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ANALYTIC CAPACITY AND QUASICONFORMAL MAPPINGS WITH $W^{1,2}$ BELTRAMI COEFFICIENT

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Abstract. We show that if ϕ is a quasiconformal mapping with compactly supported Beltrami coefficient in the Sobolev space $W^{1,2}$, then ϕ preserves sets with vanishing analytic capacity. It then follows that a compact set E is removable for bounded analytic functions if and only if it is removable for bounded quasiregular mappings with compactly supported Beltrami coefficient in $W^{1,2}$.

1. Introduction

A Beltrami coefficient is a measurable function μ such that $\|\mu\|_\infty < 1$. Given an open set $\Omega \subset \mathbb{C}$, we say that $f : \Omega \to \mathbb{C}$ is μ-quasiregular if it belongs to the Sobolev space $W^{1,2}_{loc}(\Omega)$ and satisfies the Beltrami equation

$$\overline{\partial} f(z) = \mu(z)\,\partial f(z), \qquad a.e. z \in \Omega.$$

If moreover f is a homeomorphism, then we call it μ-*quasiconformal*. For any $K \geq 1$, we say that f is K-*quasiregular* (or K-*quasiconformal* if f is homeomorphism) for some Beltrami coefficient μ satisfying $\|\mu\|_\infty \leq \frac{K-1}{K+1}$.

Several works have focussed in the question of how these mappings distort measures and capacities. For instance, Ahlfors (see [Ah1]) proved that they always preserve sets of zero area. In a remarkable paper, Astala [As] obtained deep estimates for the area distortion under K-quasiconformal mappings. More precisely, if ϕ is any (conveniently normalized) K-quasiconformal mapping, then one has the estimate

$$|\phi(E)| \leq C\,|E|^{\frac{1}{K}}$$

where the constant C depends only on K. As a consequence, the author obtained also sharp results on integrability of K-quasiconformal mappings, which in turn led to the bounds on K-quasiconformal distortion of Hausdorff dimension. Namely, for any K-quasiconformal mapping ϕ and any compact set E,

$$(1) \qquad \frac{1}{K}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \leq \frac{1}{\dim(\phi(E))} - \frac{1}{2} \leq K\left(\frac{1}{\dim(E)} - \frac{1}{2}\right).$$

Moreover, in [As] the author shows the sharpness of both inequalities.

It is well known that sometimes the regularity of the Beltrami coefficient μ is inherited by the mapping itself. For instance, when μ is a compactly supported $\mathcal{C}^\infty$ function, then every μ-quasiconformal mapping ϕ is also $\mathcal{C}^\infty$. As a consequence, ϕ is locally bilipschitz, and then some set functions like Hausdorff measures, Riesz and Bessel

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capacities, are preserved.

Nevertheless, there are other situations which, even far from $\mu \in \mathcal{C}^\infty$, give interesting improvements of equation (1). For instance, when μ belongs to the class VMO of functions of vanishing mean oscillation, then

$$\dim(\phi(E)) = \dim(E).$$

That is, the corresponding μ-quasiconformal mappings ϕ do not distort Hausdorff dimension (see for instance [Iw]). However, nothing is known on the ratio between the Hausdorff measures $\mathcal{H}^t(E)$ and $\mathcal{H}^t(\phi(E))$, for any $t \in [0,2]$.

In this context, of special interest is the assumption that μ is a compactly supported function in the Sobolev class $W^{1,2}$ (notice that this implies $\mu \in VMO$). For such Beltrami coefficients, it is shown in [CFMOZ] that the corresponding μ-quasiconformal mappings ϕ preserve sets with zero length, that is

$$\mathcal{H}^1(E) = 0 \qquad \Longleftrightarrow \qquad \mathcal{H}^1(\phi(E)) = 0, \tag{2}$$

The proof of this fact uses some BMO removability techniques, related to both the Cauchy-Riemann (i.e. $\overline{\partial}$) and the Beltrami ($\overline{\partial} - \mu\,\partial$) differential operators. The main tool is an extended version of Weyl's lemma. Recall that classical Weyl's Lemma asserts that distributional solutions to the Cauchy-Riemann equation are actually analytic functions. In the more general case of the Beltrami equation [CFMOZ, Theorem 1], an analogous result can be given provided that the Beltrami coefficient belongs to $W^{1,2}$.

Theorem. *Let μ be a compactly supported Beltrami coefficient in the Sobolev space $W^{1,2}(\mathbb{C})$. Let $f \in L^p_{loc}(\mathbb{C})$ for some $p > 2$, and suppose that*

$$\langle \overline{\partial} f - \mu\,\partial f, \varphi\rangle = 0$$

whenever $\varphi \in \mathcal{C}^\infty$ is compactly supported. Then, f is μ-quasiregular.

In [CFMOZ], similar arguments to those in (2), replacing BMO by VMO, allowed the authors to prove that if $\mu \in W^{1,2}$ is any compactly supported Beltrami coefficient, and ϕ is μ-quasiconformal, then

$$\mathcal{H}^1(E) \text{ is } \sigma\text{-finite} \qquad \Longleftrightarrow \qquad \mathcal{H}^1(\phi(E)) \text{ is } \sigma\text{-finite}. \tag{3}$$

Furthermore, these mappings ϕ are shown to map 1-rectifiable sets to 1-rectifiable sets (and purely 1-unrectifiable sets to purely 1-unrectifiable sets).

As we shall see in this paper, all these facts have interesting consequences when studying removability problems for bounded μ-quasiregular mappings, that is, the μ-quasiregular counterpart for the problem of Painlevé. Recall that a compact set E is said to be *removable* (*for bounded analytic functions*) if for any open set $\Omega \supset E$, every bounded function $f : \Omega \to \mathbb{C}$, analytic on $\Omega \setminus E$, admits an analytic extension to the whole of Ω. The *problem of Painlevé* consists of giving metric and geometric characterizations of these sets.

When studying removable sets, it is natural to talk about analytic capacity. Recall that given a compact set E, the *analytic capacity* of E is defined as

$$\gamma(E) = \sup\{|f'(\infty)|; f \in H^\infty(\mathbb{C}\setminus E), \|f\|_\infty \le 1\}.$$

Here, by $H^\infty(\Omega)$ we mean the space of bounded analytic functions on the open set Ω, and $f'(\infty) = \lim_{z\to\infty} z\,(f(\infty) - f(z))$. For a set $A \subset \mathbb{C}$ which may be non compact, one defines

$$\gamma(A) = \sup_{E\subset A \text{ compact}} \gamma(E).$$

Ahlfors [Ah2] proved that E is removable for bounded analytic functions if and only if $\gamma(E) = 0$. Furthermore, it is not difficult to show that $\gamma(E) \le C\,\mathcal{H}^1(E)$, while $\dim(E) > 1$ implies $\gamma(E) > 0$. It took long time to have a precise geometric characterization of the zero sets for γ. In [Da1], G. David proved that if E has finite length then

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad E \text{ is purely 1-unrectifiable.}$$

Later, in [To2], X. Tolsa characterized sets with vanishing analytic capacity in terms of Menger curvature (see Theorem 5 below for more details).

In this paper, as well as in [CFMOZ], our main objects of study are the removable singularities for bounded solutions to a fixed Beltrami equation. Namely, we say that a compact set E is *removable for bounded μ-quasiregular mappings*, or simply *μ-removable*, if for any open set Ω, any bounded function $f : \Omega \to \mathbb{C}$, μ-quasiregular on $\Omega \setminus E$, admits a μ-quasiregular extension to the whole of Ω. By means of Stoilow's factorization Theorem, one easily shows that E is μ-removable if and only if $\gamma(\phi(E)) = 0$ for any μ-quasiconformal mapping ϕ. In connection with this question, the following result is proved in [CFMOZ].

Theorem. *Let $\mu \in W^{1,2}(\mathbb{C})$ be a compactly supported Beltrami coefficient, and let $\phi : \mathbb{C} \to \mathbb{C}$ be a μ-quasiconformal mapping. Then,*

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E)) = 0 \tag{4}$$

for any compact set E with σ-finite $\mathcal{H}^1(E)$.

Pekka Koskela suggested us that the σ-finiteness assumption might be removed in the preceding result. In this paper we do the job.

Theorem 1. *Let $\mu \in W^{1,2}(\mathbb{C})$ be a compactly supported Beltrami coefficient, and let $\phi : \mathbb{C} \to \mathbb{C}$ be a μ-quasiconformal mapping. Then,*

$$\gamma(E) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E)) = 0$$

for any compact set E.

It follows from Theorem 1 that if $\mu \in W^{1,2}$ is compactly supported, then being removable and being μ-removable are equivalent notions.

Corollary 2. *Let $\mu \in W^{1,2}$ be a compactly supported Beltrami coefficient. Then, a compact set E is removable for bounded μ-quasiregular mappings if and only if $\gamma(E) = 0$.*

Theorem 1 implies that, given a compactly supported Beltrami coefficient $\mu \in W^{1,2}(\mathbb{C})$, the corresponding μ-quasiconformal mappings preserve the removable sets for bounded analytic functions. This fact is closely related to a question of J. Verdera [Ve1] on the preservation of removable sets under some planar homeomorphisms. More precisely, the author wondered how analytic capacity is distorted under bilipschitz mappings. Recall that a mapping $\phi : \mathbb{C} \to \mathbb{C}$ is said to be L-bilipschitz if

$$\frac{1}{L}|z-w| \leq |\phi(z)-\phi(w)| \leq L|z-w|$$

for any pair of points $z, w \in \mathbb{C}$. This question was solved in [To2]:

Theorem. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an L-bilipschitz mapping. Then,*

$$\gamma(\phi(E)) \simeq \gamma(E) \tag{5}$$

with constants that depend only on L.

Furthermore, it is shown in [To2] that any planar homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ satisfying (5) must be a bilipschitz mapping. It is well known that L-bilipschitz mappings are μ-quasiconformal for some Beltrami coefficient μ with $\|\mu\|_\infty$ depending only on L, but in general this does not imply any $W^{1,2}$ regularity for μ. In fact, there is not a precise description of the class of compactly supported Beltrami coefficients μ that produce bilipschitz μ-quasiconformal mappings. It was remarked in [CFMOZ, Example 4] that there are non bilipschitz μ-quasiconformal mappings with compactly supported $\mu \in W^{1,2}$. At the same time, the example $\mu(z) = \frac{1}{2}\,\chi_{\mathbb{D}}(z)$ gives a bilipschitz μ-quasiconformal mapping with $\mu \notin W^{1,2}$. In fact, this can be seen as a particular case of a more general result concerning smooth truncations of Hölder continuous Beltrami coefficients (see [MOV] for more details). In any case, there is no simple relation between bilipschitz μ-quasiconformal mappings and Beltrami coefficients $\mu \in W^{1,2}$.

For the proof of the Theorem 1, our main tool is the following improved version of the preceding theorem on the bilipschitz invariance of analytic capacity.

Theorem 3. *Given $E, F \subset \mathbb{C}$, let $\phi : E \to F$ be a bilipschitz homeomorphism. That is, there exists $L > 0$ such that*

$$\frac{1}{L}\,|z-w| \leq |\phi(z)-\phi(w)| \leq L\,|z-w| \tag{6}$$

whenever $z, w \in E$. Then there exists some constant C depending only on L such that

$$\frac{1}{C}\,\gamma(F) \leq \gamma(E) \leq C\gamma(F).$$

Notice that in this result we assume the mapping ϕ to be bilipschitz only on E, not in the whole complex plane. From this theorem we deduce

Corollary 4. *Assume that $\phi : \mathbb{C} \to \mathbb{C}$ is a planar homeomorphism, locally bilipschitz in a measurable set $\Omega \subset \mathbb{C}$. That is, there are constants $C > 0$ and $\delta > 0$ such that*

$$\frac{1}{C}\,|z-w| \leq |\phi(z)-\phi(w)| \leq C\,|z-w|$$

whenever $z, w \in \Omega$ *and* $|z-w| < \delta$. *Then*

$$\gamma(E \cap \Omega) = 0 \qquad \Longleftrightarrow \qquad \gamma(\phi(E \cap \Omega)) = 0$$

for any compact set $E \subset \mathbb{C}$.

As explained above, the μ-quasiconformal mappings we deal with are not bilipschitz in the whole plane, in general. However, they are locally bilipschitz on the level sets of its Jacobian determinant, modulo some *small* set of bad points. This follows from the quasisymmetry and from the fact that these μ-quasiconformal mappings belong to some second order Sobolev spaces. Moreover, it turns out that they are strongly differentiable everywhere except on a set of Hausdorff dimension 0.

The paper is structured as follows. In Section 2 we prove Theorem 3, and in Section 3 its Corollary 4. In Section 4, we use this result to prove Theorem 1.

2. Proof of Theorem 3

2.1. Analytic capacity and curvature. We need to recall the notion of curvature of a measure. Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $R(x,y,z) = \infty$, $c(x,y,z) = 0$ if x, y, z lie on the same line). If two among these points coincide, we set $c(x,y,z) = 0$. For a positive finite Borel measure σ on $\mathbb{C}$, we define the *curvature of* σ as

$$c^2(\sigma) = \iiint c(x,y,z)^2 \, d\sigma(x) d\sigma(y) d\sigma(z). \tag{7}$$

We recall the characterization of γ in terms of curvature from [To2]:

Theorem 5. *For any compact* $E \subset \mathbb{C}$ *we have*

$$\gamma(E) \simeq \sup \sigma(E),$$

where the supremum is taken over all Borel measures σ *supported on* E *such that* $\sigma(B(x,r)) \leq r$ *for all* $x \in E$, $r > 0$ *and* $c^2(\sigma) \leq \sigma(E)$.

We will prove the following result.

Theorem 6. *Let* σ *be a Borel measure supported on a compact set* $E \subset \mathbb{C}$, *such that* $\sigma(B(x,r)) \leq r$ *for all* $x \in E$, $r > 0$ *and* $c^2(\sigma) < \infty$. *Let* $\phi : E \to \phi(E)$ *be a bilipschitz mapping. There exists a positive constant* C *depending only on the bilipschitz constant of* ϕ *such that*

$$c^2(\phi_\# \sigma) \leq C\big(\sigma(E) + c^2(\sigma)\big),$$

where $\phi_\# \sigma$ *stands for the image measure of* σ *by* ϕ.

It is easy straightforward check that Theorem 3 is a direct consequence of Theorems 5 and 6. We remark that Theorem 6 was proved in [To2, Theorem 1.3] under the stronger assumption that ϕ is bilipschitz on the whole complex plane. The next Subsections 2.2, 2.3 and 2.4 deal with the proof of this result.

2.2. Additional notation and terminology. By a square we mean a square with sides parallel to the axes. Moreover, we assume the squares to be half closed - half open. The side length of a square Q is denoted by $\ell(Q)$. Given $a > 0$, aQ denotes the square concentric with Q with side length $a\ell(Q)$. The average (linear) density of a Borel measure σ on Q is

$$\theta_\sigma(Q) := \frac{\sigma(Q)}{\ell(Q)}. \tag{8}$$

We say that σ has linear growth if there exists some constant C such that

$$\sigma(B(x,r)) \le Cr \qquad \text{for all } x \in \mathbb{C},\ r > 0.$$

A square $Q \subset \mathbb{C}$ is called 4-dyadic if it is of the form $[j2^{-n}, (j+4)2^{-n}) \times [k2^{-n}, (k+4)2^{-n})$, with $j,k,n \in \mathbb{Z}$. So a 4-dyadic square with side length $4 \cdot 2^{-n}$ is made up of 16 dyadic squares with side length 2^{-n}. Given $a, b > 1$, we say that Q is (a,b)-doubling if $\sigma(aQ) \le b\sigma(Q)$. If we don't want to specify the constant b, we say that Q is a-doubling. Given two squares $Q \subset R$, we set

$$\delta_\sigma(Q,R) := \int_{R_Q \setminus 2Q} \frac{1}{|y - x_Q|}\, d\sigma(y),$$

where x_Q stands for the center of Q, and R_Q is the smallest square concentric with Q that contains R. Given a bilipschitz mapping $\phi : \mathbb{C} \to \mathbb{C}$ and a square Q, in [To2] one says that that $\phi(Q)$ is a ϕ-square, and then one defines its side length as $\ell(\phi(Q)) := \ell(Q)$. Now we only know that ϕ bilipschitz from E onto $\phi(E)$, and thus $\phi(Q)$ is not defined in general. So we have to change the notion of ϕ-square. A first attempt would consist in saying that P is a ϕ-square if $P = \phi(Q \cap E)$ for some square Q. This definition has a serious drawback: we cannot set $\ell(P) := \ell(Q)$ because it may happen that $P = \phi(Q \cap E) = \phi(R \cap E)$ for two different squares Q, R. However there is an easy solution: a ϕ-square is not a subset of $\mathbb{C}$, but a pair of the form $P = (Q, \phi(Q \cap E))$, for some square $Q \subset \mathbb{C}$. We denote $\ell(P) := \ell(Q)$. On the other hand, abusing language sometimes we will identify the ϕ-square $P = (Q, \phi(Q \cap E))$ with the set $\phi(Q \cap E)$, and so we will use notations such as $\operatorname{diam}(P)$ (notice that $\operatorname{diam}(P) \lesssim \ell(P)$), or $\phi_\#\sigma(P)$. If Q_0 is a dyadic (or 4-dyadic) square, we say that $(Q_0, \phi(Q_0 \cap E))$ is a dyadic (or 4-dyadic) ϕ-square. If $Q = (Q_0, \phi(Q_0 \cap E))$ is a ϕ-square, we denote $\lambda Q = (\lambda Q_0, \phi(\lambda Q_0 \cap E))$, for $\lambda > 0$. To simplify notation, we set $\tau := \phi_\#\sigma$ and $F := \phi(E)$. A ϕ-square Q is said to be λ-doubling if $\tau(\lambda Q) \le C\tau(Q)$ for some fixed $C \ge 1$. We also set

$$\theta_\tau(Q) := \frac{\tau(Q)}{\ell(Q)}$$

and if R is another ϕ-square which contains Q, we put

$$\delta_\tau(Q,R) := \int_{R_Q \setminus 2Q} \frac{1}{|y - x_Q|}\, d\tau(y),$$

where x_Q stands for some fixed (arbitrary) point of Q and R_Q is the smallest ϕ-square concentric with Q that contains R. That is to say, if $Q = (Q_0, \phi(Q_0 \cap E))$, $R = (R_0, \phi(R_0 \cap E))$, and S is the smallest square concentric with Q_0 that contains R_0, we set $R_Q = (S, \phi(S \cap E))$. An Ahlfors regular curve is a curve Γ such that $\mathcal{H}^1(\Gamma \cap B(x,r)) \le Cr$ for all $x \in \Gamma$, $r > 0$, and some fixed $C > 0$. We say that Γ

is a chord arc curve if it is a bilipschitz image of an interval in $\mathbb{R}$. If the bilipschitz constant of the map is L, we say that Γ is an L-chord arc curve.

2.3. The corona decomposition. Theorem 6 will be proved by means of a corona type decomposition for σ similar to the one in [To2]. In Lemma 7 below, where we prove the existence of this decomposition, we will introduce a family $\mathrm{Top}(E)$ of 4-dyadic squares (the top squares) satisfying some precise properties. Given any square $Q \in \mathrm{Top}(E)$, we denote by $\mathrm{Stop}(Q)$ the subfamily of the squares $P \in \mathrm{Top}(E)$ satisfying

(a) $P \cap 3Q \neq \varnothing$,
(b) $\ell(P) \leq \frac{1}{8}\ell(Q)$,
(c) P is maximal, in the sense that there doesn't exist another square $P' \in \mathrm{Top}(E)$ satisfying (a) and (b) which contains P.

We also denote by $Z(\sigma)$ the set of points $x \in \mathbb{C}$ such that there does not exist a sequence of $(70, 5000)$-doubling squares $\{Q_n\}_n$ centered at x with $\ell(Q_n) \to 0$ as $n \to \infty$, so that moreover $\ell(Q_n) = 2^{-k_n}$ for some $k_n \in \mathbb{Z}$. We have $\sigma(Z(\sigma)) = 0$ (see [To2, Remark 2.1]). The set of good points for Q is defined as

$$G(Q) := 3Q \cap \mathrm{supp}(\sigma) \setminus \Big[Z(\sigma) \cup \bigcup_{P\in \mathrm{Stop}(Q)} P\Big].$$

Lemma 7 (The corona decomposition)**.** *Let σ be a Borel measure supported on $E \subset \mathbb{C}$ such that $\sigma(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$ and $c^2(\sigma) < \infty$. There exists a family* $\mathrm{Top}(E)$ *of* 4*-dyadic* $(16, 5000)$*-doubling squares (called* top squares*) which satisfy the packing condition*

$$\sum_{Q\in \mathrm{Top}(E)} \theta_\sigma(Q)^2\sigma(Q) \leq C\big(\sigma(E) + c^2(\sigma)\big), \tag{9}$$

and such that for each square $Q \in \mathrm{Top}(E)$ there is a family of C_1-chord arc curves Γ_Q^i, $i = 1, \dots, N_0$, such that if we set $\Gamma_Q = \bigcup_{i=1}^{N_0} \Gamma_Q^i$, we have

(a) *$G(Q) \subset \Gamma_Q \cap E$.*
(b) *For each $P \in \mathrm{Stop}(Q)$ there exists some square $\widetilde{P}$ containing P such that $\delta_\sigma(P, \widetilde{P}) \leq C_2\theta_\sigma(Q)$ and $\widetilde{P} \cap \Gamma_Q \cap E \neq \varnothing$.*
(c) *If P is a square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \varnothing$ or there is another square $P' \in \mathrm{Stop}(Q)$ such that $P \cap P' \neq \varnothing$ and $\ell(P') \leq \ell(P)$, then $\sigma(P) \leq C_3\, \theta_\sigma(Q)\, \ell(P)$.*

Moreover, $\mathrm{Top}(E)$ *contains some* 4*-dyadic square R_0 such that $E \subset R_0$. The constants C_1, C_2, C_3, N_0 are absolute.*

Notice that the chord arc constant of the curves Γ_Q^i in the lemma is uniformly bounded above by C_1.

Proof. The proof of this lemma is very similar to the one of Main Lemma 3.1 in [To2], and so we will only describe in detail the required modifications for the proof. Notice that, in Lemma 7, Γ_Q is made up of a finite union of chord arc curves, while in [To2, Main Lemma 3.1] Γ_Q is an Ahlfors regular curve. On the other hand, in the statement (b) above we ask $\widetilde{P} \cap \Gamma_Q \cap E \neq \varnothing$, while in (b) of Main Lemma 3.1 in [To2] one asks only $\widetilde{P} \cap \Gamma_Q \neq \varnothing$. These are the only differences between both lemmas.

First we will explain the arguments to show that Γ_Q is made up of a finite number of chord arc curves. First we need some notation. Given a set $K \subset \mathbb{C}$ and a square Q, let V_Q be an infinite strip (or line in the degenerate case) of smallest possible width which contains $K \cap 3Q$, and let $w(V_Q)$ denote the width of V_Q. Denote

$$\beta_K(Q) = \frac{w(V_Q)}{\ell(Q)}.$$

Recall the following version of Jones' traveling salesman theorem [Jo]:

Theorem 8 (P. Jones)**.** *A set $K \subset \mathbb{C}$ is contained in an Ahlfors regular curve if and only if there exists some constant C_4 such that for every dyadic square Q*

$$\sum_{P \in \mathcal{D}, P \subset Q} \beta_K(P)^2 \ell(P) \leq C_4 \ell(Q).$$

The regularity constant of the curve depends on C_4.

Moreover, in [Jo] the author claims that there exists some constant $\eta > 0$ small enough such that if $\beta_K(Q) \leq \eta$ for every square $Q \in \mathcal{D}$, then K is contained in quasicircle. So we have

Theorem 9. *There exists some absolute constant $\eta > 0$ such that if, for every dyadic square Q, $K \subset \mathbb{C}$ satisfies*

$$\sum_{P \in \mathcal{D}, P \subset Q} \beta_K(P)^2 \ell(P) \leq \eta \ell(Q),$$

then K is contained in a chord arc curve.

Although the preceding result is not proved in [Jo], it follows from easy modifications of the author's arguments (we suggest the reader to look also at [GM, Chapter 10, Theorem 2.3]).
Given $R \in \operatorname{Top}(E)$, in [To2] the existence of the curve Γ_R follows from an application of Theorem 8. To this end, in [To2, Lemma 4.5] one constructs a set K which contains $G(R)$ and which, in a sense, approximates $\operatorname{supp}(\sigma)$ on the squares $P \in \operatorname{Stop}(R)$. Then one proves in [To2, Subsection 4.4] that for any square Q with $\ell(Q) \leq C_5 \ell(R)$ (where $C_5 < 1$ is some small positive constant),
(10)

$$\sum_{P \in \mathcal{D}, P \subset Q} \beta_K(P)^2 \ell(P) \leq C \theta_\sigma(R)^{-3} \iiint_{(x,y,z) \in (3Q)^3 \cap R^\ell} c(x,y,z)^2 d\sigma(x) d\sigma(y) d\sigma(z).$$

See p. 1266 of [To2] for the precise definition of R^ℓ. For the reader's convenience, let us say that the triple integral on the right hand side is some truncated version of the curvature $c^2(\sigma_{|3Q})$. Moreover, from the construction of the stopping squares, it turns out that

$$C \theta_\sigma(R)^{-3} \iiint_{(x,y,z) \in (3Q)^3 \cap R^\ell} c(x,y,z)^2 d\sigma(x) d\sigma(y) d\sigma(z) \leq C_6 \ell(Q). \tag{11}$$

¿From (10), (11), and Theorem 8 one infers the existence of the regular curve Γ_Q. However, a careful examination of the proof of Main Lemma 3.1 in [To2] shows that the constant C_6 in (11) can be taken so that $C_6 \leq \eta$ (in fact, as small as we want). To this end, one has to take the parameter ε_0 small enough in the construction of the high curvature squares (see p. 1252 of [To2]). We leave the details for the reader. Then Theorem 9 can be applied for the squares Q with $Q \cap 3R \neq \varnothing$ such that with $\ell(Q) \leq C_5\ell(R)$. As a consequence, one deduces that K is contained in a finite number N_0 of chord arc curves (we only have to cover $3R$ by a finite number of squares with side length $C_5\ell(R)$ and then we apply Theorem 9).

On the other hand, it is easy to check that the curve Γ_Q constructed in the proof of Main Lemma 3.1 satisfies $\widetilde{P} \cap \Gamma_Q \cap E \neq \varnothing$, as required above in (b). Of course, the same happens with the "new" curves Γ_Q described above since the method of construction has not changed. This is due to the fact that the set K obtained in [To2, Lemma 4.5] is contained in $\operatorname{supp}(\mu)$. This is not stated in [To2, Lemma 4.5], but it is easily seen. □

Remark. *It is easy to check that one can always assume $\widetilde{P} \subset 16Q$ in the statement (b) of Lemma 7.*

2.4. The curvature of $\phi_{\#}\sigma$. In Lemma 7 we have shown how to construct a corona type decomposition for a measure σ with linear growth and finite curvature. We will see below that ϕ sends this corona type decomposition into another corona decomposition in terms of ϕ-squares, and we will prove that that the existence of such a decomposition implies that the curvature is finite. These will be the basic ingredients for the proof of Theorem 6.

First we introduce some notation. Given a family $\operatorname{Top}(F)$ of 4-dyadic ϕ-squares and a fixed $Q \in \operatorname{Top}(F)$, we denote by $\operatorname{Stop}(Q)$ the subfamily of ϕ-squares which satisfy the properties (a), (b), (c) stated at the beginning of Subsection 2.3 (with squares replaced by ϕ-squares). The set $G(Q)$ is also defined as in Subsection 2.3, with ϕ-squares instead of squares.

Lemma 10. *Let τ be a Borel measure supported on a compact set $F \subset \mathbb{C}$. Suppose that $\tau(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$. Let $\operatorname{Top}(F)$ be a family of 4-dyadic 16-doubling ϕ-squares which contains some 4-dyadic ϕ-square R_0 such that $F = R_0$, and such that for each $Q \in \operatorname{Top}(F)$ there exists a C_7-AD regular curve Γ_Q satisfying:*

- (a) *τ-almost every point in $G(Q)$ belongs to Γ_Q.*
- (b) *For each $P \in \operatorname{Stop}(Q)$ there exists some ϕ-square $\widetilde{P}$ containing P such that $\delta_\tau(P,\widetilde{P}) \leq C\theta_\tau(Q)$ and $\widetilde{P} \cap \Gamma_Q \neq \varnothing$.*
- (c) *If P is a ϕ-square with $\ell(P) \leq \ell(Q)$ such that either $P \cap G(Q) \neq \varnothing$ or there is another ϕ-square $P' \in \operatorname{Stop}(Q)$ such that $P \cap P' \neq \varnothing$ and $\ell(P') \leq \ell(P)$, then $\tau(P) \leq C\,\theta_\tau(Q)\,\ell(P)$.*

Then,

$$c^2(\tau) \leq C \sum_{Q \in \operatorname{Top}(F)} \theta_\tau(Q)^2 \tau(Q).$$

The proof of this lemma is almost the same as the one of [To2, Main Lemma 8.1], and so we omit the details.

In order to show that ϕ transforms a corona type decomposition like the one in Lemma 7 into another like the one of the preceding lemma we need the following result of MacManus [MM]:

Theorem 11. *Any M-bilipschitz map of a subset of a line or a circle into the plane has an extension to a $C(M)$-bilipschitz map from the plane onto itself.*

We are ready to prove Theorem 6 now:

Proof of Theorem 6. We consider the measure σ and its corona type decomposition given by Lemma 7. It is straightforward to check that $\tau := \phi_{\#}\sigma$ has linear growth. We take the family $\mathrm{Top}(F) = \phi(\mathrm{Top}(E))$, and for $Q \in \mathrm{Top}(F)$ with $Q = (Q_0, \phi(Q_0 \cap E))$, we define $\mathrm{Stop}(Q) = \phi(\mathrm{Stop}(Q_0))$.
To construct a regular curve Γ_Q such as the one required in Lemma 10, we consider the union of chord arc curves $\Gamma_{Q_0} = \bigcup_{i=1}^{N_0} \Gamma_{Q_0}^i$ of Lemma 7. Notice that we cannot set $\Gamma_Q = \phi(\Gamma_{Q_0})$ because ϕ is not defined on the whole set Γ_{Q_0}. By MacManus' theorem we can solve this problem easily. For each $i = 1, \ldots, N_0$, let $\rho_i : \mathbb{R} \supset I \to \Gamma_{Q_0}^i$ be a bilipschitz parametrization of the chord arc curve $\Gamma_{Q_0}^i$. Consider the subset $\Gamma_{Q_0}^i \cap E$ and the bilipschitz map $\phi \circ \rho_i$, defined on $\rho_i^{-1}(\Gamma_{Q_0}^i \cap E) \subset \mathbb{R}$. By Theorem 11, $\phi \circ \rho_i$ has a bilipschitz extension f_i onto the whole complex plane. We consider the chord arc curve $\Gamma_i := f_i(\mathbb{R})$, and we set $\Gamma_Q := \bigcup_{i=1}^{N_0} \Gamma_i$ (and we add a finite number of segments if necessary to ensure that that Γ_Q is a regular curve).

We have to show that the assumptions of Lemma 10 hold for the family $\mathrm{Top}(F)$, their corresponding stopping squares, and the curves Γ_Q, $Q \in \mathrm{Top}(F)$. Indeed, (a) and (c) are the translation of the corresponding statements (a) and (c) of Lemma 7. On the other hand, (b) is a consequence of the fact that if $P_0 \in \mathrm{Stop}(Q_0)$ for some $Q_0 \in \mathrm{Top}(E)$, and $\widetilde{P}_0 \subset 16Q_0$ (recall Remark 2.3), and moreover we have

$$\delta_\sigma(P_0, \widetilde{P}_0) \le C_2 \theta_\sigma(Q_0),$$

then $P = (P_0, \phi(P_0 \cap E))$ and $\widetilde{P} = (\widetilde{P}_0, \phi(\widetilde{P}_0 \cap E))$ satisfy

$$\delta_\tau(P, \widetilde{P}) \lesssim \theta_\tau(Q). \tag{12}$$

To prove this estimate, recall that

$$\delta_\tau(P, \widetilde{P}) := \int_{\widetilde{P}_P \setminus 2P} \frac{1}{|y - z_P|}\, d\tau(y),$$

where z_P is some fixed point of P and $\widetilde{P}_P$ is the smallest ϕ-square concentric with P that contains $\widetilde{P}$. Then, if we set $z_{P_0} = \phi^{-1}(z_P)$, we have

$$\delta_\tau(P, \widetilde{P}) = \int_{(\widetilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|\phi(y) - \phi(z_{P_0})|}\, d\sigma(y) \simeq \int_{(\widetilde{P}_0)_{P_0} \setminus 2P_0} \frac{1}{|y - z_{P_0}|}\, d\sigma(y).$$

Since $z_{P_0} \in P_0$, from the property (c) of Lemma 7, it follows easily that

$$\begin{aligned}\int_{(\widetilde{P}_0)_{P_0}\setminus 2P_0} \frac{1}{|y - z_{P_0}|}\, d\sigma(y) &\leq \int_{(\widetilde{P}_0)_{P_0}\setminus 2P_0} \frac{1}{|y - x_{P_0}|}\, d\sigma(y) + C\theta_\sigma(Q_0)\\ &= \delta(P_0, \widetilde{P}_0) + C\theta_\sigma(Q_0) \leq C\theta_\sigma(Q_0) \simeq \theta_\tau(Q).\end{aligned}$$

(recall that x_{P_0} is the center of P_0, which may not coincide with z_{P_0}), and so (12) holds.

Thus from Lemmas 10 and 7 we infer that

$$c^2(\tau) \lesssim C \sum_{Q\in \mathrm{Top}(F)} \theta_\tau(Q)^2\tau(Q) \simeq \sum_{Q_0\in \mathrm{Top}(E)} \theta_\sigma(Q_0)^2\sigma(Q_0) \lesssim \sigma(E) + c^2(\sigma).$$

□

3. Proof of Corollary 4

This is an immediate consequence of Theorem 3. Let $F \subset E \cap \Omega$ be compact. Obviously, $\gamma(F) = 0$. Cover F by a finite number of closed balls B_i, $1 \leq i \leq N$, of diameter $\delta/2$. Since $\phi : F \cap B_i \to \phi(F \cap B_i)$ is bilipschitz, we have

$$\gamma(\phi(F \cap B_i)) \simeq \gamma(F \cap B_i) = 0$$

for all i. This implies that $\gamma(\phi(F)) = 0$ (this is a consequence of the semiadditivity of γ, but it can be proven by much simpler arguments). Since this holds for any compact set $\phi(F) \subset \phi(E \cap \Omega)$, we have $\gamma(\phi(E \cap \Omega)) = 0$. □

4. Proof of Theorem 1

In all this section, μ is a compactly supported Beltrami coefficient belonging to the Sobolev space $W^{1,2}(\mathbb{C})$ and $\phi : \mathbb{C} \to \mathbb{C}$ is a μ-quasiconformal mapping. There is no restriction if we normalize ϕ in such a way that $\phi(z) - z = O(1/|z|)$ as $|z| \to \infty$. We can assume also that $\mathrm{supp}(\mu) \subset \mathbb{D}$.

We start by giving some auxiliary results. The first one is from [CFMOZ].

Lemma 12. *Let $\mu \in W^{1,2}(\mathbb{C})$ be a compactly supported Beltrami coefficient. Let ϕ be μ-quasiconformal, and let $E \subset \mathbb{C}$. Then,*

$$\mathcal{H}^1(E) = 0 \qquad \Leftrightarrow \qquad \mathcal{H}^1(\phi(E)) = 0.$$

The following result establishes that the means of the Jacobian determinant of a quasiconformal mapping behave precisely as an incremental quotient.

Lemma 13. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. Let $z, w \in \mathbb{C}$ be such that $z \neq w$, and $D = D(z, |z - w|)$. Then,*

$$\frac{|\phi(z) - \phi(w)|}{|z - w|} \simeq \left(\frac{1}{|D|} \int_D J(z, \phi)\, dA(z) \right)^{\frac{1}{2}}$$

with constants that depend only on K.

Proof. Let $r = |w - z|$. Then,

$$\begin{aligned}|\phi(w) - \phi(z)| &\leq \max_{|\zeta - z| = r} |\phi(\zeta) - \phi(z)| \leq C_K \min_{|\zeta - z| = r} |\phi(\zeta) - \phi(z)| \\ &\leq C_K \left(\frac{|\phi(D(z,r))|}{\pi}\right)^{\frac{1}{2}} \\ &= C_K\, |z - w| \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} J(\zeta,\phi)\, dA(\zeta)\right)^{\frac{1}{2}}.\end{aligned}$$

The converse inequality can be obtained analogously. □

Recall that if E is any compact set on the plane, the $(1,p)$-*Riesz capacity* of E is defined as

$$C_{1,p}(E) = \inf \left\{ \|D\psi\|_p^p \right\}$$

where the infimum is taken over all compactly supported $\psi \in \mathcal{C}^\infty(\mathbb{C})$ with $\psi \geq \chi_E$. One obtains the same quantity if one simply assumes $\psi \in W^{1,p}(\mathbb{C})$. Recall also that if $C_{1,p}(E) = 0$ for some p with $1 < p < 2$, then $\mathcal{H}^1(E) = 0$. For more details about Riesz capacities, see [AH], for example.

Proof of Theorem 1. Notice first that ϕ is conformal out of $\mathbb{D}$, so that by normalization and Koebe's 1/4-Theorem, we have $\phi(\mathbb{D}) \subset 4\mathbb{D}$. In particular, ϕ^{-1} is conformal out of $4\mathbb{D}$.
Since $\mu \in W^{1,2}(\mathbb{C})$ is compactly supported we know (see for instance [CFMOZ, Proposition 3]) that there exists $q \in (1,2)$ such that $\phi, \phi^{-1} \in W^{2,q}_{loc}(\mathbb{C})$. Thus, $D\phi, D\phi^{-1} \in W^{1,q}_{loc}$ and by [AH, Theorem 6.2.1] both $D\phi$ and $D\phi^{-1}$ admit $C_{1,q}$-quasicontinuous representatives and one can find a decreasing sequence of open sets $U_{i+1} \subset U_i \subset \mathbb{D}$ and real numbers $r_i > 0$ such that:

- $C_{1,q}(U_i) < 2^{-i}$.
- $D\phi$ is continuous on $\overline{\mathbb{D}} \setminus U_i$.
- $\displaystyle\frac{1}{|D(z,r)|}\int_{D(z,r)} |D\phi - D\phi(z)|\, dA < 1$ for each $z \in \mathbb{D}\setminus U_i$ and each $0 < r < r_i$.

Analogously, there exists a sequence of decreasing open sets $V_{i+1} \subset V_i \subset 4\mathbb{D}$ and real numbers $\tilde{r}_i > 0$ such that

- $C_{1,q}(V_i) < 2^{-i}$.
- $D\phi^{-1}$ is continuous on $\overline{\mathbb{D}} \setminus V_i$.
- $\displaystyle\frac{1}{|D(w,r)|}\int_{D(w,r)} \left|D\phi^{-1} - D\phi^{-1}(w)\right| dA < 1$ for each $w \in 4\mathbb{D} \setminus V_i$ and each $0 < r < \tilde{r}_i$.

If we now put $Z = \cap_i U_i$, then

$$C_{1,q}(Z) \leq \lim_i C_{1,q}(U_i) = 0.$$

Therefore $\mathcal{H}^1(Z) = 0$ and by Lemma 12 also $\mathcal{H}^1(\phi(Z)) = 0$. Further, if we denote $W_i = \phi^{-1}(V_i)$ and $Z' = \cap_i W_i$, then

$$C_{1,q}(\phi(Z')) \leq \lim_i C_{1,q}(V_i) = 0$$

so that $\mathcal{H}^1(\phi(Z')) = 0$, and thus Lemma 12 implies that $\mathcal{H}^1(Z') = 0$.

Define

$$F_{i,j} = \left\{z \in E \setminus (U_i \cup W_i) : |D\phi(z)| \leq j \text{ and } |D\phi^{-1}(\phi(z))| \leq j\right\}.$$

Our first claim is that the restriction $\phi|_{F_{i,j}} : F_{i,j} \to \phi(F_{i,j})$ is locally bilipschitz. To show this, let $z \in F_{i,j}$. Then, if $|w - z| \leq r_i$,

$$\begin{aligned}
\frac{|\phi(z) - \phi(w)|}{|z - w|} &\simeq \frac{1}{D(z, |z-w|)} \int_{D(z,|z-w|)} J\phi \\
&\simeq \frac{1}{D(z, |z-w|)} \int_{D(z,|z-w|)} |D\phi|^2 \\
&\simeq \left(\frac{1}{D(z, |z-w|)} \int_{D(z,|z-w|)} |D\phi|\right)^2 \\
&\leq (1 + |D\phi(z)|)^2 \leq (1+j)^2.
\end{aligned}$$

Analogously, if we write $w' = \phi(w)$ and $z' = \phi(z)$, then if $|z' - w'| \leq \tilde{r}_i$ we get

$$\begin{aligned}
\frac{|z - w|}{|\phi(z) - \phi(w)|} &= \frac{|\phi^{-1}(w') - \phi^{-1}(z')|}{|w' - z'|} \\
&\simeq \frac{1}{D(z', |z'-w'|)} \int_{D(z',|w'-z'|)} J\phi^{-1} \\
&\simeq \left(\frac{1}{D(z', |z'-w'|)} \int_{D(z',|z'-w'|)} |D\phi^{-1}|\right)^2 \\
&\leq \left(1 + |D\phi^{-1}(z')|\right)^2 \leq (1+j)^2.
\end{aligned}$$

By the uniform continuity of ϕ, there exists $s_i \in (0, r_i)$ such that $|z - w| \leq s_i$ implies $|z' - w'| < \tilde{r}_i$ and

$$\frac{1}{C\,(1+j)^2} \leq \frac{|\phi(z) - \phi(w)|}{|z - w|} \leq C\,(1+j)^2.$$

whenever $z \in F_{i,j}$ and $|z - w| \leq s_i$, and where C is a constant that depends only on K. This means that ϕ is locally bilipschitz on $F_{i,j}$.

We now compute the analytic capacity of $\phi(E)$. First, E can be decomposed as

$$E = (E \cap (Z \cup Z')) \cup (E \setminus (Z \cup Z')).$$

Both $\phi(Z)$ and $\phi(Z')$ have vanishing analytic capacity, because they have zero length and $\gamma(F) \leq C\,\mathcal{H}^1(F)$ for any set $F \subset \mathbb{C}$. Thus, using the semiadditivity of γ, we get

$$\gamma(\phi(E)) \leq C\,\gamma(\phi(E) \setminus \phi(Z \cup Z')).$$

Now, notice that since U_i and W_i are monotonically decreasing,

$$\begin{aligned} E\setminus(Z\cup Z') &= E\setminus\big((\cap_i U_i)\cup(\cap_i W_i)\big)\\ &= E\setminus\left(\bigcap_i (U_i\cup W_i)\right)\\ &= \bigcup_i E\setminus(U_i\cup W_i). \end{aligned}$$

Also, it is easy to see that

$$E\setminus(U_i\cup W_i)\subset\bigcup_j F_{i,j}.$$

for each i. Therefore,

$$E\setminus(Z\cup Z')\subset\bigcup_{i,j}(E\cap F_{i,j}).$$

Now, since ϕ is locally bilipschitz on each $F_{i,j}$, by Corollary 4,

$$\gamma\left(\phi(E\setminus(Z\cup Z'\cup X\cup Y))\right)\le C\sum_{i,j}\gamma(\phi(E\cap F_{i,j}))=0$$

because $\gamma(E\cap F_{i,j})\le\gamma(E)=0$ for each i,j. This finishes the proof. □

Remark. *Theorem 1 also holds under the weaker assumption that ϕ is a K-quasiconformal mapping with compactly supported $\mu\in W^{1,p}(\mathbb{C})$ and $p>2K^2/(K^2+1)$, instead of $\mu\in W^{1,2}(\mathbb{C})$. The proof is very similar to the one of Theorem 1. One uses the same arguments and the fact that $\phi,\phi^{-1}\in W^{2,q}_{loc}$ for $q<q_0$, $\frac{1}{q_0}=\frac{1}{p}+\frac{K-1}{2K}$, by* [CFMOZ].

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