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A SUPPORT THEOREM FOR THE RADIATION FIELDS ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS

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1. Introduction

We prove a support theorem for the radiation fields on asymptotically Euclidean manifolds with metrics which are warped products near infinity. It generalizes to this setting the well known support theorem for the Radon transform in $\mathbb{R}^n$ [4].

The type of support theorem studied here has possible applications to the problem of reconstructing an asymptotically Euclidean manifold from the scattering matrix at all energies [13]. The reconstruction of an *asymptotically hyperbolic* manifold from the scattering matrix at all energies was studied in [14], and one would like to apply similar methods in the asymptotically Euclidean case.

An asymptotically Euclidean manifold [7, 10] is a C^∞ compact manifold X with boundary ∂X, which is equipped with a C^∞ Riemannian metric g that in a collar neighborhood of the boundary ∂X can be written as

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}, \quad \text{in} \quad [0,\epsilon) \times \partial X, \tag{1.1}$$

where x is a defining function of ∂X and h is a C^∞ one-parameter family of metrics on ∂X. The basic model of (X, g) is the radial compactification of the Euclidean $\mathbb{R}^n$, [10].

In this paper we will consider the class of metrics g which are warped product metrics near ∂X, see for example [9]. Near ∂X, g satisfies

$$g = \frac{dx^2}{x^4} + \psi(x)\frac{h_0}{x^2}, \quad x \in [0,\epsilon), \tag{1.2}$$

where $\psi \in C^\infty([0,\epsilon))$, $\psi(x) > 0$, $\psi(0) = 1$, and h_0 is a C^∞ metric on ∂X.

Let Δ_g be the Laplace operator on X, and let $u(t,z)$ satisfy the wave equation on $\mathbb{R} \times X$, i.e.

$$\begin{gathered}(D_t^2 - \Delta_g)u(t,z) = 0\\ u(0) = f_1, \quad D_t u(0) = f_2, \quad \text{with} \quad f_1, f_2 \in C_0^\infty(X).\end{gathered} \tag{1.3}$$

Here and throughout the paper, $C_0^\infty(X)$ denotes the space of $C^\infty(X)$ functions whose support does not intersect the boundary of X. The following is proved in [2, 3]:

Theorem 1.1. *Let (X, g) be an asymptotically Euclidean manifold. Let x be the boundary defining function for which* (1.1) *holds and let $z = (x, y)$, $y \in \partial X$, be the*

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corresponding boundary normal coordinates in a collar neighborhood of the boundary. If u satisfies (1.3) *then*

$$
\begin{aligned}
v_+(x,s,y) &= x^{-\frac{n-1}{2}} u(s+\frac{1}{x},x,y) \in C^\infty(\mathbb{R}_s \times [0,\epsilon)_x \times \partial X),\\
v_-(x,s,y) &= x^{-\frac{n-1}{2}} u(s-\frac{1}{x},x,y) \in C^\infty(\mathbb{R}_s \times [0,\epsilon)_x \times \partial X).
\end{aligned} \tag{1.4}
$$

Friedlander [2, 3] defined the forward and backward radiation fields respectively as

$$\mathcal{R}_+(f_1,f_2) = D_s v_+(0,s,y) \quad \text{and} \quad \mathcal{R}_-(f_1,f_2) = D_s v_-(0,s,y). \tag{1.5}$$

Lax and Phillips [8] proved that in $\mathbb{R}^n$ the forward (or backward) radiation field is the modified Radon transform, that is if

$$Rf(s,\omega) = \int_{\langle x,\omega\rangle = s} f(z)\, d\sigma, \quad d\sigma \text{ is the surface measure on } \langle x,\omega\rangle = s,$$

is the Radon transform, then

$$\mathcal{R}_+(f_1,f_2)(s,\omega) = |D_s|^{\frac{n-3}{2}} Rf_1(s,\omega) + |D_s|^{\frac{n-1}{2}} Rf_2(s,\omega).$$

Helgason's celebrated support theorem for Radon transforms [4] says that if f is a rapidly decaying function in $\mathbb{R}^n$ and $Rf(s,\omega) = 0$ for $|s| > |s_0|$, and every $\omega \in \mathbb{S}^{n-1}$, then f is supported in the ball of radius $|s_0|$. This is a result in control theory which says that the support of a function can be exactly controlled by the support of its Radon transform.

The following was proved in [12]:

Theorem 1.2. *Let* (X,g) *be an asymptotically Euclidean manifold and let* $f \in C_0^\infty(X)$. *If* $\mathcal{R}(0,f)(s,y) = 0$ *for* $s < -\frac{1}{x_0}$, $x_0 \in (0,\epsilon)$, *then* $f = 0$ *if* $x < x_0$.

This says that if there exists some $x_1 \in (0,\epsilon)$, such that $f(x,y) = 0$ for $x < x_1$, but $\mathcal{R}_+(0,f)(s,y) = 0$ for $s < -\frac{1}{x_0}$, and $x_0 > x_1$, then in fact $f(x,y) = 0$ if $x < x_0$. The purpose of this paper is to discuss the following

Question 1.1. *Let* (X,g) *be an asymptotically Euclidean manifold, and let* $\mathcal{S}(X)$ *be the space of functions in* $C^\infty(X)$ *which are smooth up to* ∂X *and vanish to infinite order at* ∂X. *If* $f \in \mathcal{S}(X)$ *and* $\mathcal{R}_+(0,f)(s,y) = 0$ *for* $s < -\frac{1}{x_0}$, $x_0 \in (0,\epsilon)$, *is it true that* $f = 0$ *if* $x < x_0$?

Due to the global nature of the operator $\mathcal{R}_+$, the analogue of Theorem 1.2 for non-compactly supported data, if true, should in principle, be harder to prove. For example, the Radon transform $Rf(s,\omega)$, consists of integrating f over a hyperplane with normal vector ω and whose distance to the origin is equal to $|s|$. Even if we restrict ourselves to very large $|s|$, this transform will take into account the values of f over large parts of the space. It is also important to point out that the assumption that f is rapidly decaying cannot be removed. We recall Helgason's example in $\mathbb{R}^2$, [4]. Let $z = x+iy \in \mathbb{C}$, and let $f(z) \in C^\infty(\mathbb{C})$ with $f(z) = z^{-k}$ if $|z| > R$, $k \in \mathbb{N}$, $k \geq 2$. Cauchy formula shows that the integral of f along any line which does not intersect the disk $|z| \leq R$ is equal to zero, so $Rf(s,\omega) = 0$ is $|s| > R$, but f is not compactly supported.

When (X,g) is *asymptotically hyperbolic* the general support theorem was proved in [14]. The main result of this paper is:

Theorem 1.3. *Let (X, g) be an asymptotically Euclidean manifold of dimension greater than or equal to 3. If g is a warped product metric near ∂X, $f \in \mathcal{S}(X)$, and $\mathcal{R}_+(0, f)(s, y) = 0$ for $s < -\frac{1}{x_0}$, then $f = 0$ if $x < x_0$.*

As in [12, 14] the proof of Theorem 1.3 is based on the unique continuation for solutions of PDEs. When the initial data is compactly supported, the proof Theorem 1.2 follows from an application of Hörmander's theorem [Theorem 28.3.4 of [5]] and Tataru's theorem [15].

2. Energy Estimates

The first step in the proof of Theorem 1.3 is to obtain estimates the solution to (1.3) up to $x = 0$ and $s = -\infty$.

The Laplacian with respect to the metric (1.2) is, in a neighborhood of ∂X, given by

$$\Delta_g = -x^4\partial_x^2 + (n-3)x^3\partial_x - x^4A(x)\partial_x + x^2\psi(x)^{-1}\Delta_{h_0},$$

where $A(x) = \partial_x \log\left(\psi(x)^{\frac{n-1}{2}}\right)$, and Δ_{h_0} is the Laplacian on ∂X with respect to the metric h_0. In what follows it is convenient to get rid of first order terms of Δ_g, so we will work with $Q = H(x)^{-1}\Delta_g H(x)$, with $H(x) = x^{\frac{n-1}{2}}(\psi(x))^{-\frac{n-1}{4}}$. We get that

$$Q = -(x^2\partial_x)^2 + x^2\phi(x)\Delta_{h_0} + x^2B(x), \text{ where}$$

$$\phi(x) = [\psi(x)]^{-1}, \quad B(x) = \frac{(n-1)(n-3)}{4} + xB_1(x), \quad B_1 \in C^\infty([0,\epsilon)).$$

If $F(x) = 1/H(x)$ and $w = F(x)u$, the wave equation (1.3) is translated into

$$\begin{gathered}(\partial_t^2 - (x^2\partial_x)^2 + x^2\phi(x)\Delta_{h_0} + x^2B(x))w = 0,\\ w(0) = F(x)f_1(x,y), \quad \partial_t w(0) = F(x)f_2(x,y).\end{gathered} \tag{2.1}$$

Instead of working with coordinates x and s, and the forward and backward radiation fields separately, it is better to work with the forward and backward radiation fields simultaneously. So we define

$$s_+ = t - \frac{1}{x} \text{ and } s_- = t + \frac{1}{x}, \quad x > 0. \tag{2.2}$$

Since we want to understand the behavior of w as $s_+ \sim -\infty$, and $s_- \sim \infty$, we compactify $\mathbb{R} \times X$ by setting

$$\mu = -\frac{1}{s_+} \text{ and } \nu = \frac{1}{s_-}.$$

We find that

$$w(\mu,\nu,y) = F\left(\frac{2\mu\nu}{\mu+\nu}\right)u\left(\frac{\mu-\nu}{\mu+\nu}, \frac{2\mu\nu}{\mu+\nu}, y\right) \tag{2.3}$$

satisfies

$$\begin{gathered}\left((\mu+\nu)^2\partial_\mu\partial_\nu - \phi\left(\frac{2\mu\nu}{\mu+\nu}\right)\Delta_{h_0}w - B\left(\frac{2\mu\nu}{\mu+\nu}\right)\right)w = 0,\\ w(\mu,\mu,y) = \widetilde{f}_1(\mu,y), \quad (\partial_\mu w)(\mu,\mu,y) = \widetilde{f}_2(\mu,y),\end{gathered} \tag{2.4}$$

where

$$\widetilde{f}_1(\mu, y) = F(\mu) f_1(\mu, y), \text{ and}$$

$$\widetilde{f}_2(\mu, y) = \frac{1}{2\mu^2} F(\mu) f_2(\mu, y) + \frac{1}{2} F'(\mu) f_1(\mu, y) + \frac{1}{2} F(\mu) f_1'(\mu, y).$$

We will work with the following weighted Sobolev spaces.

Definition 2.1. *Let $T > 0$ and $\Omega_T = (0,T) \times (0,T)$. We define*

$$H_j^s((0,T) \times \partial X) = \{f \in L^2((0,T) \times \partial X), \quad \mu^{-j} f \in H^s((0,T) \times \partial X)\},$$

$$H_j^s(\Omega_T \times \partial X) = \{f \in L^2(\Omega_T \times \partial X), \quad (\mu + \nu)^{-j} f \in H^s(\Omega_T \times \partial X)\}.$$

The norm of $f \in H_j^s((0,T) \times \partial X)$ is denoted by $||f||_{s,j}$, and if $f \in H_j^s(\Omega_T \times \partial X)$ its norm is denoted by $|||f|||_{s,j}$.

The next step is to prove

Theorem 2.2. *Let $\widetilde{f}_j(\mu, y) \in C^\infty([0,T] \times \partial X)$, $j = 1, 2$, be such that $\partial_\mu^k \widetilde{f}_j(0, y) = 0$, $k = 0, 1, 2, \ldots$. Let w satisfy (2.4) in $\Omega_T \times \partial X$. Then there exists $T_0 > 0$ such that w has a C^∞ extension up to $\overline{\Omega_T} \times \partial X$, $T \le T_0$.*

By finite speed of propagation, $w \in C^\infty(\Omega_T \times \partial X)$, we want to establish the regularity up to the closure $\overline{\Omega_T} \times \partial X$. We begin the proof with the following lemmas:

Lemma 2.3. *Let $0 < \mu \le b < T$, and $w \in C^\infty(\Omega_T)$. Then for $k \in \mathbb{N}$,*

(2.5)
$$\int_\mu^b (\mu + \nu)^{-1-k} |w(\mu, \nu)|^2 \, d\nu \le 2b\mu^{-1-k} |w(\mu, \mu)|^2 + 2b \int_\mu^b (\mu + \nu)^{-k} |\partial_\nu w(\mu, \nu)|^2 \, d\nu.$$

If $0 \le a \le \nu$ then

$$\int_a^\nu (\mu + \nu)^{-1} |w(\mu, \nu)|^2 \, d\mu \le 2|w(\nu, \nu)|^2 + 2\nu \int_a^\nu |\partial_\mu w(\mu, \nu)|^2 \, d\mu, \quad k \in \mathbb{N}. \tag{2.6}$$

Proof. For $\nu \ge \mu$ we write

$$w(\mu, \nu) = w(\mu, \mu) + \int_\mu^\nu \partial_s w(\mu, s) \, ds. \tag{2.7}$$

The Cauchy-Schwartz inequality gives

$$|w(\mu, \nu)|^2 \le 2|w(\mu, \mu)|^2 + 2(\nu - \mu) \int_\mu^\nu |\partial_s w(\mu, s)|^2 \, ds.$$

Since $\nu > 0$ and $s \le \nu$,

$$\int_\mu^b (\mu + \nu)^{-1-k} |w(\mu, \nu)|^2 \, d\nu \le$$

$$2(b - \mu)\mu^{-1-k} |w(\mu, \mu)|^2 + 2 \int_\mu^b \int_\mu^\nu (\mu + s)^{-k} |\partial_s w(\mu, s)|^2 \, ds d\nu \le$$

$$2b\mu^{-1-k} |w(\mu, \mu)|^2 + 2b \int_\mu^b (\mu + s)^{-k} |\partial_s w(\mu, s)|^2 \, ds.$$

The proof of (2.6) is very similar, and this ends the proof of the Lemma. □

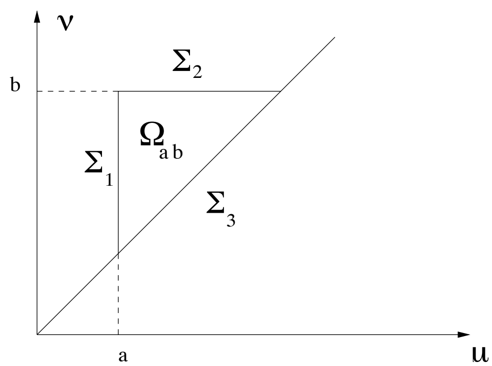

Figure 1. The region $\Omega_{a,b}$

Lemma 2.4. *Let $\Omega_{a,b} \subset \Omega_T$ be the region defined by*

$$\Omega_{a,b} = \{(\mu, \nu) : \nu \geq \mu \geq 0, \quad 0 \leq a \leq \mu, \quad \nu \leq b\}, \tag{2.8}$$

see Fig. 1. Let $F(\mu, \nu) \in L^1(\Omega_T)$ and let $a_0 \leq a \leq b$. Then

$$\int_{a_0}^{b} \left(\int_{\Omega_{ab}} F(\mu, \nu)\, d\mu d\nu \right) da = \int_{\Omega_{a_0 b}} (\mu - a_0) F(\mu, \nu)\, d\mu d\nu. \tag{2.9}$$

The proof is a straightforward application of Fubini's theorem.

Lemma 2.5. *If $v \in C^\infty(\Omega_T)$, and for some $m > 0$, $\int_0^T \mu^{-m-2} |v(\mu, \mu)|^2 d\mu$ and $\int_{\Omega_T} (\mu + \nu)^{-m} |(\partial_\nu - \partial_\mu) v|^2 d\mu d\nu$, are finite, then*

$$\begin{gathered} \int_0^T \int_0^T (\mu + \nu)^{-m-2} |v|^2 d\mu d\nu \leq \\ 2^{-m} \int_0^T \mu^{-m-1} |v(\mu, \mu)|^2\, d\mu + \frac{1}{4} \int_0^T \int_0^T (\mu + \nu)^{-m} |(\partial_\nu - \partial_\mu) v|^2 d\mu d\nu. \end{gathered} \tag{2.10}$$

Proof. To see this it is better to rotate the axes μ and ν and use coordinates

$$\begin{gathered} r = \mu + \nu, \quad \tau = \nu - \mu, \text{ so} \\ \mu = \frac{1}{2}(r - \tau), \quad \nu = \frac{1}{2}(r + \tau). \end{gathered} \tag{2.11}$$

In these coordinates $\partial_\nu - \partial_\mu = 2\partial_\tau$ and the diagonal $\mu = \nu$ becomes $\tau = 0$. Let Ω'_T be the image of Ω_T by this transformation and let $w(\tau, r) = v(\frac{r-\tau}{2}, \frac{r+\tau}{2})$. We divide Ω'_T in four regions

$$\begin{gathered} R_1 = \{0 \leq \tau \leq r,\ 0 \leq r \leq T\}, \quad R_2 = \{-r \leq \tau \leq 0, \quad 0 \leq r \leq T\} \\ R_3 = \{T \leq r \leq 2T, \tau + r \leq 2T\} \text{ and } R_4 = \{T \leq r \leq 2T, \tau - r \geq 2T\}. \end{gathered}$$

We begin with region R_1 and we write

$$w(\tau, r) = w(0, r) + \int_0^\tau \partial_s w(s, r)\, ds$$

Using the Cauchy-Schwartz inequality and the fact that $\tau/r \leq 1$, we have

$$r^{-m-2}|w(\tau,r)|^2 \leq 2r^{-m-2}|w(0,r)|^2 + 2\tau r^{-m-2}\int_0^\tau |\partial_s w(s,r)|^2\, ds \leq$$
$$2r^{-m-2}|w(0,r)|^2 + 2r^{-m-1}\int_0^\tau |\partial_s w(s,r)|^2\, ds$$

Therefore

$$\int_0^T\int_0^r r^{-m-2}|w(\tau,r)|^2\, d\tau dr \leq$$
$$2\int_0^T r^{-m-1}|w(0,r)|^2\, dr + 2\int_0^T\int_0^r r^{-m}|\partial_s w(s,r)|^2\, ds\, dr.$$

The other three regions of Ω_T' can be handled in the same way and we get

$$\int_{\Omega_T'} r^{-m-2}|w(\tau,r)|^2\, d\tau dr \leq 2\int_0^{2T} r^{-m-1}|w(0,r)|^2\, dr + 2\int_{\Omega_T'} r^{-m}|\partial_\tau w(\tau,r)|^2\, d\tau\, dr.$$

Translating this back into coordinates μ and ν we get (2.10). □

Now we prove uniform energy estimates up to $\{\mu = 0\}$, $\{\nu = 0\}$. When $n > 3$ one can choose T_0 such that if $\mu < T_0$ and $\nu < T_0$, $B > \frac{(n-1)(n-3)}{8}$. When $n = 3$, B is not necessarily positive. In this case it is convenient to work with an eigenfunction decomposition. Let ϕ_k, $k \in \mathbb{N}$, be the eigenfunctions of Δ_{h_0} and let $\{\lambda_k\}$, $k \in \mathbb{N}$, with $0 = \lambda_{k-1} \leq \lambda_k$ be the corresponding eigenvalues. Let w be the solution to (2.4) and let $w_k = \langle w, \phi_k\rangle_{L^2(\partial X)}$. When $n = 3$ and $k = 1$, w_1 satisfies

$$\left((\mu+\nu)^2\partial_\mu\partial_\nu - \frac{2\mu\nu}{\mu+\nu}B_1(\frac{2\mu\nu}{\mu+\nu})\right)w_1 = 0 \text{ in } (0,T)\times(0,T),$$
$$w(\mu,\mu) = q_1(\mu), \quad \partial_\mu w(\mu,\mu) = q_2(\mu), \tag{2.12}$$

with $q_1 = \langle \widetilde{f}_1, \phi_1\rangle$ and $q_2 = \langle \widetilde{f}_2, \phi_1\rangle$. It is clear that $||q_1||_{s,j} < \infty$ and $||q_2||_{s,j} < \infty$ for every s and j.

We have

Proposition 2.6. *Let $\Omega_T = (0,T)\times(0,T)$ and let $w_1 \in C^\infty(\Omega_T)$ satisfy (2.12) in Ω_T. Suppose that $||q_1||_{s,j} < \infty$ and $||q_2||_{s,j} < \infty$, for all s and j. Then there exists $T_0 > 0$ such that $w_1 \in C^\infty(\overline{\Omega_T})$, $T \leq T_0$.*

Proof. The proof relies on the following energy estimates:

Lemma 2.7. *Let $w_1 \in C^\infty(\Omega_T)$ satisfy (2.12) in Ω_T. There exist T_0 and a constant $C = C(T_0) > 0$ such that if $T \leq T_0$,*

$$\text{for any fixed } \mu \in [0,T], \quad \int_0^T |\partial_\nu w_1(\mu,\nu)|^2 d\nu \leq C(||q_1||^2_{1,\frac{1}{2}} + ||q_2||^2_{1,\frac{1}{2}}),$$
$$\text{and for any fixed } \nu \in [0,T], \quad \int_0^T |\partial_\mu w_1(\mu,\nu)|^2 d\mu \leq C(||q_1||^2_{1,\frac{1}{2}} + ||q_2||^2_{1,\frac{1}{2}}). \tag{2.13}$$

Proof. To prove this we first multiply (2.12) by $(\mu+\nu)^{-2}(\partial_\mu - \partial_\nu)w_1$. We obtain

$$\frac{1}{2}\partial_\nu\left((\partial_\mu w_1)^2 + 2\mu\nu(\mu+\nu)^{-3}B_1 w_1^2\right) - \frac{1}{2}\partial_\mu\left((\partial_\nu w_1)^2 + 2\mu\nu(\mu+\nu)^{-3}B_1 w_1^2\right) - \frac{1}{2}(\mu+\nu)^{-2}B_2 w_1^2 = 0,$$

where $B_2 = 4\frac{\mu-\nu}{\mu+\nu}B_1 + 4\frac{\mu\nu(\mu-\nu)}{(\mu+\nu)^2}B_1'(\frac{2\mu\nu}{\mu+\nu})$. After we integrate it in Ω_{ab} we obtain,

$$\frac{1}{2}\int_a^b \left[(\partial_\nu w_1)^2 + 2\mu\nu(\mu+\nu)^{-3}B_1|w_1|^2\right](a,\nu)d\nu - \frac{1}{2}\int_{\Omega_{ab}}(\mu+\nu)^{-2}B_2|w_1|^2 d\mu d\nu + \frac{1}{2}\int_a^b \left[(\partial_\mu w_1)^2 + 2\mu\nu(\mu+\nu)^{-3}B_1|w_1|^2\right](\mu,b)d\mu = \frac{1}{2\sqrt{2}}\int_a^b\left((q_1'-q_2)^2 + q_2^2 + \frac{1}{2}\mu^{-1}B_1(\mu)q_1^2\right)d\mu.$$

Here we used that $q_1'(\mu) = \partial_\mu w(\mu,\mu) + \partial_\nu w(\mu,\mu)$, and hence $\partial_\nu w(\mu,\mu) = q_1' - q_2$.

We apply Lemma 2.3 to show that we can pick T_0 such that for $T < T_0$,

$$\int_a^b [2\mu\nu(\mu+\nu)^{-3}|B_1||w_1|^2](a,\nu)d\nu \le 2a^{-1}bq_1^2(a) + \frac{1}{4}\int_a^b(\partial_\nu w_1)^2(a,\nu)d\nu, \text{ and}$$
$$\int_a^b [2\mu\nu(\mu+\nu)^{-3}|B_1||w_1|^2](\mu,b)d\mu \le 2b^{-1}q_1^2(b) + \frac{1}{4}\int_a^b(\partial_\nu w_1)^2(\mu,b)d\mu.$$

Therefore

(2.14)
$$\frac{1}{4}\int_a^b(\partial_\nu w_1)^2(a,\mu)d\nu + \frac{1}{4}\int_a^b(\partial_\mu w_1)^2(\mu,b)d\mu - \frac{1}{2}\int_{\Omega_{ab}}(\mu+\nu)^{-2}|B_2||w_1|^2 d\mu d\nu \le a^{-1}q_1^2(a) + b^{-1}q_1^2(b) + \frac{1}{2\sqrt{2}}\int_a^b\left((q_1'-q_2)^2 + q_2^2 + \frac{1}{2}\mu^{-1}|B_1(\mu)|q_1^2\right)d\mu.$$

If we drop the second integral from this inequality and integrate the remaining terms in a, with $a_0 \le a \le b$, and use (2.9) we get that

$$\frac{1}{4}\int_{\Omega_{a_0 b}}(\partial_\nu w_1)^2 d\nu - \int_{\Omega_{a_0 b}}\mu(\mu+\nu)^{-2}|B_2||w_1|^2 d\mu d\nu \le q_1^2(b) + \int_{a_0}^b \mu^{-1}q_1^2(\mu)d\mu + \frac{T}{2\sqrt{2}}\int_{a_0}^b\left((q_1'-q_2)^2 + q_2^2 + q_1^2\right)d\mu.$$

We can use Lemma 2.3 to show that

$$\int_{\Omega_{a_0 b}}(\mu+\nu)^{-1}|w_1|^2 d\mu d\nu \le 2T\int_{\Omega_{a_0 b}}(\partial_\nu w_1)^2 d\nu + 2T\int_{a_0}^b \mu^{-1}q_1^2(\mu)\,d\mu. \tag{2.15}$$

and therefore, if T_0 is small,

$$\frac{1}{8}\int_{\Omega_{a_0 b}}(\partial_\nu w_1)^2 d\nu \le q_1^2(b) + C(||q_1||^2_{1,\frac{1}{2}} + ||q_2||^2_{1,\frac{1}{2}}). \tag{2.16}$$

Now we substitute (2.16) and (2.15) into (2.14) and use that $||q_1||^2_{L^\infty} \leq T||q_1||^2_{1,0}$ and deduce that

$$\frac{1}{4}\int_a^b (\partial_\nu w_1)^2(a,\nu)d\nu + \frac{1}{4}\int_a^b (\partial_\mu w_1)^2(\mu,b)d\mu \leq C(||q_1||^2_{1,\frac{1}{2}} + ||q_2||^2_{1,\frac{1}{2}}).$$

By symmetry this estimate also holds in the region below the diagonal. This proves (2.13). □

Now we prove Proposition 2.6. For $\nu \geq \mu$ we write

$$w_1(\mu,\nu) = w_1(\mu,\mu) + \int_\mu^\nu \partial_s w_1(\mu,s)ds.$$

By the Cauchy-Schwartz inequality,

$$|w_1(\mu,\nu)|^2 \leq 2|w_1(\mu,\mu)|^2 + 2(\nu-\mu)\int_\mu^\nu |\partial_s w_1(\mu,s)|^2 ds,$$

and we deduce from (2.13) that if $(\mu,\nu) \in \Omega_T$, with $T < T_0$ small,

$$(\mu+\nu)^{-1}|w_1(\mu,\nu)|^2 \leq 2\mu^{-1}q_1^2(\mu) + 2\int_\mu^\nu |\partial_s w_1(\mu,s)|^2 ds \leq C(||q_1||^2_{1,\frac{1}{2}} + ||q_2||^2_{1,\frac{1}{2}}).$$

By symmetry with respect to the diagonal,

$$|w_1(\mu,\nu)| \leq C((||q_1||^2_{1,\frac{1}{2}} + ||q_2||^2_{1,\frac{1}{2}})^{\frac{1}{2}}(\mu+\nu)^{\frac{1}{2}}, \text{ in } \Omega_{T_0}.$$

From now on we will use $C(q_1,q_2)$ to denote a constant which depends on the norms $||q_1||_{s,j}$ and $||q_2||_{s,j}$ for some s and j. We then go back to equation (2.12) and deduce that if $(\mu,\nu) \in \Omega_{T_0}$,

$$|\partial_\mu\partial_\nu w_1| = |2\mu\nu(\mu+\nu)^{-3}B_1 w_1| \leq C(q_1,q_2)(\mu+\nu)^{-\frac{1}{2}},$$

thus

$$|\partial_\nu w_1(\mu,\nu)| \leq |\partial_\nu w_1(\nu,\nu)| + \int_\mu^\nu |\partial_s\partial_\nu w_1(s,\nu)|ds \leq$$

$$|\partial_\nu w_1(\nu,\nu)| + C(q_1,q_2)\int_\mu^\nu (\mu+s)^{-\frac{1}{2}}ds \leq C(q_1,q_2)(\mu+\nu)^{\frac{1}{2}}.$$

A similar argument shows that

$$|\partial_\mu w_1(\mu,\nu)| \leq C(q_1,q_q)(\mu+\nu)^{\frac{1}{2}}, \text{ if } \nu \geq \mu.$$

By symmetry these estimates hold below the diagonal. This implies that

$$|w_1(\mu,\nu)| \leq C(q_1,q_2)(\mu+\nu)^{\frac{3}{2}} \text{ in } \Omega_{T_0}.$$

We then differentiate equation (2.12) and find that

$$\partial_\mu\partial_\nu^2 w_1 = \left[\mu(2\mu-\nu)(\mu+\nu)^{-4}B_1 - 2\mu^3\nu(\mu+\nu)^{-5}B_1'\right]w_1 + 2\mu\nu(\mu+\nu)^{-1}B_1\partial_\nu w_1.$$

We deduce that

$$|\partial_\mu\partial_\nu^2 w_1| \leq C(q_1,q_2)(\mu+\nu)^{-\frac{1}{2}}$$

which implies that

$$|\partial_\mu\partial_\nu w_1(\mu,\nu)| \leq C(q_1,q_q)(\mu+\nu)^{\frac{1}{2}} \quad |\partial_\nu^2 w_1(\mu,\nu)| \leq C(q_1,q_q)(\mu+\nu)^{\frac{1}{2}}, \quad \text{in } \Omega_{T_0}.$$

A similar argument gives that

$$|\partial_\mu^2 w_1(\mu,\nu)| \leq C(q_1,q_q)(\mu+\nu)^{\frac{1}{2}}, \quad \text{in } \Omega_{T_0}.$$

hence we deduce that

$$|\partial_\mu w_1(\mu,\nu)| \leq C(q_1,q_q)(\mu+\nu)^{\frac{3}{2}}, \;\; |\partial_\nu w_1(\mu,\nu)| \leq C(q_1,q_q)(\mu+\nu)^{\frac{3}{2}}.$$

This argument can be repeated to show that

$$|\partial_\mu^j \partial_\nu^k w_1(\mu,\nu)| \leq C(q_1,q_2)(\mu+\nu)^m, \;\; j,k,m \in \mathbb{N} \text{ in } \Omega_{T_0}.$$

This implies the claim of Proposition 2.6. □

Next we study the non-degenerate cases, i.e either $n > 3$ or if $n = 3$, $\int_{\partial X} w = 0$, i.e w is orthogonal to the first eigenfunction.

Lemma 2.8. *Let $\Omega_T = (0,T) \times (0,T)$ and let $W \in C^\infty(\Omega_T \times \partial X)$ satisfy*

$$\left((\mu+\nu)^2\partial_\mu\partial_\nu - \phi(\frac{2\mu\nu}{\mu+\nu})\Delta_{h_0} - B(\frac{2\mu\nu}{\mu+\nu})\right) W = G(\mu,\nu,y) \text{ in } \Omega_T \times \partial X$$
$$W(\mu,\mu,y) = q_1(\mu,y), \;\; \partial_\mu W(\mu,\mu,y) = q_2(\mu,y). \tag{2.17}$$

If $n = 3$ we assume that $\int_{\partial X} W = 0$. Then there exists $T_0 > 0$, depending on B and ϕ, and a constant C depending on B and ϕ such that if $|||G|||_{0,2} < \infty$, and $||q_j||_{1,1} < \infty$, $j = 1,2$,

for any fixed $\mu \in [0,T]$

$$\int_0^T \int_{\partial X} \left(|\partial_\nu W|^2 + (\mu+\nu)^{-2}(|\nabla_{h_0} W|^2 + |W|^2\right)(\mu,\nu,y) d\nu d_{\mathrm{vol}_{h_0}} \leq$$
$$C(||q_1||_{1,1}^2 + ||q_2||_{1,1}^2 + |||G|||_{0,2}^2), \tag{2.18}$$

and for any fixed $\nu \in [0,T]$

$$\int_0^T \int_{\partial X} \left(|\partial_\mu W|^2 + (\mu+\nu)^{-2}(|\nabla_{h_0} W|^2 + |W|^2\right)(\mu,\nu,y) d\mu d_{\mathrm{vol}_{h_0}} \leq$$
$$C(||q_1||_{1,1}^2 + ||q_2||_{1,1}^2 + |||G|||_{0,2}^2).$$

If $||q_j||_{1,\frac{3}{2}} < \infty$, $j = 1,2$, we also have

(2.19)

$$\int_{\Omega_T \times \partial X} (\mu+\nu)^{-2}(\mu|\partial_\mu W|^2 + \nu|\partial_\nu W|^2) + (\mu+\nu)^{-3}(|\nabla_{h_0} W|^2 + |W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$CT(||q_1||_{1,\frac{3}{2}}^2 + ||q_2||_{1,\frac{3}{2}}^2 + |||G|||_{0,2}^2).$$

If $||q_j||_{1,2} < \infty$, $j = 1,2$ and $|||G|||_{0,\frac{5}{2}} < \infty$, then

$$\int_{\Omega_T \times \partial X} (\mu+\nu)^{-4}(|\nabla_{h_0} W|^2 + |W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq \tag{2.20}$$

$$C|||G|||_{0,\frac{5}{2}} + CT(||q_1||_{1,2}^2 + ||q_2||_{1,2}^2). \tag{2.21}$$

Proof. To prove these estimates we multiply (2.17) by $(\mu+\nu)^{-m}(\partial_\mu - \partial_\nu)W$, with $m \in \mathbb{R}_+$

$$\frac{1}{2}\partial_\nu\left[(\mu+\nu)^{2-m}|\partial_\mu W|^2 + (\mu+\nu)^{-m}(\phi|\nabla_{h_0}W|^2 + B|W|^2)\right] -$$
$$\frac{1}{2}\partial_\mu\left[(\mu+\nu)^{2-m}|\partial_\nu W|^2 + (\mu+\nu)^{-m}(\phi|\nabla_{h_0}W|^2 + B|W|^2)\right] +$$
$$\frac{m-2}{2}(\mu+\nu)^{1-m}((\partial_\mu W)^2 - (\partial_\nu W)^2) + \operatorname{div}_{h_0}\left[(\mu+\nu)^{-m}\phi\nabla_{h_0}W(\partial_\mu - \partial_\nu)W\right] +$$
$$(\nu-\mu)(\mu+\nu)^{-m-1}(\phi'|\nabla_{h_0}W|^2 + B'|W|^2) = (\mu+\nu)^{-m}(\partial_\mu - \partial_\nu)WG,$$

where ϕ' and B' denote the derivatives of ϕ and B.

We integrate this expression in Ω_{ab}, with $a \leq b \leq T$. Since $W(\mu,\mu,y) = q_1(\mu,y)$, $\partial_\mu q_1(\mu,y) = \partial_\mu W(\mu,\mu,y) + \partial_\nu W(\mu,\mu,y)$, and since $\partial_\mu W(\mu,\mu,y) = q_2$, we have $(\partial_\nu W)(\mu,\mu,y) = q_1'(\mu,y) - q_2(\mu,y)$. The divergence theorem gives

(2.22)
$$\frac{1}{2}\int_{\Sigma_1\times\partial X}\left[(\mu+\nu)^{2-m}|\partial_\nu W|^2 + (\mu+\nu)^{-m}(\phi|\nabla_{h_0}W|^2 + B|W|^2)\right]d\nu d\operatorname{vol}_{h_0} +$$
$$\frac{1}{2}\int_{\Sigma_2\times\partial X}\left[(\mu+\nu)^{2-m}|\partial_\mu W|^2 + (\mu+\nu)^{-m}(\phi|\nabla_{h_0}W|^2 + B|W|^2)\right]d\mu d\operatorname{vol}_{h_0} +$$
$$\frac{m-2}{2}\int_{\Omega_{ab}\times\partial X}(\mu+\nu)^{1-m}((\partial_\mu W)^2 - (\partial_\nu W)^2)d\mu d\nu d\operatorname{vol}_{h_0} +$$
$$\int_{\Omega_{ab}\times\partial X}(\nu-\mu)(\mu+\nu)^{-1-m}(\phi'|\nabla_{h_0}W|^2 + B'|W|^2)d\mu d\nu d\operatorname{vol}_{h_0} =$$
$$\int_{\Omega_{ab}\times\partial X}G(\mu+\nu)^{-m}(\partial_\mu - \partial_\nu)W d\mu d\nu d\operatorname{vol}_{h_0} +$$
$$\frac{1}{\sqrt{2}}\int_{\Sigma_3\times\partial X}(2\mu)^{-m}\left[4\mu^2((q_2)^2 + (q_1' - q_2)^2) + \left(\phi(\mu)|\nabla_{h_0}q_1|^2 + B(\mu)|q_1|^2\right)\right]d\mu d\operatorname{vol}_{h_0}$$

Here, as pictured in Fig. 1,

(2.23)
$$\Sigma_1 = \Sigma_1(a,b) = \{(a,\nu),\ a \leq \nu \leq b\},\ \ \Sigma_2 = \Sigma_2(a,b) = \{(\mu,b),\ 0 \leq a \leq \mu \leq b\} \text{ and}$$
$$\Sigma_3 = \Sigma_3(a,b) = \{(\mu,\nu),\ \mu = \nu,\ \ 0 \leq a \leq \mu \leq b\}.$$

When $n > 3$, and T is small, B is positive. When $n = 3$ this is not necessarily the case. So when $n > 3$ we guarantee that the first two integrals in (2.22) are positive. When $n = 3$, $B(0) = 0$. But we assumed that in this case $\int_{\partial X} W d\operatorname{vol}_{h_0} = 0$ and therefore $\int_{\partial X}|\nabla_{h_0}W|^2 d\operatorname{vol}_{h_0} \geq \lambda_2\int_{\partial X}|W|^2 d\operatorname{vol}_{h_0}$. Since $\lambda_2 > 0$, if T_0 is small the term in $B|W|^2$ can be absorbed by the term in $|\nabla_{h_0}W|^2$. So we may assume that the second integral is positive. We drop it from (2.22) and integrate the remaining terms in the variable a, which determines Σ_1, with $a_0 \leq a \leq b$. Using (2.9) we obtain, for small T,

(2.24)

$$\frac{1}{2}\int_{\Omega_{a_0b}\times\partial X}\left[(\mu+\nu)^{2-m}-(m-2)\mu(\mu+\nu)^{1-m}\right]|\partial_\nu W|^2 d\mu d\nu d\,\mathrm{vol}_{h_0}+$$
$$\frac{1}{2}\int_{\Omega_{a_0b}\times\partial X}(m-2)(\mu+\nu)^{1-m}(\mu-a_0)|\partial_\mu W|^2 d\mu d\nu d\,\mathrm{vol}_{h_0}$$
$$+\frac{1}{4}\int_{\Omega_{a_0b}\times\partial X}(\mu+\nu)^{-m}(\phi|\nabla_{h_0}W|^2+B|W|^2)d\mu d\nu d\,\mathrm{vol}_{h_0}\leq$$
$$T(||q_1||^2_{1,\frac{m}{2}}+||q_2||^2_{1,\frac{m}{2}})+\int_{\Omega_{a_0b}\times\partial X}G(\mu-a_0)(\mu+\nu)^{-m}(\partial_\mu-\partial_\nu)W d\mu d\nu d\,\mathrm{vol}_{h_0}\,.$$

When $m=2$ this gives

$$\int_{\Omega_{a_0b}\times\partial X}\left[|\partial_\nu W|^2+(\mu+\nu)^{-2}(\phi|\nabla_{h_0}W|^2+B|W|^2)\right]d\mu d\nu d\,\mathrm{vol}_{h_0}\leq$$

(2.25)

$$CT(||q_1||^2_{1,1}+||q_2||^2_{1,1})+CT\int_{\Omega_{a_0b}\times\partial X}|G|(\mu+\nu)^{-2}(\partial_\mu-\partial_\nu)W d\mu d\nu d\,\mathrm{vol}_{h_0}\,.$$

Now we drop the first integral in (2.22) and integrate the remaining terms in b with $a_0\leq b\leq T$. Using (2.25) we obtain

$$\int_{\Omega_{a_0T}\times\partial X}\left[|\partial_\mu W|^2+(\mu+\nu)^{-2}(\phi|\nabla_{h_0}W|^2+B|W|^2)\right]d\mu d\nu d\,\mathrm{vol}_{h_0}\leq$$
$$CT(||q_1||^2_{1,1}+||q_2||^2_{1,1})+CT\int_{\Omega_{a_0T}\times\partial X}|G|(\mu+\nu)^{-2}|(\partial_\mu-\partial_\nu)W|d\mu d\nu d\,\mathrm{vol}_{h_0}\,.$$

We now set $b=T$ in (2.25) and add these estimates

$$\int_{\Omega_{a_0T}\times\partial X}\left[|\partial_\nu W|^2+|\partial_\mu W|^2+(\mu+\nu)^{-2}(\phi|\nabla_{h_0}W|^2+B|W|^2)\right]d\mu d\nu d\,\mathrm{vol}_{h_0}\leq$$
$$CT(||q_1||^2_{1,1}+||q_2||^2_{1,1})+CT\int_{\Omega_{a_0T}\times\partial X}|G|(\mu+\nu)^{-2}(\partial_\mu-\partial_\nu)W d\mu d\nu d\,\mathrm{vol}_{h_0}\,.$$

By the Cauchy-Schwartz inequality

$$\int_{\Omega_{a_0T}\times\partial X}|G|(\mu+\nu)^{-2}(\partial_\mu-\partial_\nu)W d\mu d\nu d\,\mathrm{vol}_{h_0}\leq\frac{1}{2}|||G|||^2_{0,2}+$$
$$\int_{\Omega_{a_0T}\times\partial X}(|\partial_\mu W|^2+|\partial_\nu W|^2)d\mu d\nu d\,\mathrm{vol}_{h_0}$$

If one takes T small one gets

(2.26)

$$\int_{\Omega_{a_0T}\times\partial X}\left[|\partial_\nu W|^2+|\partial_\mu W|^2+(\mu+\nu)^{-2}(\phi|\nabla_{h_0}W|^2+B|W|^2)\right]d\mu d\nu d\,\mathrm{vol}_{h_0}\leq$$
$$CT(||q_1||^2_{1,1}+||q_2||^2_{1,1}+|||G|||^2_{0,2}).$$

Then (2.22) with $m = 2$ and (2.26) show that for any $a_0 \in [0, T]$, $\Sigma_1 = \Sigma_1(a_0, T)$ and $\Sigma_2 = \Sigma_2(a_0, T)$

$$\int_{\Sigma_1 \times \partial X} \left[|\partial_\nu W|^2 + (\mu+\nu)^{-2}\left(\phi|\nabla_{h_0} W|^2 + B|W|^2\right)\right] d\nu d\,\mathrm{vol}_{h_0} +$$
$$\int_{\Sigma_2 \times \partial X} \left[|\partial_\mu W|^2 + (\mu+\nu)^{-2}\left(\phi|\nabla_{h_0} W|^2 + B|W|^2\right)\right] d\mu d\,\mathrm{vol}_{h_0} \leq$$
$$CT(||q_1||^2_{1,1} + ||q_2||^2_{1,1} + |||G|||^2_{0,2}).$$

If one reverses the roles of (μ, ν) and uses the same argument, one finds that such an inequality also holds below the diagonal. Putting these estimates together proves (2.18).

When $m = 3$ equation (2.24) gives

$$\frac{1}{2}\int_{\Omega_{a_0 b} \times \partial X} (\mu+\nu)^{-2}[(\mu - a_0)|\partial_\mu W|^2 + \nu|\partial_\nu W|^2] d\mu d\nu d\,\mathrm{vol}_{h_0} +$$
$$\frac{1}{4}\int_{\Omega_{a_0 b} \times \partial X} (\mu+\nu)^{-3}(\phi|\nabla_{h_0} W|^2 + B|W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$T(||q_1||^2_{1,\frac{3}{2}} + ||q_2||^2_{1,\frac{3}{2}}) + \int_{\Omega_{a_0 b} \times \partial X} |G(\mu - a_0)(\mu+\nu)^{-3}(\partial_\mu - \partial_\nu)W| d\mu d\nu d\,\mathrm{vol}_{h_0}.$$

Using that $\nu \geq \mu$ and the Cauchy-Schwartz inequality we find that

$$\int_{\Omega_{a_0 b} \times \partial X} |G(\mu - a_0)(\mu+\nu)^{-3}(\partial_\mu - \partial_\nu)W| d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$2T|||G|||^2_{0,2} + \frac{1}{4}\int_{\Omega_{a_0 b}} (\mu - a_0)(\mu+\nu)^{-2}(|\partial_\mu W|^2 + |\partial_\nu W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$2T|||G|||^2_{0,2} + \frac{1}{4}\int_{\Omega_{a_0 b}} (\mu+\nu)^{-2}[(\mu - a_0)|\partial_\mu W|^2 + \nu|\partial_\nu W|^2] d\mu d\nu d\,\mathrm{vol}_{h_0}.$$

Thus we obtain

$$\int_{\Omega_{a_0 b} \times \partial X} (\mu+\nu)^{-2}[(\mu - a_0)|\partial_\mu W|^2 + \nu|\partial_\nu W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} +$$
$$\int_{\Omega_{a_0 b} \times \partial X} (\mu+\nu)^{-3}(\phi|\nabla_{h_0} W|^2 + B|W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$CT(||q_1||^2_{1,\frac{3}{2}} + ||q_2||^2_{1,\frac{3}{2}} + |||G|||^2_{0,2}).$$

Since C does not depend on a_0, letting $a_0 \to 0$ and $b = T$ gives (2.19) in the region Ω_{0T}. By symmetry this estimate holds below the diagonal and we obtain (2.19).

Now we consider the case $m = 4$. We deduce from (2.24) that for small T

(2.27)
$$\frac{1}{2}\int_{\Omega_{a_0 b}\times\partial X} \left[(\nu+\mu)^{-3}((\nu-\mu)|\partial_\nu W|^2 + 2(\mu-a_0)|\partial_\mu W|^2\right)\, d\mu d\nu d\,\mathrm{vol}_{h_0} +$$
$$\frac{1}{4}\int_{\Omega_{a_0 b}\times\partial X} (\mu+\nu)^{-4}(\phi|\nabla_{h_0} W|^2 + B|W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$T(||q_1||_{1,2}^2 + ||q_2||_{1,2}^2) + \int_{\Omega_{a_0 b}\times\partial X} |G|(\mu-a_0)(\mu+\nu)^{-4}|(\partial_\mu - \partial_\nu)W| d\mu d\nu d\,\mathrm{vol}_{h_0}\,.$$

Again we use the Cauchy-Schwartz inequality and get that

$$\int_{\Omega_{a_0 b}\times\partial X} |G|(\mu-a_0)(\mu+\nu)^{-4}||(\partial_\mu-\partial_\nu)W\, d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$2|||G|||_{0,\frac{5}{2}}^2 + \frac{1}{4}\int_{\Omega_{a_0 b}} \mu^2(\mu+\nu)^{-3}(|\partial_\mu W|^2 + |\partial_\nu W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$2|||G|||_{0,\frac{5}{2}}^2 + \frac{1}{4}\int_{\Omega_{a_0 b}} (\mu+\nu)^{-2}(\mu|\partial_\mu W|^2 + \nu|\partial_\nu W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0},$$

and from (2.19) we get that

$$\int_{\Omega_{a_0 b}\times\partial X} |G|\mu(\mu+\nu)^{-4}||(\partial_\mu-\partial_\nu)W| d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$C|||G|||_{0,\frac{5}{2}}^2 + CT(||q_1||_{1,2}^2 + ||q_2||_{1,2}^2 + |||G|||_{0,2}^2).$$

By substituting this into (2.27) we find that

$$\frac{1}{2}\int_{\Omega_{a_0 b}\times\partial X} \left[(\nu+\mu)^{-3}((\nu-\mu)|\partial_\nu W|^2 + 2(\mu-a_0)|\partial_\mu W|^2\right)\, d\mu d\nu d\,\mathrm{vol}_{h_0} +$$

(2.28)
$$\frac{1}{2}\int_{\Omega_{a_0 b}\times\partial X} (\mu+\nu)^{-4}(\phi|\nabla_{h_0} W|^2 + B|W|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq$$
$$T(||q_1||_{1,2}^2 + ||q_2||_{1,2}^2 + |||G|||_{0,2}^2) + |||G|||_{0,\frac{5}{2}}^2$$

Letting $a_0 \to 0$, and $b = T$ (2.28) gives (2.21) in Ω_{0T}. By symmetry this also holds below the diagonal, and we get (2.21). This proves Lemma 2.8. □

Now we are ready to prove Theorem 2.2.

Proof. We will concentrate on the cases not covered by Proposition 2.6. So we assume that if $n = 3$, $\int_{\partial X} w\, d\,\mathrm{vol}_{h_0} = 0$.

We can then apply Lemma 2.8 to equation (2.17) with $G = 0$. From (2.21) we get that the solution w to equation (2.4) satisfies

$$|||w|||_{0,2} \leq C(\widetilde{f}_1, \widetilde{f}_2). \tag{2.29}$$

Since Δ_{h_0} commutes with the equation, we also have

$$|||\Delta_{h_0}^k w|||_{0,2} \leq C(\widetilde{f}_1, \widetilde{f}_2), \quad k = 0, 1, ... \tag{2.30}$$

Now we differentiate equation (2.4) with respect to the vector field $\partial_\mu - \partial_\nu$. Let $W_j = (\partial_\mu - \partial_\nu)^j w$. We find that W_1 satisfies, in $\Omega_T \times \partial X$,

$$\left((\mu+\nu)^2\partial_\mu\partial_\nu - \phi(\frac{2\mu\nu}{\mu+\nu})\Delta_{h_0} - B(\frac{2\mu\nu}{\mu+\nu})\right) W_1 = 2(\nu-\mu)(\mu+\nu)^{-1}(\phi'\Delta_{h_0} - B')w$$
$$W_1(\mu,\mu,y) = Q_1(\mu,y), \quad \partial_\mu W_1(\mu,\mu,y) = Z_1(\mu,y)$$

where $Q_1, Z_1 \in C^\infty([0,T])$, depend on $\widetilde{f}_1$ and $\widetilde{f}_2$ and satisfy

$$\partial_\mu^k Q_1(0,y) = 0, \quad \partial_\mu^k Z_1(0,y) = 0, \qquad k = 0,1,2,...$$

Let $G_1 = 2(\nu-\mu)(\mu+\nu)^{-1}(\phi'\Delta_{h_0} w - B'w)$. In view of (2.29) and (2.30), $|||G_1|||_{0,2} \leq C(\widetilde{f}_1,\widetilde{f}_2)$. Then (2.19) in Lemma 2.8 guarantees that $|||W_1|||_{0,\frac{3}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2)$. So Lemma 2.5 implies that

$$|||w|||_{0,\frac{5}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2), \quad |||\Delta_{h_0}^k w|||_{0,\frac{5}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2), \quad k \in \mathbb{N}_0. \tag{2.31}$$

But then $|||G_1|||_{0,\frac{5}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2)$ and Lemma 2.8 guarantees that

$$|||W_1|||_{0,2} \leq C(\widetilde{f}_1,\widetilde{f}_2). \tag{2.32}$$

Then Lemma 2.5 gives that

$$|||w|||_{0,3} \leq C(\widetilde{f}_1,\widetilde{f}_2), \quad |||\Delta_{h_0}^k w|||_{0,3} \leq C(\widetilde{f}_1,\widetilde{f}_2), \quad k \in \mathbb{N}_0. \tag{2.33}$$

Now we differentiate (2.4) again with respect to $\partial_\mu - \partial_\nu$. We find that W_2 satisfies the following equation in $\Omega_T \times \partial X$.

$$\begin{gathered}\left((\mu+\nu)^2\partial_\mu\partial_\nu - \phi(\frac{2\mu\nu}{\mu+\nu})\Delta_{h_0} - B(\frac{2\mu\nu}{\mu+\nu})\right) W_2 = \\ 2(\nu-\mu)(\mu+\nu)^{-1}(\phi'\Delta_{h_0} - B')W_1 + \\ \left[-4(\mu+\nu)^{-1}(\phi'\Delta_{h_0} - B') + 4(\mu-\nu)^2(\mu+\nu)^{-2}(\phi''\Delta_{h_0} - B'')\right] w \\ W_2(\mu,\mu,y) = Q_2(\mu,y), \quad \partial_\mu W_1(\mu,\mu,y) = Z_2(\mu,y),\end{gathered} \tag{2.34}$$

where Q_2 and Z_2 vanish to infinite order at $\mu = 0$. Let G_2 be the right hand side of (2.34). In view of (2.32) and (2.33), $|||G_2|||_{0,2} \leq C(\widetilde{f}_1,\widetilde{f}_2)$, but then it follows from (2.19) in Lemma 2.8 that $|||W_2|||_{0,\frac{3}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2)$. Then Lemma 2.5 implies that $|||W_1|||_{0,\frac{5}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2)$, and so $|||w|||_{0,\frac{7}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2)$. This implies that in fact $|||G_2|||_{0,\frac{5}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2)$, and therefore $|||W_2|||_{0,2} \leq C(\widetilde{f}_1,\widetilde{f}_2)$. By Lemma 2.5 $|||W_1|||_{0,3} \leq C(\widetilde{f}_1,\widetilde{f}_2)$. Now we differentiate the equation again and repeat the argument. We find that

$$|||(\partial_\mu - \partial_\nu)^j \Delta_{h_0}^k w|||_{0,\frac{5}{2}} \leq C(\widetilde{f}_1,\widetilde{f}_2), \quad j,k \in \mathbb{N}_0$$

and by Lemma 2.5 we conclude that

$$|||w|||_{0,j} \leq C(\widetilde{f}_1,\widetilde{f}_2), \quad |||\Delta_{h_0}^k w|||_{0,j} \leq C(\widetilde{f}_1,\widetilde{f}_2), \quad j,k \in \mathbb{N}_0 \tag{2.35}$$

We still need to get regularity in ∂_μ and ∂_ν separately. We then go back to equation (2.22) and apply it to the solution w to (2.4). In this case $G = 0$ and we obtain

(2.36)
$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_1\times\partial X} \left[(\mu+\nu)^{2-m}|\partial_\nu w|^2 + (\mu+\nu)^{-m}(\phi|\nabla_{h_0} w|^2 + B|w|^2)\right] d\nu d\,\mathrm{vol}_{h_0} + \\
&\frac{1}{2}\int_{\Sigma_2\times\partial X} \left[(\mu+\nu)^{2-m}|\partial_\mu w|^2 + (\mu+\nu)^{-m}(\phi|\nabla_{h_0} w|^2 + B|W|^2)\right] d\mu d\,\mathrm{vol}_{h_0} + \\
&\frac{m-2}{2}\int_{\Omega_{ab}\times\partial X} (\mu+\nu)^{1-m}((\partial_\mu w)^2 - (\partial_\nu w)^2) d\mu d\nu d\,\mathrm{vol}_{h_0} + \\
&\int_{\Omega_{ab}\times\partial X} (\nu-\mu)(\mu+\nu)^{-1-m}(\phi'|\nabla_{h_0} w|^2 + B'|w|^2) d\mu d\nu d\,\mathrm{vol}_{h_0} = \\
&\int_{\Sigma_3\times\partial X} \frac{(2\mu)^{-m}}{\sqrt{2}}\left[4\mu^2((\widetilde{f}_2)^2 + (\widetilde{f}_2 - \widetilde{f}_1')^2) + (\phi(\mu)|\nabla_{h_0}\widetilde{f}_1|^2 + B(\mu)|\widetilde{f}_1|^2)\right] d\mu d\,\mathrm{vol}_{h_0}
\end{aligned}$$

The term

$$\begin{aligned}
&\int_{\Omega_{ab}\times\partial X} (\mu+\nu)^{1-m}((\partial_\mu w)^2 - (\partial_\nu w)^2) d\mu d\nu d\,\mathrm{vol}_{h_0} = \\
&\int_{\Omega_{ab}\times\partial X} (\mu+\nu)^{1-m}((\partial_\mu - \partial_\nu)w)((\partial_\mu + \partial_\nu)w) d\mu d\nu d\,\mathrm{vol}_{h_0} \leq \\
&2\int_{\Omega_{ab}\times\partial X} (\mu+\nu)^{2-2m}|(\partial_\mu - \partial_\nu)w|^2 d\mu d\nu d\,\mathrm{vol}_{h_0} + \\
&2\int_{\Omega_{ab}\times\partial X} |(\partial_\mu + \partial_\nu)w|^2 d\mu d\nu d\,\mathrm{vol}_{h_0} \leq C(\widetilde{f}_1, \widetilde{f}_2),
\end{aligned}$$

and from (2.35) we have that

$$\left|\int_{\Omega_{ab}\times\partial X} (\nu-\mu)(\mu+\nu)^{-1-m}(\phi'|\nabla_{h_0} w|^2 + B'|w|^2) d\mu d\nu d\,\mathrm{vol}_{h_0}\right| \leq C(\widetilde{f}_1, \widetilde{f}_2).$$

Therefore we conclude that

(2.37)
$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_1\times\partial X} (\mu+\nu)^{-m}\left[(\mu+\nu)^2|\partial_\nu w|^2 + (\phi|\nabla_{h_0} w|^2 + B|w|^2)\right] d\nu d\,\mathrm{vol}_{h_0} + \\
&\frac{1}{2}\int_{\Sigma_2\times\partial X} (\mu+\nu)^{-m}\left[(\mu+\nu)^2|\partial_\mu w|^2 + (\phi|\nabla_{h_0} w|^2 + B|W|^2)\right] d\mu d\,\mathrm{vol}_{h_0} \leq C(\widetilde{f}_1, \widetilde{f}_2).
\end{aligned}$$

Now we write, for $m > 2$,

$$\begin{aligned}
&|(\mu+\nu)^{1-m} w(\mu,\nu)| \leq 2\mu^{1-m}|w(\mu,\mu)| + 2(\mu+\nu)^{2-m}\int_\mu^\nu |\partial_s w(\mu,s)|^2 ds \leq \\
&2\mu^{1-m}|w(\mu,\mu)| + 2\int_\mu^\nu (\mu+s)^{2-m}|\partial_s w(\mu,s)|^2 ds \leq C(\widetilde{f}_1, \widetilde{f}_2).
\end{aligned} \tag{2.38}$$

Now we argue as we did in the case of w_1, and conclude that $w \in C^\infty(\overline{\Omega_T} \times \partial X)$. This ends the proof of Theorem 2.2. □

To prove Theorem 1.3 we need the following:

Lemma 2.9. *Suppose that $f \in \mathcal{S}(X)$ and $\mathcal{R}_+(0,f)(s,y) = 0$ for $s < s_0 << 0$. Let u be the solution to* (2.1) *and let w be the function defined by u in* (2.3)*. Then*

$$\partial_\mu^k w(0,\nu,y) = 0 \text{ if } \nu \leq -\frac{1}{s_0}, \quad \partial_\nu^k w(\mu,0,y) = 0 \text{ if } \mu \leq -\frac{1}{s_0}, \quad k \in \mathbb{N}_0. \tag{2.39}$$

Proof. Suppose $\mathcal{R}_+(0,f)(s,y) = 0$ for $s < s_0$. Since the initial data is of the form $(0,f)$, the solution u of (2.1) is odd in time, and therefore $\mathcal{R}_-(0,f)(s,y) = 0$ if $s > -s_0$. Since w is smooth up to $\{\mu = 0\}$ and $\{\nu = 0\}$, this means that

$$\partial_\nu w(0,\nu,y) = 0 \quad \text{if } \nu \leq -\frac{1}{s_0}, \text{ and } \partial_\mu w(\mu,0,y) = 0 \quad \text{if } \mu \leq -\frac{1}{s_0}.$$

We know from (2.38) that w vanishes to infinite order at $\{\mu = \nu = 0\}$. Therefore

$$w(0,\nu,y) = 0 \quad \text{if } \nu \leq -\frac{1}{s_0}, \quad w(\mu,0,y) = 0 \quad \text{if } \mu \leq -\frac{1}{s_0}.$$

From the equation (2.4), we conclude that

$$\begin{aligned} &\text{if } \mu \neq 0, \;\; \partial_\mu \partial_\nu w(\mu,0,y) = \mu^{-2}(\phi(0)\Delta_{h_0} + B(0))w(\mu,0) = 0 \\ &\text{if } \nu \neq 0, \;\; \partial_\mu \partial_\nu w(0,\nu,y) = \nu^{-2}(\phi(0)\Delta_{h_0} + B(0))w(0,\nu) = 0. \end{aligned}$$

Using that w vanishes to infinite order at $\{\mu = \nu = 0\}$, we deduce that

$$\partial_\mu w(0,\nu,y) = 0 \quad \text{if } \nu \leq -\frac{1}{s_0}, \quad \partial_\nu w(\mu,0,y) = 0 \quad \text{if } \mu \leq -\frac{1}{s_0}.$$

Now we differentiate equation (2.4) with respect to μ and we find that

$$\begin{aligned} (\mu+\nu)^2 \partial_\mu^2 \partial_\nu w + \phi \Delta_{h_0} \partial_\mu w + B \partial_\mu w + 2(\mu+\nu)\partial_\mu \partial_\nu w + \\ \nu^2(\mu+\nu)^{-2}\Delta_{h_0} w + \nu^2(\mu+\nu)^{-2} B' w = 0 \end{aligned}$$

Hence, if $\nu \neq 0$, $\partial_\mu^2 \partial_\nu w(0,\nu,y) = 0$. Since w is smooth, and vanishes to infinite order at $\mu = \nu = 0$, it follows that $\partial_\mu^2 w(0,\nu,y) = 0$ if $\nu \leq -\frac{1}{s_0}$. By symmetry, $\partial_\nu^2 w(\mu,0,y) = 0$ if $\mu \leq -\frac{1}{s_0}$. Since away from $\{\mu = \nu = 0\}$ the coefficients of (2.4) are smooth, we can repeat the argument to prove that all derivatives of w vanish at $\{\mu = 0, \;\; \nu \leq -\frac{1}{s_0}\} \cup \{\nu = 0, \;\; \mu \leq -\frac{1}{s_0}\}$. □

3. Carleman Estimates

Let w be a solution to (2.4) with $\widetilde{f}_1 = 0$ and $\widetilde{f}_2 = \widetilde{f} = \frac{1}{2\mu^2} F(\mu) f(\mu,y)$, and let

$$w_k(\mu,\nu) = \langle w(\mu,\nu,y), \phi_k(y) \rangle_{L^2(\partial X, d\,\mathrm{vol}_{h_0})}, \tag{3.1}$$

where ϕ_k, $k = 1,2,...$ are the eigenfunctions of Δ_{h_0} with eigenvalue λ_k, where $0 = \lambda_1 < \lambda_2$, and $\lambda_k \leq \lambda_{k+1}$, for $k > 1$, and $\lambda_k \to \infty$. Then $w_k \in C^\infty([0,T] \times [0,T])$ satisfies

$$\begin{aligned} &\left((\mu+\nu)^2 \partial_\mu \partial_\nu + F_k(\mu,\nu)\right) w_k = 0 \text{ in } [0,T] \times [0,T], \\ &w_k(\mu,\mu) = 0, \;\; \partial_\mu w_k(\mu,\mu) = \widetilde{f}_k(\mu) = \langle \widetilde{f}, \phi_k \rangle, \;\; j = 1,2. \end{aligned}$$

Here $F_k(\mu,\nu) = \lambda_k \phi(\mu,\nu) + B(\mu,\nu)$.

By assumption w vanishes to infinite order at $\{\mu = 0\}$ and $\{\nu = 0\}$. Thus so does $w_k(\mu,\nu)$, $k = 1,2,...$ We will prove that under these assumptions there exists $\epsilon > 0$, independent of k such that $w_k(\mu,\nu)=0$ if $\mu\leq\epsilon$ and $\nu\leq\epsilon$. In particular $\widetilde{f}_k(\mu)=0$, $k=0,1,2,...$ if $\mu\leq\epsilon$, and hence $f(x)=0$ if $x\leq\epsilon$.

It is convenient to work with coordinates r and τ defined in (2.11). The main ingredient of the proof is

Lemma 3.1. *Let U be a neighborhood of $(0,0)$. Let*

$$P_k=r^2\partial_r^2-r^2\partial_\tau^2+F_k(\tau,r),$$

where $F_k(r,\tau)=\lambda_k\phi(\frac{r^2-\tau^2}{2r})-B(\frac{r^2-\tau^2}{2r})$. Then there exists $C>0$, independent of k, and $\gamma_0=\gamma_0(k)$ such that for every $\gamma>\gamma_0$, and every $u\in C_0^\infty(U)$, which is supported in $\{(\tau,r):\ \ r\geq 0$ and $-r\leq\tau\leq r\}$,

$$\|r^{-\gamma-2}P_ku\|^2\geq C\left(\gamma^2\|r^{-1}\partial_r r^{-\gamma}u\|^2+\gamma^2\|r^{-1-\gamma}\partial_\tau u\|^2+\gamma^4\|r^{-\gamma-2}u\|^2\right). \tag{3.2}$$

Here $||\cdot||$ denotes the $L^2(U)$ norm.

Proof. Let $P_{\gamma,k}=r^{-\gamma-2}P_kr^\gamma$. Hence

$$P_{\gamma,k}=r^{-2}P_k+2\gamma r^{-1}\partial_r+\gamma(\gamma-1)r^{-2}.$$

The support of u is contained in $\{r\geq 0\}$ and $\{r\geq\tau\geq -r\}$. So we write

$$u=r^\gamma v,\ \text{ and } r^{-\gamma-2}P_ku=P_{\gamma,k}v.$$

We have

$$\begin{gathered}\|P_{\gamma,k}v\|^2=\|r^{-2}P_kv\|^2+4\gamma^2\|r^{-1}\partial_rv\|^2+\gamma^2(\gamma-1)^2\|r^{-2}v\|^2+\\ 4\gamma\langle r^{-2}P_kv,r^{-1}\partial_rv\rangle+2\gamma(\gamma-1)\langle r^{-2}P_kv,r^{-2}v\rangle+4\gamma^2(\gamma-1)\langle r^{-1}\partial_rv,r^{-2}v\rangle.\end{gathered} \tag{3.3}$$

Now we integrate by parts to compute the inner products

$$\langle r^{-2}P_kv,r^{-1}\partial_r\rangle,\ \ \langle r^{-2}P_kv,r^{-2}v\rangle \text{ and } \langle r^{-1}\partial_rv,r^{-2}v\rangle.$$

We begin with

$$\langle r^{-1}\partial_rv,r^{-2}v\rangle=\frac{1}{2}\int r^{-3}\partial_rv^2drd\tau=\frac{3}{2}\|r^{-2}v\|^2.$$

$\langle r^{-2}P_kv,r^{-1}\partial_rv\rangle$ has two terms:

$$\begin{gathered}\langle(\partial_r^2-\partial_\tau^2)v,r^{-1}\partial_rv\rangle=\frac{1}{2}\int r^{-1}\partial_r(\partial_rv)^2drd\tau-\int r^{-1}\partial_\tau^2\partial_rvdrd\tau=\\ \frac{1}{2}\|r^{-1}\partial_rv\|^2+\frac{1}{2}\|r^{-1}\partial_\tau v\|^2,\end{gathered}$$

and

$$\begin{gathered}\langle r^{-2}F_kv,r^{-1}\partial_rv\rangle=\frac{1}{2}\int r^{-3}F_k\partial_rv^2drd\tau=\frac{1}{2}\int(3r^{-4}F_k-r^{-3}\partial_rF_k)v^2drd\tau\geq\\ -C(\lambda_k+1)\|r^{-2}v\|^2.\end{gathered}$$

$\langle r^{-2}P_kv,r^{-2}v\rangle$ also has two terms:

$$\langle(\partial_r^2-\partial_\tau^2)v,r^{-2}v\rangle=3\|r^{-2}v\|^2-\|r^{-1}\partial_rv\|^2+\|r^{-1}\partial_\tau v\|^2.$$

and

$$\langle r^{-2}F_kv,r^{-2}v\rangle\geq -C(\lambda_k+1)\|r^{-2}v\|^2.$$

Putting these estimates together we find that

$$||P_{\gamma,k}v||^2 \geq (2\gamma^2+4\gamma)||r^{-1}\partial_r v||^2 + 2(\gamma^2+\gamma)||r^{-1}\partial_\tau v||^2 +$$
$$(\gamma^2(\gamma-1)^2 + 6\gamma^2(\gamma-1) + 6\gamma(\gamma-1) - 2C(\lambda_k+1)(\gamma^2+\gamma))||r^{-2}v||^2.$$

Thus, if $\gamma_0 >> C(\lambda_k+1)$,

$$||P_{\gamma,k}v||^2 \geq C(\gamma^2||r^{-1}\partial_r v||^2 + \gamma^2||r^{-1}\partial_\tau v||^2 + \gamma^4||r^{-2}v||^2).$$

Since $v = r^{-\gamma}u$ and $P_{k,\gamma} = r^{-\gamma-2}P_k r^\gamma$ this implies (3.2). □

4. Proof of Theorem 1.3

Since $\mathcal{R}(0,f)(s,y) = 0$ for $s \leq s_0 = -\frac{1}{x_0}$, we know from Lemma 2.9 that the solution w to (2.4) vanishes to infinite order at the wedge $\{\mu = 0, 0 \leq \nu \leq x_0\} \cup \{\nu = 0, 0 \leq \mu \leq x_0\}$. Let w_k be as defined in (3.1). Then w_k also vanishes at this wedge, and after changing coordinates (μ,ν) to (r,τ), we can apply Lemma 3.1 to $\chi(r)w_k$, with $\chi \in C^\infty(-T,T)$, $T < x_0$, such that $\chi(r) = 1$ if $|r| < \frac{T}{4}$ and $\chi(r) = 0$ if $|r| > \frac{T}{2}$. As a consequence of (3.2),

$$||r^{-\gamma-2}P_k\chi w_k||^2 \geq C\gamma^4||r^{-\gamma-2}\chi w_k||^2.$$

Since $P_k w_k = 0$,

$$P_k\chi(r)w_k = r^2\chi''(r)w_k + 2r^2\chi'(r)\partial_r w_k,$$

and therefore $P_k\chi(r)w_k$ is supported in $\frac{T}{4} \leq r \leq \frac{T}{2}$. Thus

$$||r^{-\gamma-2}P_k\chi w_k||^2 \leq C(\chi,w_k)\left(\frac{T}{4}\right)^{-2\gamma-4}.$$

But since $\chi(r) = 1$ if $r < T/4$,

$$||r^{-\gamma-2}\chi w_k||^2 \geq ||r^{-\gamma-2}w_k||^2_{L^2(B(0,\frac{T}{4}))} \geq \left(\frac{T}{4}\right)^{-2\gamma-4}||w_k||^2_{L^2(B(0,\frac{T}{4}))},$$

where $||w_k||_{L^2(B(0,\frac{T}{4}))}$ is the L^2 norm of w_k on the ball centered at $(0,0)$ and radius $\frac{T}{4}$. So we have

$$C(\chi,w_k) \geq C\gamma^4||w_k||^2_{L^2(B(0,\frac{T}{4}))}, \quad \gamma \geq \gamma_0.$$

We let $\gamma \to \infty$ and get that $||w_k||_{L^2(B(0,\frac{T}{4}))} = 0$. Since T does not depend on k we conclude that $w = 0$ in $B(0,\frac{T}{4})$. In particular this shows that $f(x) = 0$ if $x \leq \frac{T}{4}$. Therefore f is compactly supported. Then Theorem 1.3 follows from Theorem 1.2.

Notice that the weight in the Carleman estimate depends on the eigenvalue λ_k. The bigger the eigenvalue, the larger the parameter γ has to be. □

It is useful to understand this construction in $\mathbb{R}^n$. If one views $r = \mu+\nu$ as the distance in polar coordinates and $\tau = \mu-\nu$ as time, equation (2.4) reduces to the wave equation in $\mathbb{R}^n$, and this shows unique continuation across the surface of the light cone. It shows that a C^∞ solution of the wave equation which vanishes inside the light cone must be equal to zero near the tip of the cone . This had been established by Friedlander [1] using properties of the analytic wave front set of the solution of the wave equation.

If g is not warped this proof fails because the eigenfunctions will depend on x. In this case perhaps one can hope to adapt the methods of Tataru [15], Hörmander [6] and Robbiano-Zuilly [11], but it is rather unclear if one can do that because their methods require the operator to be analytic in some variables, and the surface not characteristic. On the other hand, if g is not warped, it is possible that the support theorem could be true, even if the unique continuation for the analogue of (2.4) is false.

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