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GIVENTAL'S LAGRANGIAN CONE AND S^1-EQUIVARIANT GROMOV–WITTEN THEORY

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Abstract. In the approach to Gromov–Witten theory developed by Givental, genus-zero Gromov–Witten invariants of a manifold X are encoded by a Lagrangian cone in a certain infinite-dimensional symplectic vector space. We give a construction of this cone, in the spirit of S^1-equivariant Floer theory, in terms of S^1-equivariant Gromov–Witten theory of $X \times \mathbb{P}^1$. This gives a conceptual understanding of the "dilaton shift": a change-of-variables which plays an essential role in Givental's theory.

1. Introduction

It has long been understood that it is a good idea to arrange Gromov–Witten invariants into generating functions which reflect their origins in physics: many operations in Gromov–Witten theory which seem complicated when viewed at the level of individual invariants correspond to the application of rather simpler differential operators to these generating functions. A recent insight of Givental is that such differential operators, which can themselves appear quite complicated, are often the quantizations of very simple linear symplectic transformations of a certain symplectic vector space. This point of view — Givental's *quantization formalism* [22,23] — has been a crucial ingredient in several recent advances in the subject. These include the proof of the Virasoro conjecture for toric Fano manifolds [21], the computation of twisted Gromov–Witten invariants [7,42], the proof of a Quantum Hirzebruch–Riemann–Roch theorem relating quantum extraordinary cohomology to quantum cohomology [8,9], and the construction of integrable hierarchies controlling the total descendant potentials of certain Frobenius manifolds [24,26,39].

The symplectic vector space associated to the Gromov–Witten theory of an almost-Kähler manifold X is the space of Laurent series

$$\mathcal{H} = H^\bullet(X) \otimes \mathbb{C}(\!(z^{-1})\!)$$

equipped with the symplectic form

$$\Omega(f,g) = \mathrm{Res}_{z=0} \Big(f(-z), g(z)\Big)\, dz.$$

Here $(\cdot,\cdot)$ is the Poincaré pairing on $H^\bullet(X)$. Generating functions for Gromov–Witten invariants — the genus-g Gromov–Witten potentials of X and the total descendant potential of X — are regarded as functions on $\mathcal{H}_+ = H^\bullet(X)[z]$ via a change of

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variables, called the *dilaton shift*, described in equation 6 below. Genus-zero Gromov–Witten invariants are encoded by a certain Lagrangian submanifold $\mathcal{L}$ of $\mathcal{H}$, defined in section 2.3 below. This submanifold $\mathcal{L}$ has a very tightly-constrained geometry: it is a Lagrangian cone ruled by a finite-dimensional family of isotropic subspaces [7, 23].

We currently lack a conceptual understanding of why the quantization formalism is so effective. It makes sense, therefore, to look for a geometric interpretation of the ingredients of the formalism — of the symplectic vector space $\mathcal{H}$, the submanifold $\mathcal{L}$, and the dilaton shift. In this paper we give a simple and geometric construction of the submanifold $\mathcal{L}$ in terms of the S^1-equivariant Gromov–Witten theory of the space $X \times \mathbb{P}^1$. This gives rise to the dilaton shift in a natural way. Our construction suggests that $\mathcal{H}$ should be thought of as the S^1-equivariant Floer homology of the loop space of X; this is discussed further in Section 3 below.

The idea of the construction is as follows. There is an "evaluate at infinity" map

$$\mathrm{ev}_\infty : (X \times \mathbb{P}^1)^{op}_{0,n,(d,1)} \to X$$

from an open set in the moduli space of stable maps to $X \times \mathbb{P}^1$ of bidegree $(d, 1)$ from genus-zero curves with n marked points. This open set is the locus of stable maps $f : \Sigma \to X \times \mathbb{P}^1$ such that the preimage $f^{-1}(X \times \{\infty\})$ consists of a single unmarked smooth point — so there are no bubbles or marked points over $\infty \in \mathbb{P}^1$ — and the map ev_∞ records the point of X mapped to by $f^{-1}(X \times \{\infty\})$. Although ev_∞ is not proper, it is equivariant with respect to the S^1-action on $(X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}$ coming from the S^1-action of weight -1 on the second factor of $X \times \mathbb{P}^1$ and the trivial S^1-action on X. This allows us to define a push-forward

$$(\mathrm{ev}_\infty)_\star : H^\bullet_{S^1}\left((X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}\right) \otimes \mathbb{C}(\!(z^{-1})\!) \to H^\bullet(X) \otimes \mathbb{C}(\!(z^{-1})\!),$$

where $H^\bullet_{S^1}(pt) = \mathbb{C}[z]$: the restriction of the map ev_∞ to S^1-fixed sets is proper, so we can define the push-forward using fixed-point localization. To push an equivariant class forward along ev_∞ we first restrict it to the S^1-fixed set in $(X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}$, then cap with the virtual fundamental class of the fixed set, then divide by the S^1-equivariant Euler class of the virtual normal bundle, and then push forward (in the usual sense) along the map ev_∞ from the S^1-fixed set to X. One can think of this operation as a virtual push-forward in S^1-equivariant Borel–Moore–Tate homology; it is defined only over the field of fractions $\mathbb{C}(z)$ of $H^\bullet_{S^1}(pt)$, and not over $H^\bullet_{S^1}(pt)$ itself, because we need to divide by the Euler class of the virtual normal bundle. The Lagrangian cone $\mathcal{L}$ is the image of a certain class

$$(-z) \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \frac{Q^d}{n!} \prod_{i=1}^{i=n} \mathrm{ev}_i^\star \, \mathbf{t}(\psi_i) \in \bigoplus_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} H^\bullet_{S^1}\left((X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}\right), \tag{1}$$

defined in detail in Section 2 below, under this push-forward.

The dilaton shift arises here in the following way: the S^1-fixed set in the space $(X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}$ can almost always be identified with the space $X_{0,n+1,d}$ of degree-d stable maps to X from genus-zero curves with $n + 1$ marked points. There are two exceptions to this, however: when $(n, d) = (0, 0)$ and when $(n, d) = (1, 0)$, the moduli space $X_{0,n+1,d}$ is empty but the S^1-fixed set is a copy of X. It is the contributions to the push-forward of (1) coming from these exceptional fixed loci which give rise to the dilaton shift. In the notation of Section 2, the push-forward of (1) is

$$\underbrace{-z + \mathbf{t}(z)}_{\substack{\text{contributions from}\\ \text{exceptional fixed loci}}} \quad + \\ \sum_{\substack{d \in H_2(X;\mathbb{Z})\\ n \geq 0}} \frac{Q^d}{n!} \underbrace{(\mathrm{ev}_{n+1})_\star \left[[X_{0,n+1,d}]^{vir} \cap \left(\prod_{i=1}^{i=n} \mathrm{ev}_i^\star \, \mathbf{t}(\psi_i) \right) \cdot \frac{1}{-z - \psi_{n+1}} \right]}_{\text{contribution from } X_{0,n+1,d}}.$$

This makes the change of variables (6) seem very natural.

We should emphasize that none of the geometric ingredients here are new. The observation that a product of two copies of the J-function — a certain generating function for genus-zero Gromov–Witten invariants — can be computed by fixed-point localization on the graph space $X \times \mathbb{P}^1$ was, or was equivalent to, a crucial step in proving mirror theorems for toric varieties [3, 19, 20, 30–32]. The equivariant push-forward described above was introduced by Braverman [4] in order to extract one copy of the J-function of a flag manifold from the corresponding graph space. The content of this paper is the observation that when Braverman's construction is extended to "big quantum cohomology" and to include gravitational descendants, the dilaton shift emerges automatically.

Experts in the subject may wish to stop reading here, as what follows is routine. Section 2 contains an introduction to Givental's quantization formalism. The details of the construction of $\mathcal{L}$ are in Theorem 1 and Section 3. The localization theorem which we need does not appear to have been written down anywhere, so we prove it in the Appendix.

2. Givental's Quantization Formalism

We begin by describing the quantization formalism. We fix notation for Gromov–Witten invariants in section 2.1 and discuss the framework for working with higher-genus invariants in section 2.2. The latter section is not logically necessary: the reader who is familiar with Givental's approach or uninterested in the surrounding context should skip to section 2.3, where the genus-zero picture is described.

2.1. Gromov–Witten Invariants. Let X be a smooth projective variety[1]. The Gromov–Witten invariants of X are certain intersection numbers in moduli spaces of stable maps (see *e.g.* [16, 34, 36, 38, 41]). Let $X_{g,n,d}$ denote the moduli space of stable maps to X of degree $d \in H_2(X;\mathbb{Z})$ from curves of genus g with n marked points, and let $[X_{g,n,d}]^{vir}$ be its virtual fundamental class [1, 2, 37]. The moduli space comes equipped with evaluation maps

$$\mathrm{ev}_i : X_{g,n,d} \to X \qquad\qquad i \in \{1, \ldots, n\}$$

[1]A virtual localization theorem has recently been established in the symplectic category [5, 27], and so the constructions here can now be extended to the case of almost-Kähler target manifolds X.

and universal cotangent line bundles

$$L_i \to X_{g,n,d} \qquad\qquad i \in \{1, \ldots, n\}$$

at each marked point. We denote the first Chern class of L_i by ψ_i. Gromov–Witten invariants are intersection numbers of the form

$$\int_{[X_{g,n,d}]^{vir}} \prod_{i=1}^{i=n} \mathrm{ev}_i^\star(\alpha_i) \cdot \psi_i^{k_i}, \tag{2}$$

where $\alpha_1, \ldots, \alpha_n$ are cohomology classes on X and $k_1, \ldots, k_n$ are non-negative integers. If any of the k_i are non-zero, such invariants are called gravitational descendants.

The *genus-g descendant potential of* X is a generating function for Gromov–Witten invariants:

$$\mathcal{F}_X^g(t_0, t_1, \ldots) = \sum_{d \in H_2(X;\mathbb{Z})} \sum_{n \geq 0} \frac{Q^d}{n!} \int_{[X_{g,n,d}]^{vir}} \prod_{i=1}^{i=n} \mathrm{ev}_i^\star \mathbf{t}(\psi_i).$$

Here $t_0, t_1, \ldots$ are cohomology classes on X; $\mathbf{t}(\psi) = t_0 + t_1\psi + t_2\psi^2 + \ldots$, so that

$$\mathrm{ev}_i^\star \mathbf{t}(\psi_i) = \mathrm{ev}_i^\star(t_0) + \mathrm{ev}_i^\star(t_1) \cdot \psi_i + \mathrm{ev}_i^\star(t_2) \cdot \psi_i^2 + \ldots; \tag{3}$$

and Q^d is the representative of d in the Novikov ring [38, III 5.2.1], which is a certain completion of the group ring of $H_2(X;\mathbb{Z})$. If we pick a basis $\{\phi_1, \ldots, \phi_N\}$ for $H^\bullet(X;\mathbb{C})$ and write

$$t_i = t_i^1 \phi_1 + \ldots + t_i^N \phi_N \tag{4}$$

then

$$\mathcal{F}_X^g(t_0, t_1, \ldots) = \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \sum_{\substack{k_1, \ldots, k_n \\ \alpha_1, \ldots, \alpha_n}} \frac{Q^d t_{k_1}^{\alpha_1} \ldots t_{k_n}^{\alpha_n}}{n!} \int_{[X_{g,n,d}]^{vir}} \prod_{i=1}^{i=n} \mathrm{ev}_i^\star(\phi_{\alpha_i}) \cdot \psi_i^{k_i},$$

so we can regard $\mathcal{F}_X^g$ as a formal power series with Taylor coefficients given by Gromov–Witten invariants of X. The *total descendant potential of* X

$$\mathcal{D}_X(t_0, t_1, \ldots) = \exp\left(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_X^g(t_0, t_1, \ldots) \right)$$

is a generating function for Gromov–Witten invariants of all genera.

2.2. The quantization formalism. Consider the space

$$\mathcal{H} = H^\bullet(X)(\!(z^{-1})\!)$$

of cohomology-valued Laurent series, equipped with the symplectic form

$$\Omega(f, g) = \mathrm{Res}_{z=0} \Big(f(-z), g(z) \Big)\, dz.$$

Here and from now on we work over a ground ring Λ which is the tensor product of the Novikov ring with $\mathbb{C}$: we take cohomology with coefficients in Λ, the Poincaré pairing $(\cdot, \cdot)$ and the symplectic form are Λ-valued, and so on. The space $\mathcal{H}$ is the direct sum of Lagrangian subspaces

$$\mathcal{H}_+ = H^\bullet(X)[z], \qquad\qquad \mathcal{H}_- = z^{-1} H^\bullet(X)[\![z^{-1}]\!].$$

A general element of $\mathcal{H}$ takes the form

$$\sum_{k=0}^{\infty}\sum_{\mu=1}^{N} q_k^{\mu}\,\phi_\mu z^k + \sum_{l=0}^{\infty}\sum_{\nu=1}^{N} p_l^{\nu}\,\phi^{\nu}(-z)^{-1-l} \tag{5}$$

where $\{\phi_1,\ldots,\phi_N\}$ is our basis for $H^{\bullet}(X)$, we set $g_{\alpha\beta}=(\phi_\alpha,\phi_\beta)$, define $g^{\alpha\beta}$ to be the (α,β)-entry of the matrix inverse to that with (α,β)-entry $g_{\alpha\beta}$, and raise indices using $g^{\alpha\beta}$:

$$\phi^{\nu}=\sum_{\lambda=1}^{N} g^{\nu\lambda}\phi_\lambda.$$

Equation (5) defines Darboux co-ordinates $\{q_k^{\mu},p_l^{\nu}\}$ on $\mathcal{H}$ which are compatible with the polarization $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$.

To each linear infinitesimal symplectic transformation $A\in\mathfrak{sp}(\mathcal{H})$ we associate a differential operator — the *quantization* of A — constructed as follows. The quadratic Hamiltonian $h_A : f\mapsto \frac{1}{2}\Omega(Af,f)$ can be written as a linear combination of quadratic monomials in the Darboux co-ordinates $\{q_k^{\mu},p_l^{\nu}\}$. We set

$$\widehat{q_k^{\mu}q_l^{\nu}}=\frac{q_k^{\mu}q_l^{\nu}}{\hbar}, \qquad \widehat{p_k^{\mu}q_l^{\nu}}=q_l^{\nu}\frac{\partial}{\partial q_k^{\mu}}, \qquad \widehat{p_k^{\mu}p_l^{\nu}}=\hbar\frac{\partial}{\partial q_k^{\mu}}\frac{\partial}{\partial q_l^{\nu}},$$

and extend by linearity, defining the quantization $\widehat{A}$ of A to equal $\widehat{h_A}$. The quantized operator $\widehat{A}$ acts on certain[2] formal power series in the variables q_k^{α}, where $\alpha\in\{1,\ldots,N\}$ and $k\geq 0$.

Let

$$q_k=\sum_{\lambda=1}^{N} q_k^{\lambda}\phi_\lambda \qquad\qquad k=0,1,2,\ldots,$$

and

$$\mathbf{q}(z)=q_0+q_1z+q_2z^2+\ldots.$$

Quantized infinitesimal symplectic transformations $\widehat{A}$ act on certain formal functions of $\mathbf{q}(z)$ — in other words, on certain formal power series in the q_k^{α} — whereas the total descendant potential $\mathcal{D}_X(t_0,t_1,\ldots)$ is a formal function of

$$\mathbf{t}(z)=t_0+t_1z+t_2z^2+\ldots$$

— or in other words, a formal power series in the variables t_k^{α} from (4). We let quantized operators $\widehat{A}$ act on the total descendant potential $\mathcal{D}_X(t_0,t_1,\ldots)$ via the identification

$$\mathbf{q}(z)=\mathbf{t}(z)-z. \tag{6}$$

This change of variables is called the *dilaton shift*.

[2]Since the symplectic space $\mathcal{H}$ is infinite-dimensional, quantizations $\widehat{A}$ in general contain infinite sums of differential operators. The application of a general quantized infinitesimal symplectic transformation to a general formal power series in the variables q_k^{α} is not well-defined, but it is easy to check that the operations used in the Example below do in fact make sense.

This framework allows one to express many operations which arise in Gromov–Witten theory in terms of the quantizations of very simple linear symplectic transformations of $\mathcal{H}$. One example of this occurs in the Gromov–Witten theory of a point.

Example: The Virasoro Conjecture. Let X be a point. The corresponding symplectic space is

$$\mathcal{H} = \mathbb{C}(\!(z^{-1})\!), \qquad \Omega(f,g) = \mathrm{Res}_{z=0}\, f(-z)g(z)\,dz,$$

and Darboux co-ordinates $\{q_k, p_l\}$ on $(\mathcal{H}, \Omega)$ are given by

$$\ldots \frac{p_2}{(-z)^3} + \frac{p_1}{(-z)^2} + \frac{p_0}{(-z)} + q_0 + q_1 z + q_2 z^2 + \ldots.$$

The quadratic Hamiltonians corresponding to the linear infinitesimal symplectic transformations

$$\begin{aligned} l_n : \mathcal{H} &\longrightarrow \mathcal{H} \\ f &\longmapsto z^{-1/2}\left(z\frac{d}{dz}z\right)^{n+1} z^{-1/2} f \qquad\qquad n \geq -1 \end{aligned}$$

are

$$-\sum_{k\geq 1} p_{k-1}q_k - \frac{1}{2}q_0^2 \qquad\qquad n=-1$$

$$-\sum_{k\geq 0} \frac{\Gamma(k+n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} q_k p_{k+n} + \sum_{l=0}^{l=n-1} (-1)^l \frac{\Gamma(n-l+\frac{1}{2})}{\Gamma(-l-\frac{1}{2})} p_l p_{n-1-l} \qquad\qquad n \geq 0$$

The quantizations $\widehat{l_n}$ are the differential operators

$$\frac{\partial}{\partial t_0} - \sum_{k\geq 1} t_k \frac{\partial}{\partial t_{k-1}} + \frac{t_0^2}{2\hbar} \qquad\qquad n=-1$$

$$\begin{aligned} \frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \frac{\partial}{\partial t_{n+1}} &- \sum_{k\geq 0} \frac{\Gamma\left(k+n+\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} t_k \frac{\partial}{\partial t_{k+n}} \\ &- \frac{\hbar}{2}\sum_{l=0}^{l=n-1} (-1)^{l+1} \frac{\Gamma\left(n-l+\frac{1}{2}\right)}{\Gamma\left(-l-\frac{1}{2}\right)} \frac{\partial}{\partial t_l}\frac{\partial}{\partial t_{n-1-l}} \qquad\qquad n \geq 0. \end{aligned}$$

Note that the dilaton shift (6) plays an essential role here, as without it these differential operators would not be quadratic in p_α and q_β. The Virasoro Conjecture for Gromov–Witten invariants of a point (see *e.g.* [17]) asserts that

$$\left(\widehat{l_n} - \frac{\delta_{n,0}}{16}\right)\mathcal{D}_{pt}(t_0, t_1, \ldots) = 0, \qquad\qquad n \geq -1.$$

This is equivalent [11] to Witten's Conjecture [43], proved by Kontsevich [35].

2.3. The genus-zero picture. So far we have considered a formalism for working with Gromov–Witten invariants of all genera. This involves quantized symplectic transformations applied to generating functions for the invariants. The semi-classical limit of this framework involves *unquantized* symplectic transformations applied to certain Lagrangian submanifolds of $\mathcal{H}$. This is how the Lagrangian submanifold $\mathcal{L}$ from the Introduction enters the theory.

It is easy to see that if

$$\mathcal{D}(s) = \exp\left(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g(s)\right)$$

is a one-parameter family of formal power series in the variables q_k^μ such that

$$\frac{d}{ds}\mathcal{D}(s) = \widehat{A}\,\mathcal{D}(s)$$

for some $A \in \mathfrak{sp}\,(\mathcal{H})$, then the formal germ of a Lagrangian submanifold of $\mathcal{H}$ given in Darboux co-ordinates (5) by

$$p_l^\nu = \frac{\partial \mathcal{F}^0(s)}{\partial q_l^\nu}$$

evolves with s under the Hamiltonian flow of h_A. We thus consider the formal germ of a Lagrangian submanifold $\mathcal{L}$ defined by

$$p_l^\nu = \frac{\partial \mathcal{F}^0_X}{\partial q_l^\nu}, \tag{7}$$

where we regard $\mathcal{F}^0_X(t_0, t_1, \ldots)$ as a formal power series in the q_l^ν via the dilaton shift (6). The formal germ $\mathcal{L}$ is defined for $\mathbf{q}(z)$ near $-z$. It corresponds, under the identification of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $T^\star\mathcal{H}_+ = \mathcal{H}_+ \oplus \mathcal{H}_+^\vee$ coming from the polarization, to the graph of the differential of the genus-zero descendant potential $\mathcal{F}^0_X$. $\mathcal{L}$ therefore encodes genus-zero Gromov–Witten invariants of X. A general point of $\mathcal{L}$ takes the form

$$\mathbf{q}(z) + \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \frac{Q^d}{n!} \left(\mathrm{ev}_{n+1}\right)_\star \left[[X_{0,n+1,d}]^{vir} \cap \left(\prod_{i=1}^{i=n} \mathrm{ev}_i^\star\, \mathbf{t}(\psi_i)\right) \cdot \frac{1}{-z - \psi_{n+1}} \right]. \tag{8}$$

To see this, expand $\frac{1}{-z-\psi_{n+1}}$ as a power series in z^{-1} and compare (8) with (5) and (7).

3. The Localization Calculation

We begin this section by giving a precise definition of the virtual push-forward described in the Introduction. We then state Theorem 1. The proof of Theorem 1, which is a straightforward application of the virtual localization result of Graber and Pandharipande [25], is contained in section 3.2.

3.1. A virtual push-forward. Given schemes Y and Z with $\mathbb{C}^\times$-action[3], an equivariant map $f : Y \to Z$ such that the induced map on fixed sets is proper gives a push-forward

$$f_\star : H^{\mathbb{C}^\times}_{\bullet,BM}(Y) \otimes \mathbb{C}(z) \to H^{\mathbb{C}^\times}_{\bullet,BM}(Z) \otimes \mathbb{C}(z) \tag{9}$$

in $\mathbb{C}^\times$-equivariant[4] Borel–Moore homology [4]. $\mathbb{C}(z)$ here is the field of fractions of $H^\bullet_{\mathbb{C}^\times}(pt) = \mathbb{C}[z]$. The localization theorem (see [13] and the Appendix) implies that the maps

$$(i_Y)_\star : H^{\mathbb{C}^\times}_{\bullet,BM}(Y^{\mathbb{C}^\times}) \to H^{\mathbb{C}^\times}_{\bullet,BM}(Y), \qquad (i_Z)_\star : H^{\mathbb{C}^\times}_{\bullet,BM}(Z^{\mathbb{C}^\times}) \to H^{\mathbb{C}^\times}_{\bullet,BM}(Z)$$

induced by the inclusions $i_Y : Y^{\mathbb{C}^\times} \to Y$, $i_Z : Z^{\mathbb{C}^\times} \to Z$ of $\mathbb{C}^\times$-fixed sets become isomorphisms after tensoring with $\mathbb{C}(z)$. The push-forward (9) is defined to be the composition

$$\begin{tikzcd}
H^{\mathbb{C}^\times}_{\bullet,BM}(Y) \otimes \mathbb{C}(z) \arrow[r, dashed, "f_\star"] \arrow[d, "((i_Y)_\star)^{-1}"'] & H^{\mathbb{C}^\times}_{\bullet,BM}(Z) \otimes \mathbb{C}(z) \\
H^{\mathbb{C}^\times}_{\bullet,BM}(Y^{\mathbb{C}^\times}) \otimes \mathbb{C}(z) \arrow[r] & H^{\mathbb{C}^\times}_{\bullet,BM}(Z^{\mathbb{C}^\times}) \otimes \mathbb{C}(z) \arrow[u, "(i_Z)_\star"']
\end{tikzcd}$$

where the bottom horizontal arrow is the usual proper push-forward. When the map f is proper, (9) agrees with the usual push-forward.

If Y and Z are smooth $\mathbb{C}^\times$-varieties and $f : Y \to Z$ is equivariant and proper on fixed sets as before then this construction gives a push-forward in equivariant cohomology

$$f_\star : H^\bullet_{\mathbb{C}^\times}(Y) \otimes \mathbb{C}(z) \to H^\bullet_{\mathbb{C}^\times}(Z) \otimes \mathbb{C}(z)$$

which raises degree by $2\dim_{\mathbb{C}}(Z) - 2\dim_{\mathbb{C}}(Y)$. This is by definition the composition

$$\begin{tikzcd}
H^\bullet_{\mathbb{C}^\times}(Y) \otimes \mathbb{C}(z) \arrow[r, dashed, "f_\star"] \arrow[d] & H^\bullet(Z) \otimes \mathbb{C}(z) \\
H^{\mathbb{C}^\times}_{\bullet,BM}(Y) \otimes \mathbb{C}(z) \arrow[r] & H^{\mathbb{C}^\times}_{\bullet,BM}(Z) \otimes \mathbb{C}(z) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are Poincaré duality and the bottom horizontal arrow is the push-forward (9).

In the case we wish to consider, Y will be an open subset of a moduli space of stable maps. This need not be smooth, but it it does carry a $\mathbb{C}^\times$-equivariant perfect obstruction theory: it is “virtually smooth”. Given a $\mathbb{C}^\times$-scheme Y equipped with a $\mathbb{C}^\times$-equivariant perfect obstruction theory, a smooth $\mathbb{C}^\times$-variety Z, and an equivariant map $f : Y \to Z$ which is proper on fixed sets, we define the *virtual push-forward*

$$f_\star : H^\bullet_{\mathbb{C}^\times}(Y) \otimes \mathbb{C}(z) \to H^\bullet_{\mathbb{C}^\times}(Z) \otimes \mathbb{C}(z)$$

as follows. The obstruction theory determines a virtual fundamental class [1, 2, 37] in the equivariant Chow group $A^{\mathbb{C}^\times}_{\mathrm{vdim}(Y)}(Y)$, where $\mathrm{vdim}(Y)$ is the virtual dimension,

[3]We have switched from S^1-actions to $\mathbb{C}^\times$-actions in order to make use of the virtual localization result [25].

[4]Equivariant Borel–Moore homology is discussed in the Appendix.

and hence (via the cycle map) gives a class in equivariant Borel–Moore homology

$$[Y]^{vir} \in H^{\mathbb{C}^\times}_{2\,\mathrm{vdim}(Y),BM}(Y).$$

The virtual push-forward is defined to be the composition

$$\begin{tikzcd} H^\bullet_{\mathbb{C}^\times}(Y) \otimes \mathbb{C}(z) \arrow[r, dashed, "f_\star"] \arrow[d, "{[Y]^{vir}\cap}"'] & H^\bullet(Z) \otimes \mathbb{C}(z) \\ H^{\mathbb{C}^\times}_{\bullet,BM}(Y) \otimes \mathbb{C}(z) \arrow[r] & H^{\mathbb{C}^\times}_{\bullet,BM}(Z) \otimes \mathbb{C}(z) \arrow[u] \end{tikzcd}$$

where the left-hand vertical arrow is cap product with the class $[Y]^{vir}$, the right-hand vertical arrow is Poincaré duality, and the bottom horizontal arrow is the push-forward (9). The virtual push-forward raises degree by $2\dim_{\mathbb{C}}(Z) - 2\,\mathrm{vdim}(Y)$. Once appropriate definitions are in place, the construction extends without change to the case (which we will need below) in which Y is a Deligne–Mumford quotient stack rather than a scheme; see the Appendix for details.

The virtual localization result of Graber and Pandharipande [25] implies that, under a mild technical hypothesis[5],

$$[Y]^{vir} = (i_Y)_\star \left[\sum \frac{[Y_j]^{vir}}{\mathbf{e}\left(N_j^{vir}\right)} \right] \in H^{\mathbb{C}^\times}_{\bullet,BM}(Y) \otimes \mathbb{C}(z). \tag{10}$$

The sum here is over components Y_j of the $\mathbb{C}^\times$-fixed locus in Y. The virtual fundamental classes $[Y_j]^{vir}$ and virtual normal bundles N_j^{vir} are determined by the obstruction theory; $\mathbf{e}$ here denotes the $\mathbb{C}^\times$-equivariant Euler class. If we write f_j for the restriction of $f : Y \to Z$ to the $\mathbb{C}^\times$-fixed component Y_j then (10) implies that we can write the virtual push-forward of $\alpha \in H^\bullet_{\mathbb{C}^\times}(Y) \otimes \mathbb{C}(z)$ as

$$f_\star(\alpha) = \sum (f_j)_\star \left[\frac{[Y_j]^{vir} \cap \alpha|_{Y_j}}{\mathbf{e}(N_j^{vir})} \right]. \tag{11}$$

Consider now the open subset $(X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}$ of the moduli space $(X \times \mathbb{P}^1)_{0,n,(d,1)}$ consisting of those stable maps $f : \Sigma \to X \times \mathbb{P}^1$ such that the preimage $f^{-1}(X \times \{\infty\})$ is a single unmarked smooth point x_∞. Consider the $\mathbb{C}^\times$-action on moduli space coming from the trivial $\mathbb{C}^\times$-action on X and the $\mathbb{C}^\times$-action of weight -1 on $\mathbb{P}^1$. The space $(X \times \mathbb{P}^1)_{0,n,(d,1)}$ carries a canonical $\mathbb{C}^\times$-equivariant perfect obstruction theory, and so the $\mathbb{C}^\times$-invariant open subset $(X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}$ does too. The "evaluate at infinity" map

$$\mathrm{ev}_\infty : (X \times \mathbb{P}^1)^{op}_{0,n,(d,1)} \longrightarrow X$$

which sends the stable map $f : \Sigma \to X \times \mathbb{P}^1$ to $f(x_\infty)$ is $\mathbb{C}^\times$-equivariant and proper on fixed sets. The virtual push-forwards along the maps ev_∞ assemble to give a map

$$\mathrm{Ev}_\infty : \bigoplus_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} H^\bullet_{S^1}\left((X \times \mathbb{P}^1)^{op}_{0,n,(d,1)} \right) \longrightarrow \mathcal{H}.$$

[5]They require that Y admit a $\mathbb{C}^\times$-equivariant embedding into a non-singular Deligne–Mumford stack. This is the case when Y is an open subset of a moduli stack of stable maps to a $\mathbb{C}^\times$-variety: see Appendices A and C of [25].

We are now ready to state our result.

Theorem 1. *$\mathcal{L}$ is the image under Ev_∞ of the class*

$$(-z) \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \frac{Q^d}{n!} \prod_{i=1}^{i=n} \mathrm{ev}_i^\star \mathbf{t}(\psi_i) \in \bigoplus_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} H^\bullet_{S^1}\Big((X \times \mathbb{P}^1)^{op}_{0,n,(d,1)} \Big).$$

3.2. The Proof of Theorem 1. This is a straightfoward application of the formula (11) for the virtual push-forward. The calculations are similar to, but easier than, those occurring in section 4 of [25].

Case 1: $(n,d) \notin \{(0,0),(1,0)\}$. The $\mathbb{C}^\times$-fixed locus in $(X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}$ consists of stable maps from nodal curves such that exactly one component of the curve is mapped with degree 1 to $\{x_\infty\} \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$, and the rest of the curve is mapped to $X \times \{0\}$. We identify the fixed locus with the moduli space $X_{0,n+1,d}$ of $(n+1)$-pointed stable maps to X: the component mapped to $\{x_\infty\} \times \mathbb{P}^1$ is attached at the $(n+1)$st marked point. The $\mathbb{C}^\times$-fixed part of the perfect obstruction theory on $(X \times \mathbb{P}^1)^{op}_{0,n,(d,1)}$ coincides with the usual perfect obstruction theory on $X_{0,n+1,d}$, and the virtual normal bundle to the fixed locus is

$$\mathbb{C}_{(-1)} \oplus \big(L_{n+1} \otimes \mathbb{C}_{(-1)} \big)$$

where $\mathbb{C}_{(-1)}$ denotes the trivial bundle over $X_{0,n+1,d}$ with $\mathbb{C}^\times$-weight -1. Thus

$$(\mathrm{ev}_\infty)_\star \left[(-z) \prod_{i=1}^{i=n} \mathrm{ev}_i^\star \mathbf{t}(\psi_i) \right] = (\mathrm{ev}_{n+1})_\star \left[[X_{0,n+1,d}]^{vir} \cap \left(\prod_{i=1}^{i=n} \mathrm{ev}_i^\star \mathbf{t}(\psi_i) \right) \cdot \frac{1}{-z - \psi_{n+1}} \right] \tag{12}$$

Case 2: $(n,d) = (1,0)$. We have

$$\big(X \times \mathbb{P}^1 \big)^{op}_{0,n,(d,1)} \cong X \times \mathbb{C}$$

and the $\mathbb{C}^\times$-fixed locus here is a copy of X. The virtual fundamental class on X determined by the $\mathbb{C}^\times$-fixed part of the perfect obstruction theory is the usual fundamental class of X. The restriction to the fixed locus of the universal cotangent line bundle L_1 is the trivial bundle $\mathbb{C}_{(1)}$ over X of $\mathbb{C}^\times$-weight 1, and the virtual normal bundle is the trivial bundle $\mathbb{C}_{(-1)}$ of weight -1. Thus

$$(\mathrm{ev}_\infty)_\star \Big[(-z) \cdot \mathrm{ev}_1^\star \mathbf{t}(\psi_1) \Big] = \mathbf{t}(z). \tag{13}$$

Case 3: $(n,d) = (0,0)$. Here

$$\big(X \times \mathbb{P}^1 \big)^{op}_{0,0,(0,1)} \cong X$$

and there is no moving part of the obstruction theory. The virtual fundamental class induced on the fixed locus X is the usual fundamental class of X, and

$$(\mathrm{ev}_\infty)_\star \Big[-z \Big] = -z. \tag{14}$$

Combining (12), (13), and (14), we find that the image of the class from Theorem 1 under Ev_∞ is

$$-z+\mathbf{t}(z)+ \sum_{\substack{d\in H_2(X;\mathbb{Z})\\ n\geq 0}} \frac{Q^d}{n!} (\mathrm{ev}_{n+1})_\star \left[[X_{0,n+1,d}]^{vir} \cap \left(\prod_{i=1}^{i=n} \mathrm{ev}_i^\star \, \mathbf{t}(\psi_i) \right) \cdot \frac{1}{-z-\psi_{n+1}} \right].$$

This coincides with our expression (8) for a general point of $\mathcal{L}$. The proof is complete. □

Remark 1. We see from the proof of Theorem 1 that one should regard the factor of $-z$ occurring in the statement as the $\mathbb{C}^\times$-equivariant Euler class of $R^\bullet\pi_\star \, \mathrm{ev}_{n+1}^\star \, \mathbb{C}_{(-1)}$, where $\pi : X_{g,n+1,d} \to X_{g,n,d}$ is the universal family over the moduli space of stable maps and $\mathbb{C}_{(-1)}$ is the trivial bundle of $\mathbb{C}^\times$-weight -1 over X. Such a "twist by the Euler class" roughly corresponds to considering the Gromov–Witten theory of a hypersurface [7]. If we regard our study of $\left(X \times \mathbb{P}^1\right)^{op}_{0,n,(d,1)}$ as a proxy for studying the Gromov–Witten theory of $X\times\mathbb{C}$ then the two ingredients of our construction push in opposite directions: we end up, roughly speaking, thinking of X as an "equivariant hypersurface" in $X \times \mathbb{C}_{(-1)}$. The dilaton shift arises exactly from the difference between the two notions of stability here: stability as a map to X and stability as a graph in $X \times \mathbb{C}$.

Remark 2. Our construction of $\mathcal{L}$ bears a striking resemblance to the "fundamental Floer cycle" — the semi-infinite cycle in loop space consisting of loops which bound holomorphic discs — in the heuristic picture relating quantum cohomology to the S^1-equivariant Floer homology of loop space outlined in [18]. This suggests that one should regard $\mathcal{H}$ as the S^1-equivariant Floer homology of the loop space of X. Other evidence for this comes from comparing the symplectic transformation in [7, Theorem 1] with the calculations in [18, Section 4], and from the beautiful recent work of Costello [10]. As mentioned above, the graph space $\left(X \times \mathbb{P}^1\right)_{0,n,(d,1)}$ plays a key role in many proofs of toric mirror symmetry [3, 19, 20, 28–32], where it links Floer-theoretic predictions to rigorous calculations in Gromov–Witten theory. It would be interesting to understand exactly how S^1-equivariant Floer homology relates to our picture.

Appendix: $\mathbb{C}^\times$-Equivariant Borel–Moore Homology

In [4] Braverman used a sheaf-theoretic definition of equivariant Borel–Moore homology, in the spirit of [33]. We will take a different point of view, regarding Borel–Moore homology as the homology theory of singular chains with locally finite support. This meshes more readily with constructions of the virtual fundamental class. We collect the properties of non-equivariant Borel–Moore homology that we will need in section A1 and describe the equivariant theory, constructed by Edidin and Graham in [12], in section A2. In section A3 we discuss the Borel–Moore homology of certain quotient stacks. Since the precise form of the localization theorem for $\mathbb{C}^\times$-equivariant Borel–Moore homology which we used in section 3.1 does not appear to have been written down anywhere, we prove it in section A4; it was undoubtedly already well-known.

A1. Borel–Moore homology. Good introductions to Borel–Moore homology can be found in [15, chapter 19], [14, Appendix B], [6, section 2.6], and [40, Appendix C]. We work with the definition from [15]: if a space X is embedded as a closed subspace of $\mathbb{R}^n$ then

$$H_{i,BM}(X) := H^{n-i}(\mathbb{R}^n, \mathbb{R}^n - X). \tag{15}$$

All homology and cohomology groups are taken with complex coefficients throughout. Properties of Borel–Moore homology include:

BM1 There are *cap products*

$$H^j(X) \otimes H_{k,BM}(X) \to H_{k-j,BM}(X).$$

See [15, section 19.1].

BM2 If X is a smooth variety of dimension n then $H_{2n,BM}(X)$ is freely generated by the *fundamental class* $[X] \in H_{2n,BM}(X)$, and

$$[X]\cap : H^k(X) \to H_{2n-k,BM}(X)$$

is an isomorphism. This is *Poincaré duality.* See [15, section 19.1].

BM3 There is a *Künneth formula*

$$H_{k,BM}(X \times Y) = \bigoplus_{i+j=k} H_{i,BM}(X) \otimes H_{j,BM}(Y).$$

This follows immediately from definition (15) and the Künneth formula for relative homology.

BM4 There are *covariant push-forwards* for proper maps $f : X \to Y$,

$$f_\star : H_{k,BM}(X) \to H_{k,BM}(Y).$$

See [15, section 19.1].

BM5 There are *contravariant pull-backs* for open embeddings $j : U \to Y$,

$$j^\star : H_{k,BM}(Y) \to H_{k,BM}(U).$$

See [15, section 19.1].

BM6 There is a *long exact sequence*

$$\ldots \to H_{i+1,BM}(U) \to H_{i,BM}(X) \xrightarrow{i_\star} H_{i,BM}(Y) \xrightarrow{j^\star} H_{i,BM}(U) \to \ldots$$

where $j : U \to Y$ is an open embedding and $i : X \to Y$ is the closed embedding of the complement X to U in Y. See [15, section 19.1].

BM7 If X is a scheme of dimension n then $H_{i,BM}(X) = 0$ for $i > 2n$. This is part of Lemma 19.1.1 in [15].

BM8 For any scheme X there is a *cycle map*

$$\mathrm{cl} : A_k(X) \to H_{2k,BM}(X)$$

which is covariant for proper maps and compatible with Chern classes. See [15, section 19.1].

BM9 For any l.c.i. morphism of schemes $f : Y \to X$ of codimension d there is a *Gysin map*

$$f^\star : H_{k,BM}(X) \to H_{k-2d,BM}(Y).$$

Such maps are functorial and compatible with the cycle class. When Y is a vector bundle over X of rank d, $f^\star$ is the *Thom isomorphism* $H_{k,BM}(X) \to H_{k+2d,BM}(Y)$. See [15, Example 19.2.1].

A2. Equivariant Borel–Moore homology. Given a g-dimensional linear algebraic group G acting in a reasonable way[6] on an scheme X of dimension n, Edidin and Graham [12] define the G-equivariant Borel–Moore homology groups of X as

$$H^G_{i,BM}(X) := H_{i+2l-2g,BM}(X_G).$$

Here X_G is the *mixed space* $(X \times U)/G$, where U is an open set in an l-dimensional representation V of G such that the action of G on U is free and the real codimension of $V - U$ in V is more than $2n - i + 1$.

One can see that this is well-defined using Bogomolov's *double filtration argument* [12, Definition-Proposition 1 and Section 2.8]. Suppose that V_1 and V_2 are representations of G respectively of dimensions l_1 and l_2 and containing open sets U_1 and U_2 such that the G-action on each U_j is free and the real codimension of $V_j - U_j$ in V_j is more than $2n - i + 1$. Then $V_1 \oplus V_2$ contains an open set W on which G acts freely and which contains both $U_1 \oplus V_2$ and $V_1 \oplus U_2$. The dimension of

$$(X \times W)/G - (X \times (U_1 \oplus V_2))/G$$

is less than $2l_1 + 2l_2 - 2g + i - 1$, so

$$H_{i+2l_1+2l_2-2g,BM}((X \times W)/G) = H_{i+2l_1+2l_2-2g,BM}((X \times (U_1 \oplus V_2))/G)$$

by **BM6** and **BM7**. But $(X \times (U_1 \oplus V_2))/G$ is a vector bundle of rank l_2 over $(X \times U_1)/G$, so

$$H_{i+2l_1+2l_2-2g,BM}((X \times W)/G) = H_{i+2l_1-2g,BM}((X \times U_1)/G)$$

by **BM9**. Similarly,

$$H_{i+2l_1+2l_2-2g,BM}((X \times W)/G) = H_{i+2l_2-2g,BM}((X \times U_2)/G).$$

If the real codimension of the open set U in the representation V is c then $\pi_j(U) = 0$ for $j < c-1$, so the mixed spaces X_G are algebraic approximations to the Borel space $(X \times EG)/G$. Combining the construction above with the discussion in section A1 immediately yields[7] the following properties:

[6] We sidestep a technical issue here. Edidin and Graham work with algebraic spaces, rather than schemes. This is because the quotient of an algebraic space by a free action of an algebraic group is an algebraic space, but the quotient of a scheme by a free action of of an algebraic group need not be a scheme. We would like the mixed space X_G to be a scheme, because we want to use properties of the Borel–Moore homology of schemes listed in section A1. Proposition 23 in [12] gives conditions on the group action sufficient to ensure that X_G is a scheme: we will consider only actions of G on X which satisfy these hypotheses, calling such actions *reasonable*. In view of the construction of the moduli space of stable maps as a stack quotient given in [16], it suffices for the purposes of this paper to consider only reasonable actions. Another, perhaps more satisfactory, approach would be to develop a Borel–Moore homology theory for algebraic spaces — much as is done for intersection theory in section 6.1 of [12] — but as we do not need to do this, we won't.

[7] This is entirely parallel to section 2.3 of [12].

EBM1 There are *cap products*

$$H_G^j(X) \otimes H_{k,BM}^G(X) \to H_{k-j,BM}^G(X).$$

EBM2 If X is a smooth variety of dimension n then there is a *Poincaré duality* isomorphism

$$H_G^k(X) \to H_{2n-k,BM}^G(X).$$

EBM3 If the action of G on X is trivial then

$$H_{k,BM}^G(X) = \bigoplus_{i+j=k} H_{i,BM}(X) \otimes H_{j,BM}^G(pt).$$

EBM4 There are *covariant push-forwards* for proper G-equivariant maps $f : X \to Y$,

$$f_\star : H_{k,BM}^G(X) \to H_{k,BM}^G(Y).$$

EBM5 There are *contravariant pull-backs* for G-equivariant open embeddings $j : U \to Y$,

$$j^\star : H_{k,BM}^G(Y) \to H_{k,BM}^G(U).$$

EBM6 There is a *long exact sequence*

$$\ldots \to H_{i+1,BM}^G(U) \to H_{i,BM}^G(X) \xrightarrow{i_\star} H_{i,BM}^G(Y) \xrightarrow{j^\star} H_{i,BM}^G(U) \to \ldots$$

where $j : U \to Y$ is a G-equivariant open embedding and $i : X \to Y$ is the G-equivariant closed embedding of the complement X to U in Y.

EBM7 We have $H_{i,BM}^G(X) = 0$ for $i > 2n$.

EBM8 There is a *cycle map*

$$\mathrm{cl} : A_k^G(X) \to H_{2k,BM}^G(X)$$

which is covariant for proper maps and compatible with G-equivariant Chern classes.

EBM9 There are *Gysin maps*

$$f^\star : H_{k,BM}^G(X) \to H_{k-2d,BM}^G(Y)$$

for G-equivariant l.c.i. morphisms $f : Y \to X$ of codimension d. These are functorial and compatible with the cycle class. When Y is a G-equivariant vector bundle over X of rank d, $f^\star$ is the *Thom isomorphism* $H_{k,BM}^G(X) \to H_{k+2d,BM}^G(Y)$.

A3. Borel–Moore homology groups for quotient stacks. In this section, we define ordinary and $\mathbb{C}^\times$-equivariant Borel–Moore homology groups for certain quotient stacks, following [12, section 5] and [25, Appendix C]. This allows us to consider the $\mathbb{C}^\times$-equivariant Borel–Moore homology of moduli spaces of stable maps.

Non-equivariant Borel–Moore homology. Given a quotient stack of the form $[X/G]$, where X is a scheme with a reasonable action of the g-dimensional linear algebraic group G, we define the Borel–Moore homology groups of $[X/G]$ to be

$$H_{i,BM}([X/G]) := H^G_{i+2g,BM}(X).$$

We can see that this is well-defined using the argument of [13, Proposition 16]. Suppose that $[X/G] \cong [Y/H]$ as quotient stacks, where G (respectively H) acts reasonably on the scheme X (respectively Y). Let V_1 be an l_1-dimensional representation of G containing an open set U_1 on which the G-action is free, and let $X_G = (X \times U_1)/G$. Let V_2 be an l_2-dimensional representation of H containing an open set U_2 on which the H-action is free, and let $Y_H = (Y \times U_2)/H$. The diagonal of a quotient stack is representable, so the fiber product

$$Z = X_G \times_{[X/G]} Y_H$$

is a scheme. But Z fibers over X_G with fiber U_2 and over Y_H with fiber U_1, so

$$H_{i+2l_1,BM}(X_G) = H_{i+2l_1+2l_2,BM}(Z) = H_{i+2l_2,BM}(Y_H).$$

$\mathbb{C}^\times$*-equivariant Borel–Moore homology.* Here we follow Appendix C of [25]. We define the $\mathbb{C}^\times$-equivariant Borel–Moore homology groups of a quotient stack X by setting

$$H^{\mathbb{C}^\times}_{i,BM}(X) := H_{i+2l-2,BM}([(X \times U)/\mathbb{C}^\times]) \tag{16}$$

where U is an open set in an l-dimensional representation of $\mathbb{C}^\times$ as above. In other words, we follow the prescription described in section A2 but construct the mixed space $X_{\mathbb{C}^\times}$ as a stack quotient. In the case where X is the quotient of a scheme Y by a reasonable and proper action of a linear algebraic group G such that the $\mathbb{C}^\times$-action on X descends from a reasonable action of $G \times \mathbb{C}^\times$ on Y, we can use the constructions described earlier in this section to define the right-hand side of (16). In applications to moduli stacks of stable maps, we need only consider quotients of this form where $G = PGL$ [16].

A4. Localization in $\mathbb{C}^\times$-equivariant Borel–Moore homology. This section contains the proof of the localization theorem which we used in section 3.1. In summary: the argument given by Graber and Pandharipande in Appendix C of [25] works for Borel–Moore homology too.

Theorem. *Suppose that the stack X is the quotient of a scheme Y by a reasonable and proper action of a connected reductive group G, and that X is equipped with a $\mathbb{C}^\times$-action which descends from a reasonable action of $G \times \mathbb{C}^\times$ on Y. Then the push-forward*

$$i_\star : H^{\mathbb{C}^\times}_{\bullet,BM}(X^{\mathbb{C}^\times}) \to H^{\mathbb{C}^\times}_{\bullet,BM}(X)$$

along the inclusion $i : X^{\mathbb{C}^\times} \to X$ of the $\mathbb{C}^\times$-fixed stack becomes an isomorphism after tensoring with the field of fractions $\mathbb{C}(z)$ of $H^\bullet_{S^1}(pt)$.

Proof. In view of **EBM6** if suffices to show that the $\mathbb{C}^\times$-equivariant Borel–Moore homology groups of $X - X^{\mathbb{C}^\times}$ vanish after localization. But $\mathbb{C}^\times$ acts without fixed points on $X - X^{\mathbb{C}^\times}$, so $X - X^{\mathbb{C}^\times}$ is the quotient of a scheme Z by a reasonable and proper action of $G \times \mathbb{C}^\times$ and

$$H^{\mathbb{C}^\times}_{\bullet,BM}(X - X^{\mathbb{C}^\times}) = H_{\bullet,BM}([Z/(G \times \mathbb{C}^\times)]).$$

But these groups are non-zero in only finitely many degrees, since they are isomorphic to Borel–Moore homology groups of the coarse quotient. They therefore vanish after localization. □

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