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SMOOTH HYPERSURFACE SECTIONS CONTAINING A GIVEN SUBSCHEME OVER A FINITE FIELD

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1. Introduction

Let $\mathbb{F}_q$ be a finite field of $q = p^a$ elements. Let X be a smooth quasi-projective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. N. Katz asked for a finite field analogue of the Bertini smoothness theorem, and in particular asked whether one could always find a hypersurface H in $\mathbb{P}^n$ such that $H \cap X$ is smooth of dimension $m-1$. A positive answer was proved in [Gab01] and [Poo04] independently. The latter paper proved also that in a precise sense, a positive fraction of hypersurfaces have the required property.

The classical Bertini theorem was extended in [Blo70, KA79] to show that the hypersurface can be chosen so as to contain a prescribed closed smooth subscheme Z, provided that the condition $\dim X > 2 \dim Z$ is satisfied. (The condition arises naturally from a dimension-counting argument.) The goal of the current paper is to prove an analogous result over finite fields. In fact, our result is stronger than that of [KA79] in that we do not require $Z \subseteq X$, but weaker in that we assume that $Z \cap X$ be smooth. (With a little more work and complexity, we could prove a version for a non-smooth intersection as well, but we restrict to the smooth case for simplicity.) One reason for proving our result is that it is used by [SS07].

Let $S = \mathbb{F}_q[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. Let $S_d \subseteq S$ be the $\mathbb{F}_q$-subspace of homogeneous polynomials of degree d. For each $f \in S_d$, let H_f be the subscheme $\operatorname{Proj}(S/(f)) \subseteq \mathbb{P}^n$. For the rest of this paper, we fix a closed subscheme $Z \subseteq \mathbb{P}^n$. For $d \in \mathbb{Z}_{\geq 0}$, let I_d be the $\mathbb{F}_q$-subspace of $f \in S_d$ that vanish on Z. Let $I_{\text{homog}} = \bigcup_{d \geq 0} I_d$. We want to measure the density of subsets of I_{homog}, but under the definition in [Poo04], the set I_{homog} itself has density 0 whenever $\dim Z > 0$; therefore we use a new definition of density, relative to I_{homog}. Namely, we define the *density* of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ by

$$\mu_Z(\mathcal{P}) := \lim_{d \to \infty} \frac{\#(\mathcal{P} \cap I_d)}{\# I_d},$$

if the limit exists. For a scheme X of finite type over $\mathbb{F}_q$, define the zeta function [Wei49]

$$\zeta_X(s) = Z_X(q^{-s}) := \prod_{\text{closed } P \in X} \left(1 - q^{-s \deg P}\right)^{-1} = \exp\left(\sum_{r=1}^{\infty} \frac{\# X(\mathbb{F}_{q^r})}{r} q^{-rs} \right);$$

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the product and sum converge when $\operatorname{Re}(s) > \dim X$.

Theorem 1.1. *Let X be a smooth quasi-projective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$. Let Z be a closed subscheme of $\mathbb{P}^n$. Assume that the scheme-theoretic intersection $V := Z \cap X$ is smooth of dimension ℓ. (If V is empty, take $\ell = -1$.) Define*

$$\mathcal{P} := \{\, f \in I_{\mathrm{homog}} : H_f \cap X \text{ is smooth of dimension } m-1 \,\}.$$

(i) *If $m > 2\ell$, then*

$$\mu_Z(\mathcal{P}) = \frac{\zeta_V(m+1)}{\zeta_V(m-\ell)\, \zeta_X(m+1)} = \frac{1}{\zeta_V(m-\ell)\, \zeta_{X-V}(m+1)}.$$

In this case, in particular, for $d \gg 1$, there exists a degree-d hypersurface H containing Z such that $H \cap X$ is smooth of dimension $m-1$.

(ii) *If $m \leq 2\ell$, then $\mu_Z(\mathcal{P}) = 0$.*

The proof will use the closed point sieve introduced in [Poo04]. In fact, the proof is parallel to the one in that paper, but changes are required in almost every line.

2. Singular points of low degree

Let $\mathcal{I}_Z \subseteq \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of Z, so $I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d))$. Tensoring the surjection

$$\mathcal{O}^{\oplus(n+1)} \to \mathcal{O}$$
$$(f_0, \dots, f_n) \mapsto x_0 f_0 + \cdots + x_n f_n$$

with $\mathcal{I}_Z$, twisting by $\mathcal{O}(d)$, and taking global sections shows that $S_1 I_d = I_{d+1}$ for $d \gg 1$. Fix c such that $S_1 I_d = I_{d+1}$ for all $d \geq c$.

Before proving the main result of this section (Lemma 2.3), we need two lemmas.

Lemma 2.1. *Let Y be a finite subscheme of $\mathbb{P}^n$. Let*

$$\phi_d \colon I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \to H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$$

be the map induced by the map of sheaves $\mathcal{I}_Z \to \mathcal{I}_Z \cdot \mathcal{O}_Y$ on $\mathbb{P}^n$. Then ϕ_d is surjective for $d \geq c + \dim H^0(Y, \mathcal{O}_Y)$,

Proof. The map of sheaves $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y$ on $\mathbb{P}^n$ is surjective so $\mathcal{I}_Z \to \mathcal{I}_Z \cdot \mathcal{O}_Y$ is surjective too. Thus ϕ_d is surjective for $d \gg 1$.

Enlarging $\mathbb{F}_q$ if necessary, we can perform a linear change of variable to assume $Y \subseteq \mathbb{A}^n := \{x_0 \neq 0\}$. Dehomogenization (setting $x_0 = 1$) identifies S_d with the space S'_d of polynomials in $\mathbb{F}_q[x_1, \dots, x_n]$ of total degree $\leq d$. and identifies ϕ_d with a map

$$I'_d \to B := H^0(\mathbb{P}^n, \mathcal{I}_Z \cdot \mathcal{O}_Y).$$

By definition of c, we have $S'_1 I'_d = I'_{d+1}$ for $d \geq c$. For $d \geq b$, let B_d be the image of I'_d in B, so $S'_1 B_d = B_{d+1}$ for $d \geq c$. Since $1 \in S'_1$, we have $I'_d \subseteq I'_{d+1}$, so

$$B_c \subseteq B_{c+1} \subseteq \cdots .$$

But $b := \dim B < \infty$, so $B_j = B_{j+1}$ for some $j \in [c, c+b]$. Then

$$B_{j+2} = S'_1 B_{j+1} = S'_1 B_j = B_{j+1}.$$

Similarly $B_j = B_{j+1} = B_{j+2} = \dots$, and these eventually equal B by the previous paragraph. Hence ϕ_d is surjective for $d \geq j$, and in particular for $d \geq c + b$. □

Lemma 2.2. *Suppose $\mathfrak{m} \subseteq \mathcal{O}_X$ is the ideal sheaf of a closed point $P \in X$. Let $Y \subseteq X$ be the closed subscheme whose ideal sheaf is $\mathfrak{m}^2 \subseteq \mathcal{O}_X$. Then for any $d \in \mathbb{Z}_{\geq 0}$.*

$$\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) = \begin{cases} q^{(m-\ell)\deg P}, & \text{if } P \in V, \\ q^{(m+1)\deg P}, & \text{if } P \notin V. \end{cases}$$

Proof. Since Y is finite, we may now ignore the twisting by $\mathcal{O}(d)$. The space $H^0(Y, \mathcal{O}_Y)$ has a two-step filtration whose quotients have dimensions 1 and m over the residue field κ of P. Thus $\#H^0(Y, \mathcal{O}_Y) = (\#\kappa)^{m+1} = q^{(m+1)\deg P}$. If $P \in V$ (or equivalently $P \in Z$), then $H^0(Y, \mathcal{O}_{Z\cap Y})$ has a filtration whose quotients have dimensions 1 and ℓ over κ; if $P \notin V$, then $H^0(Y, \mathcal{O}_{Z\cap Y}) = 0$. Taking cohomology of

$$0 \to \mathcal{I}_Z \cdot \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_{Z\cap Y} \to 0$$

on the 0-dimensional scheme Y yields

$$\begin{aligned} \#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) &= \frac{\#H^0(Y, \mathcal{O}_Y)}{\#H^0(Y, \mathcal{O}_{Z\cap Y})} \\ &= \begin{cases} q^{(m+1)\deg P}/q^{(\ell+1)\deg P}, & \text{if } P \in V, \\ q^{(m+1)\deg P}, & \text{if } P \notin V. \end{cases} \end{aligned}$$

□

If U is a scheme of finite type over $\mathbb{F}_q$, let $U_{<r}$ be the set of closed points of U of degree $< r$. Similarly define $U_{>r}$.

Lemma 2.3 (Singularities of low degree)**.** *Let notation and hypotheses be as in Theorem 1.1, and define*

$$\mathcal{P}_r := \{\, f \in I_{\text{homog}} : H_f \cap X \text{ is smooth of dimension } m-1 \text{ at all } P \in X_{<r} \,\}.$$

Then

$$\mu_Z(\mathcal{P}_r) = \prod_{P \in V_{<r}} \left(1 - q^{-(m-\ell)\deg P}\right) \cdot \prod_{P \in (X-V)_{<r}} \left(1 - q^{-(m+1)\deg P}\right).$$

Proof. Let $X_{<r} = \{P_1, \dots, P_s\}$. Let $\mathfrak{m}_i$ be the ideal sheaf of P_i on X. let Y_i be the closed subscheme of X with ideal sheaf $\mathfrak{m}_i^2 \subseteq \mathcal{O}_X$, and let $Y = \bigcup Y_i$. Then $H_f \cap X$ is singular at P_i (more precisely, not smooth of dimension $m-1$ at P_i) if and only if the restriction of f to a section of $\mathcal{O}_{Y_i}(d)$ is zero.

By Lemma 2.1, $\mu_Z(\mathcal{P})$ equals the fraction of elements in $H^0(\mathcal{I}_Z \cdot \mathcal{O}_Y(d))$ whose restriction to a section of $\mathcal{O}_{Y_i}(d)$ is nonzero for every i. Thus

$$\begin{aligned} \mu_Z(\mathcal{P}_r) &= \prod_{i=1}^{s} \frac{\#H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) - 1}{\#H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i})} \\ &= \prod_{P \in V_{<r}} \left(1 - q^{-(m-\ell)\deg P}\right) \cdot \prod_{P \in (X-V)_{<r}} \left(1 - q^{-(m+1)\deg P}\right), \end{aligned}$$

by Lemma 2.2. □

Corollary 2.4. *If $m > 2\ell$, then*

$$\lim_{r\to\infty} \mu_Z(\mathcal{P}_r) = \frac{\zeta_V(m+1)}{\zeta_X(m+1)\,\zeta_V(m-\ell)}.$$

Proof. The products in Lemma 2.3 are the partial products in the definition of the zeta functions. For convergence, we need $m - \ell > \dim V = \ell$, which is equivalent to $m > 2\ell$. □

Proof of Theorem 1.1(ii). We have $\mathcal{P} \subseteq \mathcal{P}_r$. By Lemma 2.3,

$$\mu_Z(\mathcal{P}_r) \le \prod_{P\in V_{<r}} \left(1 - q^{-(m-\ell)\deg P}\right),$$

which tends to 0 as $r \to \infty$ if $m \le 2\ell$. Thus $\mu_Z(\mathcal{P}) = 0$ in this case. □

From now on, we assume $m > 2\ell$.

3. Singular points of medium degree

Lemma 3.1. *Let $P \in X$ is a closed point of degree e, where $e \le \frac{d-c}{m+1}$. Then the fraction of $f \in I_d$ such that $H_f \cap X$ is not smooth of dimension $m-1$ at P equals*

$$\begin{cases} q^{-(m-\ell)e}, & \text{if } P \in V, \\ q^{-(m+1)e}, & \text{if } P \notin V. \end{cases}$$

Proof. This follows by applying Lemma 2.1 to the Y in Lemma 2.2, and then applying Lemma 2.2. □

Define the upper and lower densities $\overline{\mu}_Z(\mathcal{P})$, $\underline{\mu}_Z(\mathcal{P})$ of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ as $\mu_Z(\mathcal{P})$ was defined, but using lim sup and lim inf in place of lim.

Lemma 3.2 (Singularities of medium degree)**.** *Define*

$$\mathcal{Q}_r^{\text{medium}} := \bigcup_{d\ge 0} \{\, f \in I_d : \text{there exists } P \in X \text{ with } r \le \deg P \le \frac{d-b}{m+1} \text{ such that } H_f \cap X \text{ is not smooth of dimension } m-1 \text{ at } P \,\}.$$

Then $\lim_{r\to\infty} \overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}}) = 0$.

Proof. By Lemma 3.1, we have

$$\begin{aligned} \frac{\#(\mathcal{Q}_r^{\text{medium}} \cap I_d)}{\# I_d} &\le \sum_{\substack{P\in Z \\ r\le \deg P \le \frac{d-b}{m+1}}} q^{-(m-\ell)\deg P} + \sum_{\substack{P\in X-Z \\ r\le \deg P \le \frac{d-b}{m+1}}} q^{-(m+1)\deg P} \\ &\le \sum_{P\in Z_{\ge r}} q^{-(m-\ell)\deg P} + \sum_{P\in (X-Z)_{\ge r}} q^{-(m+1)\deg P}. \end{aligned}$$

Using the trivial bound that an m-dimensional variety has at most $O(q^{em})$ closed points of degree e, as in the proof of [Poo04, Lemma 2.4], we show that each of the two sums converges to a value that is $O(q^{-r})$ as $r \to \infty$, under our assumption $m > 2\ell$. □

4. Singular points of high degree

Lemma 4.1. *Let P be a closed point of degree e in $\mathbb{P}^n - Z$. For $d \geq c$, the fraction of $f \in I_d$ that vanish at P is at most $q^{-\min(d-c,e)}$.*

Proof. Equivalently, we must show that the image of ϕ_d in Lemma 2.1 for $Y = P$ has $\mathbb{F}_q$-dimension at least $\min(d-c, e)$. The proof of Lemma 2.1 shows that as d runs through the integers $c, c+1, \ldots$, this dimension increases by at least 1 until it reaches its maximum, which is e. □

Lemma 4.2 (Singularities of high degree off V)**.** *Define*

$$\mathcal{Q}^{\text{high}}_{X-V} := \bigcup_{d \geq 0} \{\, f \in\ I_d : \exists P \in (X - V)_{> \frac{d-c}{m+1}} \text{ such that } H_f \cap X \text{ is not smooth of dimension } m-1 \text{ at } P \,\}$$

Then $\overline{\mu}_Z(\mathcal{Q}^{\text{high}}_{X-V}) = 0$.

Proof. It suffices to prove the lemma with X replaced by each of the sets in an open covering of $X - V$, so we may assume X is contained in $\mathbb{A}^n = \{x_0 \neq 0\} \subseteq \mathbb{P}^n$, and that $V = \emptyset$. Dehomogenize by setting $x_0 = 1$, to identify $I_d \subseteq S_d$ with subspaces of $I'_d \subseteq S'_d \subseteq A := \mathbb{F}_q[x_1, \ldots, x_n]$.

Given a closed point $x \in X$, choose a system of local parameters $t_1, \ldots, t_n \in A$ at x on $\mathbb{A}^n$ such that $t_{m+1} = t_{m+2} = \cdots = t_n = 0$ defines X locally at x. Multiplying all the t_i by an element of A vanishing on Z but nonvanishing at x, we may assume in addition that all the t_i vanish on Z. Now $dt_1, \ldots, dt_n$ are a $\mathcal{O}_{\mathbb{A}^n,x}$-basis for the stalk $\Omega^1_{\mathbb{A}^n/\mathbb{F}_q,x}$. Let $\partial_1, \ldots, \partial_n$ be the dual basis of the stalk $\mathcal{T}_{\mathbb{A}^n/\mathbb{F}_q,x}$ of the tangent sheaf. Choose $s \in A$ with $s(x) \neq 0$ to clear denominators so that $D_i := s\partial_i$ gives a global derivation $A \to A$ for $i = 1, \ldots, n$. Then there is a neighborhood N_x of x in $\mathbb{A}^n$ such that $N_x \cap \{t_{m+1} = t_{m+2} = \cdots = t_n = 0\} = N_x \cap X$, $\Omega^1_{N_x/\mathbb{F}_q} = \oplus_{i=1}^n \mathcal{O}_{N_x} dt_i$, and $s \in \mathcal{O}(N_u)^*$. We may cover X with finitely many N_x, so we may reduce to the case where $X \subseteq N_x$ for a single x. For $f \in I'_d \simeq I_d$, $H_f \cap X$ *fails* to be smooth of dimension $m-1$ at a point $P \in U$ if and only if $f(P) = (D_1 f)(P) = \cdots = (D_m f)(P) = 0$.

Let $\tau = \max_i(\deg t_i)$, $\gamma = \lfloor (d-\tau)/p \rfloor$, and $\eta = \lfloor d/p \rfloor$. If $f_0 \in I'_d$, $g_1 \in S'_\gamma$, $\ldots$, $g_m \in S'_\gamma$, and $h \in I'_\eta$ are selected uniformly and independently at random, then the distribution of

$$f := f_0 + g_1^p t_1 + \cdots + g_m^p t_m + h^p$$

is uniform over I'_d, because of f_0. We will bound the probability that an f constructed in this way has a point $P \in X_{> \frac{d-c}{m+1}}$ where $f(P) = (D_1 f)(P) = \cdots = (D_m f)(P) = 0$. We have $D_i f = (D_i f_0) + g_i^p s$ for $i = 1, \ldots, m$. We will select $f_0, g_1, \ldots, g_m, h$ one at a time. For $0 \leq i \leq m$, define

$$W_i := X \cap \{D_1 f = \cdots = D_i f = 0\}.$$

Claim 1: For $0 \leq i \leq m-1$, conditioned on a choice of $f_0, g_1, \ldots, g_i$ for which $\dim(W_i) \leq m-i$, the probability that $\dim(W_{i+1}) \leq m-i-1$ is $1 - o(1)$ as $d \to \infty$. (The function of d represented by the $o(1)$ depends on X and the D_i.)

Proof of Claim 1: This is completely analogous to the corresponding proof in [Poo04].

Claim 2: Conditioned on a choice of $f_0, g_1, \ldots, g_m$ for which W_m is finite, $\operatorname{Prob}(H_f \cap W_m \cap X_{>\frac{d-c}{m+1}} = \emptyset) = 1 - o(1)$ as $d \to \infty$.

Proof of Claim 2: By Bézout's theorem as in [Ful84, p. 10], we have $\# W_m = O(d^m)$. For a given point $P \in W_m$, the set H^{bad} of $h \in I'_\eta$ for which H_f passes through P is either $\emptyset$ or a coset of $\ker(\operatorname{ev}_P : I'_\eta \to \kappa(P))$, where $\kappa(P)$ is the residue field of P, and ev_P is the evaluation-at-P map. If moreover $\deg P > \frac{d-c}{m+1}$, then Lemma 4.1 implies $\# H^{\text{bad}} / \# I'_\eta \le q^{-\nu}$ where $\nu = \min\left(\eta, \frac{d-c}{m+1}\right)$. Hence

$$\operatorname{Prob}(H_f \cap W_m \cap X_{>\frac{d-c}{m+1}} \neq \emptyset) \le \# W_m q^{-\nu} = O(d^m q^{-\nu}) = o(1)$$

as $d \to \infty$, since ν eventually grows linearly in d. This proves Claim 2.

End of proof: Choose $f \in I_d$ uniformly at random. Claims 1 and 2 show that with probability $\prod_{i=0}^{m-1}(1 - o(1)) \cdot (1 - o(1)) = 1 - o(1)$ as $d \to \infty$, $\dim W_i = m - i$ for $i = 0, 1, \ldots, m$ and $H_f \cap W_m \cap X_{>\frac{d-c}{m+1}} = \emptyset$. But $H_f \cap W_m$ is the subvariety of X cut out by the equations $f(P) = (D_1 f)(P) = \cdots = (D_m f)(P) = 0$, so $H_f \cap W_m \cap X_{>\frac{d-c}{m+1}}$ is exactly the set of points of $H_f \cap X$ of degree $> \frac{d-c}{m+1}$ where $H_f \cap X$ is not smooth of dimension $m - 1$. Thus $\overline{\mu}_Z(\mathcal{Q}^{\text{high}}_{X-V}) = 0$. $\square$

Lemma 4.3 (Singularities of high degree on V)**.** *Define*

$$\mathcal{Q}^{\text{high}}_V := \bigcup_{d \ge 0} \{ f \in I_d : \exists P \in V_{>\frac{d-c}{m+1}} \text{ such that } H_f \cap X \text{ is not smooth of dimension } m-1 \text{ at } P \}.$$

Then $\overline{\mu}_Z(\mathcal{Q}^{\text{high}}_V) = 0$.

Proof. As before, we may assume $X \subseteq \mathbb{A}^n$ and we may dehomogenize. Given a closed point $x \in X$, choose a system of local parameters $t_1, \ldots, t_n \in A$ at x on $\mathbb{A}^n$ such that $t_{m+1} = t_{m+2} = \cdots = t_n = 0$ defines X locally at x, and $t_1 = t_2 = \cdots = t_{m-\ell} = t_{m+1} = t_{m+2} = \cdots = t_n = 0$ defines V locally at x. If $\mathfrak{m}_w$ is the ideal sheaf of w on $\mathbb{P}^n$, then $\mathcal{I}_Z \to \frac{\mathfrak{m}_w}{\mathfrak{m}_w^2}$ is surjective, so we may adjust $t_1, \ldots, t_{m-\ell}$ to assume that they vanish not only on V but also on Z.

Define ∂_i and D_i as in the proof of Lemma 4.2. Then there is a neighborhood N_x of x in $\mathbb{A}^n$ such that $N_x \cap \{t_{m+1} = t_{m+2} = \cdots = t_n = 0\} = N_x \cap X$, $\Omega^1_{N_x/\mathbb{F}_q} = \oplus_{i=1}^n \mathcal{O}_{N_x} dt_i$, and $s \in \mathcal{O}(N_u)^*$. Again we may assume $X \subseteq N_x$ for a single x. For $f \in I'_d \simeq I_d$, $H_f \cap X$ *fails* to be smooth of dimension $m-1$ at a point $P \in V$ if and only if $f(P) = (D_1 f)(P) = \cdots = (D_m f)(P) = 0$.

Again let $\tau = \max_i(\deg t_i)$, $\gamma = \lfloor (d - \tau)/p \rfloor$, and $\eta = \lfloor d/p \rfloor$. If $f_0 \in I'_d$, $g_1 \in S'_\gamma$, $\ldots$, $g_{\ell+1} \in S'_\gamma$, are chosen uniformly at random, then

$$f := f_0 + g_1^p t_1 + \cdots + g_{\ell+1}^p t_{\ell+1}$$

is a random element of I'_d, since $\ell + 1 \le m - \ell$.

For $i = 0, \ldots, \ell + 1$, the subscheme

$$W_i := V \cap \{D_1 f = \cdots = D_i f = 0\}$$

depends only on the choices of $f_0, g_1, \dots, g_i$. The same argument as in the previous proof shows that for $i = 0, \dots, \ell$, we have

$$\operatorname{Prob}(\dim W_i \le \ell - i) = 1 - o(1)$$

as $d \to \infty$. In particular, W_ℓ is finite with probability $1 - o(1)$.

To prove that $\overline{\mu}_Z(\mathcal{Q}_V^{\text{high}}) = 0$, it remains to prove that conditioned on choices of $f_0, g_1, \dots, g_\ell$ making $\dim W_\ell$ finite,

$$\operatorname{Prob}(W_{\ell+1} \cap V_{> \frac{d-c}{m+1}} = \emptyset) = 1 - o(1).$$

By Bézout's theorem, $\# W_\ell = O(d^\ell)$. The set H^{bad} of choices of $g_{\ell+1}$ making $D_{\ell+1} f$ vanish at a given point $P \in W_\ell$ is either empty or a coset of $\ker(\operatorname{ev}_P : S'_\gamma \to \kappa(P))$. Lemma 2.5 of [Poo04] implies that the size of this kernel (or its coset) as a fraction of $\# S'_\gamma$ is at most $q^{-\nu}$ where $\nu := \min\left(\gamma, \frac{d-c}{m+1}\right)$. Since $\# W_\ell q^\nu = o(1)$ as $d \to \infty$, we are done. □

5. Conclusion

Proof of Theorem 1.1(i). We have

$$\mathcal{P} \subseteq \mathcal{P}_r \subseteq \mathcal{P} \cup \mathcal{Q}_r^{\text{medium}} \cup \mathcal{Q}_{X-V}^{\text{high}} \cup \mathcal{Q}_V^{\text{high}},$$

so $\overline{\mu}_Z(\mathcal{P})$ and $\underline{\mu}_Z(\mathcal{P})$ each differ from $\mu_Z(\mathcal{P}_r)$ by at most $\overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}}) + \overline{\mu}_Z(\mathcal{Q}_{X-V}^{\text{high}}) + \overline{\mu}_Z(\mathcal{Q}_V^{\text{high}})$. Applying Corollary 2.4 and Lemmas 3.2, 4.2, and 4.3, we obtain

$$\mu_Z(\mathcal{P}) = \lim_{r \to \infty} \mu_Z(\mathcal{P}_r) = \frac{\zeta_V(m+1)}{\zeta_V(m-\ell)\, \zeta_X(m+1)}.$$

□

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References

[Blo70] S. Bloch, 1970. Ph.D. thesis, Columbia University.

[Ful84] W. Fulton, *Introduction to intersection theory in algebraic geometry*, CBMS Regional Conference Series in Mathematics, vol. 54, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1984.MR735435 (85j:14008)

[Gab01] O. Gabber, *On space filling curves and Albanese varieties*, Geom. Funct. Anal. **11** (2001), no. 6, 1192–1200.MR1878318 (2003g:14034)

[KA79] S. L. Kleiman and Allen B. Altman, *Bertini theorems for hypersurface sections containing a subscheme*, Comm. Algebra **7** (1979), no. 8, 775–790.MR529493 (81i:14007)

[Poo04] B. Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127. MR **2144974 (2006a:**14035)

[SS07] S. Saito and Kanetomo Sato, *Finiteness theorem on zero-cycles over p-adic fields* (April 11, 2007). `arXiv:math.AG/0605165`.

[Wei49] A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. **55** (1949), 497–508.MR0029393 (10,592e)

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