oplus M_4(k)^{(2)}$ as algebras; so A is not a twisting of $k\mathbb{A}_5$. We shall prove that these are the only simple Hopf algebras that can arise as twistings of groups of order 60.

For the rest of this subsection, G will be a group of order 60, $J \in kG \otimes kG$ a twist, and $A = (kG)^J$ a nontrivial deformation. Since A is not trivial, the minimal subgroup H associated to J must be of order 4. Then necessarily $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and J corresponds to the only nontrivial 2-cocycle on H .

We assume first that G is not simple.

Lemma 4.9. *Suppose A is simple. Then $G \cong D_3 \times D_5$.*

Proof. First note that since G is not simple, then G contains a unique subgroup of order 5. Second, the subgroup H cannot be contained in a normal subgroup P of G , since otherwise, the Hopf subalgebra $kP^J \subset (kG)^J$ would be normal. Similarly, we may assume that $Z(G) = 1$.

Let $S \trianglelefteq G$ of order 5. We may assume that $G' = G/S$ has a normal subgroup T of order 3, since otherwise G would contain a normal subgroup of order 20, thus containing H . Then $N = \pi^{-1}(T) \trianglelefteq G$ is of order 15; so G is a semidirect product $N \rtimes H$. Because $Z(G) = 1$, $G \cong D_3 \times D_5$. \square

Theorem 4.10. *Let $|G| = 60$ and let $J \in kG \otimes kG$ be a twist such that $A = (kG)^J$ is not cocommutative and simple. Then either*

- (i) $G = \mathbb{A}_5$ and A is isomorphic to the Hopf algebra in Example 2.7; or
- (ii) $G = D_3 \times D_5$ and A is isomorphic to the self-dual Hopf algebra in Theorem 4.5.

Since the Hopf algebra in (i) is not self-dual, this gives three simple examples of semisimple Hopf algebras in dimension 60.

Proof. We use Lemma 4.9. Note that the subgroups of order 4 in G are conjugated, and $|H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, k^*)| = 2$. Hence every pair of nontrivial twists in G gives rise to isomorphic Hopf algebras. \square

4.3. Twisting deformations of groups of order 36. The main construction of this section gives an example of a nontrivial semisimple Hopf algebra A of dimension 36 that is simple as a Hopf algebra. This is thus the smallest semisimple Hopf algebra which is not semisolvable and the unique simple case in dimension 36 [16]. The Hopf algebra A is a twisting of the group $D_3 \times D_3$, and we have $A \cong A^{*\text{cop}} \cong k^{(4)} \oplus M_2(k)^{(4)} \oplus M_4(k)$ as algebras.

Theorem 4.11. *Let G be a group of order 36 and let $J \in kG \otimes kG$ be a twist such that $(kG)^J$ is simple. Then $G \cong D_3 \times D_3$ and $(kG)^J \cong A$.*

Proof. We may assume $Z(G) = 1$. If J is a non-degenerate twist, then necessarily $G = \mathbb{Z}_3 \times \mathbb{A}_4$ or $G = \mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_6)$. This contradicts the assumption $Z(G) = 1$; hence, we may assume that J is not minimal. Moreover, the minimal subgroup H of G has order 4 or 9 and it is not cyclic. Also, H is not contained in any normal subgroup. This leads to consider the case $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G \cong D_3 \times D_3$. Moreover all subgroups of order 4 are conjugated and the conjugation action must preserve nontrivial twists. \square

5. Twisting of nilpotent groups

We have shown simple Hopf algebras obtained as twisting of supersolvable groups. We now prove that this cannot arise from nilpotent groups.

Let H be a finite dimensional Hopf algebra over k .

Definition 5.1. [12]. A *lower normal series* for H is a series of proper Hopf subalgebras $H_n = k \subset H_{n-1} \subset \cdots \subset H_1 \subset H_0 = H$, where H_{i+1} is normal in H_i , for all i . The *factors* are the quotients $\overline{H}_i = H_i/H_i H_{i+1}^+$.

An *upper normal series* is inductively defined as follows. Let $H_{(0)} = H$. Let H_i be a normal Hopf subalgebra of H_{i-1} and define $H_{(i)} = H_{(i)}/H_{(i)} H_i^+$. Assume that $H_n = H_{(n-1)}$, for some positive integer n such that $H_{(n)} = k$. The *factors* are the Hopf subalgebras H_i of the quotients $H_{(i)}$.

Let G be a finite group and let $A = (kG)^J$ be a twisting.

Lemma 5.2. *Let $Z \subset G$ be a central subgroup. Then $kZ \subset A$ is a central Hopf subalgebra and $A/A(kZ)^+ \cong (kG/Z)^{\overline{J}}$.*

Proof. Since $A = kG$ as algebras, kZ is central and $\Delta^J(a) = a \otimes a$. Let $\pi : kG^J \rightarrow k(G/Z)^{\overline{J}}$ be the Hopf algebra map induced by the projection $G \rightarrow G/Z$. Since $kZ \subset A^{\text{cop}}$ and $\dim A = \dim A^{\text{cop}} \dim \pi(A)$, $kZ = A^{\text{cop}}$. \square

Theorem 5.3. *Suppose G is nilpotent. Then*

- (1) *A has an upper normal series with factors $k\mathbb{Z}_p$, $p \mid \dim A$, prime.*
- (2) *A has a lower normal series whose factors are cocommutative.*

In particular, A is semisolvable in the sense of [12].

Proof. (1). Since G is nilpotent, $Z(G) \neq 1$. Let $Z \subset Z(G)$ be a subgroup of order p , p prime. Let $H_1 = kZ \cong k\mathbb{Z}_p$ and $H_{(1)} = A/AH_1^+$. By Lemma 5.2, $H_{(1)} \cong k(G/Z)^{\overline{J}}$. Since G/Z is nilpotent, (1) follows by induction on $|G|$.

(2). Let $H \subset G$ be the subgroup such that $J \in kH \otimes kH$ is minimal. Since every subgroup of a nilpotent group is subnormal [22], we have $H_0 = H \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$. Then

$$(5.1) \quad kH_0 = kH^J \triangleleft kH_1^J \triangleleft \cdots \triangleleft kH_n^J = kG^J,$$

is part of a lower normal series of A with factors $k[H_{i+1}/H_i]$, since $J \in kH_i$, for all i . Since H is nilpotent, there is an upper normal series $H_{(0)} = kH^J \twoheadrightarrow H_{(1)} \cdots \twoheadrightarrow H_{(s)} = k$. Moreover, $kH^J \cong (kH^J)^*$ because kH^J is minimal. Thus the dual of this series, that is, $k \hookrightarrow H_{(1)}^* \hookrightarrow \cdots \hookrightarrow (kH^J)^* \cong kH^J$, is a lower normal series for kH^J that completes the series (5.1). \square

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