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PROPER HOLOMORPHIC DISKS IN THE COMPLEMENT OF VARIETIES IN $\mathbb{C}^2$

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Abstract. We prove that for any complete pluripolar set $X \subset \mathbb{C}^2$ (in particular for any analytic subset) there exists a proper holomorphic embedding $\varphi\colon \triangle \hookrightarrow \mathbb{C}^2$ of the open unit disk $\triangle \subset \mathbb{C}$ such that $\varphi(\triangle) \cap X = \varnothing$. It follows that the same holds true in $\mathbb{C}^n$ for any $n > 1$.

1. Introduction

In [4] the authors proved that there are proper holomorphic disks in $\mathbb{C}^2$ that avoid the set $\{zw = 0\}$. Furthermore they said that it would be interesting to know whether there could be such disks avoiding any finite set of complex lines. This question was solved in [3] where it was proved that in any Stein manifold there are proper holomorphic disks in the complement of any closed complete pluripolar set.

We show that there exist properly *embedded* disks satisfying this more general property:

Theorem 1. *Let X be any closed complete pluripolar subset of $\mathbb{C}^2$ and let $\triangle$ denote the unit disk in $\mathbb{C}$. Then there exists a proper holomorphic embedding $\varphi\colon \triangle \hookrightarrow \mathbb{C}^2$ such that $\varphi(\triangle) \cap X = \varnothing$.*

Since any analytic subset of a Stein manifold is a complete pluripolar set, this proves the existence of properly embedded disks in the complement of analytic subsets of $\mathbb{C}^2$. It should be remarked that the corresponding result for $\mathbb{C}$ instead of the disk is false since Kobayashi hyperbolicity of $\mathbb{C}^2 \setminus X$ is an obstruction. In fact it is known that any analytic subset X of $\mathbb{C}^2$ can be embedded in a different way $f\colon X \hookrightarrow \mathbb{C}^2$ into $\mathbb{C}^2$ such that the complement $\mathbb{C}^2 \setminus f(X)$ is Kobayashi hyperbolic (see [1, 2]). In such a situation there is not even a non-constant holomorphic map from $\mathbb{C}$ into that complement. An easier example is the following:

Example 2. Let X be the union of the following three lines in $\mathbb{C}^2_{z,w}$:

$$l_1 = \{w = 0\}, \qquad l_2 = \{w = 1\}, \qquad l_3 = \{z = w\}.$$

If a holomorphic map $\varphi\colon \mathbb{C} \to \mathbb{C}^2$ avoids l_1 and l_2 it is of the form $\varphi(\theta) = \big(f(\theta), c\big)$ since the projection $\pi_w \circ \varphi$ is a map from $\mathbb{C}$ into $\mathbb{C} \setminus \{0, 1\}$ and thus constant $= c$. For φ to be an embedding means that $f(\theta) = a\theta + b$, $a \neq 0$. Therefore the image of φ meets l_3.

Note that in this example $\mathbb{C}^2 \setminus X$ is not Kobayashi hyperbolic. The maps $\theta \mapsto (\exp\theta + c, c)$ provide non-degenerate holomorphic maps from $\mathbb{C}$ into $\mathbb{C}^2 \setminus X$ if $c \neq 0, 1$.

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Hyperbolicity of $\mathbb{C}^2 \setminus X$ is the reason why additional interpolation on discrete (or even finite) sets is not possible in general for embeddings as in our theorem.

2. Construction

Recall the following (simplified) definition from [6]:

Definition 3. *Given a smooth real curve $\Gamma = \{\gamma(t); t \in [0,\infty)$ or $t \in (-\infty,\infty)\}$ in $\mathbb{C}^2$ without self-intersection, we say that Γ has the* nice projection property *if there is a holomorphic automorphism $\alpha \in \mathrm{Aut}_{hol}(\mathbb{C}^2)$ of $\mathbb{C}^2$ such that, if $\beta(t) = \alpha(\gamma(t))$, $\Gamma' = \alpha(\Gamma)$, and $\pi_1 \colon \mathbb{C}^2 \to \mathbb{C}$ denotes the projection onto the first coordinate, then the following hold:*

(i) $\lim_{|t|\to\infty} |\pi_1(\beta(t))| = \infty$, *and*
(ii) *There is an $M \in \mathbb{R}$ such that for all $R \geq M$ we have that $\mathbb{C} \setminus (\pi_1(\Gamma') \cup \overline{\triangle}_R)$ does not contain any relatively compact connected components.*

Note that if a curve Γ has the nice projection property and $\Theta \in \mathrm{Aut}_{hol}(\mathbb{C}^2)$, then $\Theta(\Gamma)$ has the nice projection property. To see this let α be as in the definition and consider the composition $\alpha' := \alpha \circ \Theta^{-1}$.

The reason for introducing this notion is the following lemma from [7]:

Lemma 4. *Let Γ be a curve having the nice projection property, let $K \subset \mathbb{C}^2 \setminus \Gamma$ be a polynomially convex compact set, and let $\varepsilon > 0$. Then for any $R \in \mathbb{R}$ there exists a $\Phi \in \mathrm{Aut}_{hol}(\mathbb{C}^2)$ such that:*

(i) $\|\Phi(z) - z\| < \varepsilon$ *for all $z \in K$, and*
(ii) $\Phi(\Gamma) \subset \mathbb{C}^2 \setminus \mathbb{B}_R$.

Starting with an embedded surface in $\mathbb{C}^2$ with such a boundary Γ, one can apply the lemma to create a *proper* embedding of the surface by carrying the boundary inductively to infinity (see also [6]).

Let W denote the set $W := \overline{\triangle} \setminus \{1\}$ and let Γ denote the set $\Gamma := \{z \in W; |z| = 1\}$. We will say that a subset $\tilde{W} \subset W$ is $\mathfrak{b}$-*nice* if $\tilde{W}$ has a smooth boundary, and if there is a disk D centered at 1 such that $W \cap D = \tilde{W} \cap D$. We let $\tilde{\Gamma}$ denote $\partial\tilde{W} \cap W$. Note that if $\varphi(W)$ is an embedding such that $\varphi(\Gamma)$ has the nice projection property, then $\varphi(\tilde{\Gamma})$ has the nice projection property. This is because the two embedded curves are the same near infinity.

The following lemma will provide us with the inductive step in our construction:

Lemma 5. *Let X be a closed complete pluripolar subset of $\mathbb{C}^2$ and let $\varphi \colon W \hookrightarrow \mathbb{C}^2$ be a smooth embedding, holomorphic on the interior, such that the following hold for some integer N:*

(i) $\lim_{j\to\infty} \|\varphi(z_j)\| = \infty$ *for all $\{z_j\} \subset W$ with $z_j \to 1$,*
(ii) *$\varphi(\Gamma)$ has the nice projection property,*
(iii) $\varphi(W) \cap X \cap \overline{\mathbb{B}}_N = \varnothing$,
(iv) $\varphi(\Gamma) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_{N+1}$, *and*
(v) *$\varphi(W)$ intersects $\partial\mathbb{B}_N$ transversally.*

Let S denote the set $S := \varphi^{-1}(\varphi(W) \cap \overline{\mathbb{B}}_N)$, let V be a connected component of S and let $\varepsilon > 0$ (by (iv) *we have that $S \subset \triangle$). Then there exists a $\mathfrak{b}$-nice subset $\tilde{W}$ of W with $V \subset\subset \tilde{W}$, $\tilde{W} \cap (S \setminus V) = \varnothing$, and an embedding $\tilde{\varphi} \colon \tilde{W} \hookrightarrow \mathbb{C}^2$ (smooth, and holomorphic on the interior) such that:*

(a) $\lim_{j\to\infty} \|\tilde{\varphi}(z_j)\| = \infty$ *for all* $\{z_j\} \subset \tilde{W}$ *with* $z_j \to 1$,
(b) $\tilde{\varphi}(\tilde{\Gamma})$ *has the nice projection property,*
(c) $\tilde{\varphi}(\tilde{W}) \cap X \cap \overline{\mathbb{B}}_{N+1} = \varnothing$,
(d) $\tilde{\varphi}(\tilde{\Gamma}) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_{N+2}$,
(e) $\tilde{\varphi}(\tilde{W})$ *intersects* $\partial\mathbb{B}_{N+1}$ *transversally,*
(f) $\|\tilde{\varphi} - \varphi\|_V < \varepsilon$, *and*
(g) $\tilde{\varphi}(\tilde{W} \setminus V) \subset \mathbb{C}^2 \setminus \mathbb{B}_{N-\varepsilon}$.

Proof. It is not hard to see that (i) and (iv) implies that there exist positive real numbers $0 < r, \delta < 1$ such that the set $A_r := \{z \in W; |z| \geq r\}$ satisfies

$$(*) \qquad \varphi(A_r) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_{N+1+\delta}.$$

It follows that the set $P := \varphi^{-1}\big(\varphi(W) \cap (X \cap \overline{\mathbb{B}}_{N+1})\big)$ has a connected complement since the total intersection set $Z := \varphi^{-1}\big(\varphi(W) \cap X\big)$ is a complete pluripolar set in W and since $P \subset Z \cap (W \setminus A_r)$. The set S is clearly also contained in $W \setminus A_r$ and by (v) it is a finite disjoint union of smoothly bounded sets.

Now for an arbitrarily small neighborhood $\mathcal{N}$ of P we have that $\operatorname{dist}\big(\varphi\big((\overline{W \setminus A_r}) \setminus \mathcal{N}\big), X \cap \overline{\mathbb{B}}_{N+1}\big) > 0$. Since we also have $(*)$ we get that

$$\operatorname{dist}\big(\varphi(W \setminus \mathcal{N}), X \cap \overline{\mathbb{B}}_{N+1}\big) > 0.$$

This means that we may choose a $\mathfrak{b}$-nice domain $\tilde{W} \subset W \setminus \big(P \cup (S \setminus V)\big)$ such that $V \subset\subset \tilde{W}$ and such that $\operatorname{dist}\big(\varphi(\tilde{W}), X \cap \overline{\mathbb{B}}_{N+1}\big) > 0$. Note that $\varphi(\tilde{\Gamma}) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$ and that $\varphi(\tilde{\Gamma})$ has the nice projection property since $\tilde{\Gamma}$ is the same as Γ near 1.

Since $K := \overline{\mathbb{B}}_N \cup (X \cap \overline{\mathbb{B}}_{N+1})$ is polynomially convex (for the proof remark that the plurisubharmonic convex hull and the polynomial convex hull in $\mathbb{C}^n$ are the same and then use the same idea of proof as in [5, Lemma 2]) there is an open neighborhood Ω of K such that $\overline{\Omega}$ is polynomially convex and such that $\overline{\Omega} \cap \varphi(\tilde{\Gamma}) = \varnothing$. Thus by Lemma 4 (see also [6]) there exists a $\Phi \in \operatorname{Aut}_{hol}(\mathbb{C}^2)$ such that $\|\Phi - \operatorname{Id}\|_{\overline{\Omega}} < \varepsilon$ and such that $\Phi\big(\varphi(\tilde{\Gamma})\big) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_{N+2}$. By possibly having to decrease ε we may assume that $\Phi\big(\varphi(\tilde{W})\big) \cap (X \cap \overline{\mathbb{B}}_{N+1}) = \varnothing$ and so we may put $\tilde{\varphi} := \Phi \circ \varphi$. The conditions (a), (c), (d), and (f) are then immediate. Since Φ is an automorphism we have that $\tilde{\varphi}(\tilde{\Gamma})$ has the nice projection property and so we get (b). Condition (g) follows since we chose $\tilde{W}$ such that $\varphi(\tilde{W} \setminus V) \subset \mathbb{C}^2 \setminus \mathbb{B}_N$ and because $\|\Phi - \operatorname{Id}\|_{\overline{\Omega}} < \varepsilon$. Finally, consider the case where the intersection of $\tilde{\varphi}(\tilde{W})$ with any sphere $\partial\mathbb{B}_\rho$ is not transversal. In that case there is a point $z \in \tilde{W}$ with $\|\tilde{\varphi}(z)\| = \rho$ and $\langle \tilde{\varphi}(z), \mathrm{d}\tilde{\varphi}(z) \rangle = 0$. Hence the set of problematic points is analytic and thus discrete in $\tilde{W}$. So there exist ρ's arbitrarily close to 1 such that the intersection of $\tilde{\varphi}(\tilde{W})$ with $\partial\mathbb{B}_{\rho(N+1)}$ is transversal. Thus there are arbitrarily small linear perturbations of $\tilde{\varphi}$ that give us (e), and the other properties are clearly preserved. $\square$

Proof of Theorem 1. We will inductively construct an increasing sequence of simply connected sets in the unit disk along with a corresponding sequence of holomorphic embeddings.

To start the induction we embed the disk into $\mathbb{C}^2$ as follows: Start by letting $f_1 \colon \overline{\triangle} \hookrightarrow \mathbb{C}^2$ be the map $z \mapsto (3z, 0)$. We may of course assume that $f(\overline{\triangle}) \cap X = \varnothing$. For $\delta > 0$ let f_δ denote the rational map $f_\delta \colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $(z, w) \mapsto (z, w + \frac{\delta}{z-3})$. Let W be as in Lemma 5. If we put $\varphi_1 := f_\delta \circ f_1 \colon W \to \mathbb{C}^2$ for a small enough δ it is not hard to verify that all conditions in Lemma 5 are satisfied with $N = 1$ (to get the nice projection property, project to the w-axis). Let $U_1 := \varphi_1(W) \cap \overline{\mathbb{B}}_1$ — a set we may assume to be connected and (automatically) simply connected — and choose $\varepsilon_1 > 0$ such that if $\psi \colon \overline{U}_1 \to \mathbb{C}^2$ is any holomorphic map with $\|\psi - \varphi_1\|_{\overline{U}_1} < \varepsilon_1$ then ψ is an embedding and $\psi(\overline{U}_1) \cap X = \varnothing$. Choose ε_1 such that $\varepsilon_1 < 2^{-2}$.

Assume that we have constructed/chosen the following objects with the listed properties:

(1) Smoothly bounded simply connected domains $U_j \subset \triangle$ and $\mathfrak{b}$-nice domains W_j such that $U_1 \subset\subset U_2 \subset\subset \cdots \subset\subset U_N \subset\subset W_N \subset W_{N-1} \subset \cdots \subset W_1 \subset \overline{\triangle}$,
(2) Holomorphic embeddings $\varphi_j \colon W_j \hookrightarrow \mathbb{C}^2$ such that $\varphi_j(U_j) \subset \mathbb{B}_j$ and $\|\varphi_j(z)\| = j$ for all $z \in \partial U_j$,
(3) $\varphi_j(U_j \setminus U_{j-1}) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_{j-1-2^{-j}}$,
(4) $\varphi_j(\overline{U}_j) \cap X = \varnothing$, and
(5) The pair (φ_N, W_N) satisfies the condition in Lemma 5.

(Technically W_N is not the same as in Lemma 5, but by the Riemann Mapping Theorem it does not make a difference.) Additionally, assume that we have inductively chosen a sequence $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_N > 0$ with $\varepsilon_j < 2^{-j-1}$ and assured that

(6) If $\psi \colon \overline{U}_j \to \mathbb{C}^2$ is a holomorphic map with $\|\psi - \varphi_j\|_{\overline{U}_j} < \varepsilon_j$ then ψ is an embedding and $\psi(\overline{U}_j) \cap X = \varnothing$, and
(7) $\|\varphi_j - \varphi_{j-1}\|_{\overline{U}_{j-1}} < \varepsilon_{j-1} 2^{-j}$.

We now show how to get U_{N+1}, φ_{N+1}, W_{N+1}, and ε_{N+1} so that we have (1)–(7) with $N+1$ in place of N.

To apply Lemma 5 we let $\varphi := \varphi_N$, $W := W_N$, $V := \overline{U}_N$, and $\varepsilon := \varepsilon_N 2^{-N-1}$. Let φ_{N+1} and W_{N+1} denote the objects corresponding to $\tilde{\varphi}$ and $\tilde{W}$ in the conclusion of the lemma. We get immediately then that the pair (φ_{N+1}, W_{N+1}) satisfy the conditions in the lemma, i.e. we have (5). In particular this means that

$$\varphi_{N+1}(W_{N+1}) \cap (\overline{\mathbb{B}}_{N+1} \cap X) = \varnothing.$$

Since $\overline{U}_N = V$ by assumption we also get that $\|\varphi_{N+1} - \varphi_N\|_{\overline{U}_N} < \varepsilon = \varepsilon_N 2^{-N-1}$, i.e. we get (7). To define U_{N+1} we consider the set $S := \varphi_{N+1}^{-1}\big(\varphi_{N+1}(W_{N+1}) \cap \overline{\mathbb{B}}_{N+1}\big)$. Note first that $U_N \subset S$ since we just established (7). This means that we may define U_{N+1} to be the interior of the connected component of S that contains U_N. By (e) we have that U_{N+1} is smoothly bounded, and we get (1), (2), and (4). Since $U_{N+1} \setminus U_N \subset W_{N+1} \setminus U_N$ we get (3) from Lemma 5 (g). Finally we choose ε_{N+1} small enough to get (6).

To finish the proof we construct a sequence (U_j, φ_j) according to the above procedure. We define $U := \cup_{j=1}^{\infty} U_j$. Then U is an increasing union of simply connected domains and so U is itself simply connected. By the Riemann Mapping Theorem, U is conformally equivalent to the unit disk. We define a map $\psi\colon U \to \mathbb{C}^2$ by

$$\psi(z) = \lim_{j\to\infty} \varphi_j(z).$$

To see that this is well defined we consider a point $z \in \overline{U}_k$: For $m > n \geq k$ we have by (7) that

$$\|\varphi_m(z) - \varphi_n(z)\| \leq \sum_{i=n+1}^{m} \|\varphi_i(z) - \varphi_{i-1}(z)\| \leq \sum_{i=n+1}^{m} \varepsilon_{i-1} 2^{-i} < \varepsilon_n < 2^{-n-1}.$$

This shows that $\{\varphi_j(z)\}$ is a Cauchy sequence and so ψ defines a holomorphic map from U into $\mathbb{C}^2$. It also shows that

$$(**) \qquad \|\psi - \varphi_k\|_{\overline{U}_k} < \varepsilon_k,$$

so it follows from (6) that ψ is an embedding and that $\psi(U) \cap X = \varnothing$.

To see that ψ is proper, consider a point $z \in U_{k+1} \setminus \overline{U}_k$ for some k. By (3) we have that $\|\varphi_{k+1}(z)\| \geq k - 2^{-k-1}$, and so by $(**)$ we get that $\|\psi(z)\| \geq k - 2^{-k-1} - \varepsilon_k > k - 2^{-k}$. This means that $\|\psi(z)\| > k - 2^{-k}$ for all $z \in U \setminus U_k$, hence ψ is proper. $\square$

3. Concluding remarks

We note that if $n > 2$ the analogous result to our main theorem holds with $\mathbb{C}^n$ instead of $\mathbb{C}^2$ (if the codimension of X is bigger than one, this is an easy consequence of transversality).

Corollary 6. *Let X be any closed complete pluripolar subset of $\mathbb{C}^n$, $n \geq 2$, and let $\triangle$ denote the unit disk in $\mathbb{C}$. Then there exists a proper holomorphic embedding $\varphi\colon \triangle \hookrightarrow \mathbb{C}^n$ such that $\varphi(\triangle) \cap X = \varnothing$.*

The construction in the proof of Theorem 1 works in this case also. On the other hand one can simply take a 2-dimensional $\mathbb{C}$-linear subspace C of $\mathbb{C}^n$ such that $X \cap C \neq C$ and apply Theorem 1 to embed $\triangle$ into $C \setminus (X \cap C)$.

As pointed out in the introduction there is no way to add interpolation conditions at two or more points due to hyperbolicity obstructions, but it is a trivial addition to the proof in order to assure interpolation at one point, i.e. we can assure that the origin of the disk passes through a prescribed point in the complement of the pluripolar set. In fact, we believe that it is possible to to make the image of the embedding containing a prescribed discrete subset A of $\mathbb{C}^n$ (with $A \cap X = \varnothing$).

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