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SUBCRITICAL L^p BOUNDS ON SPECTRAL CLUSTERS FOR LIPSCHITZ METRICS

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Abstract. We establish asymptotic bounds on the L^p norms of spectrally localized functions in the case of two-dimensional Dirichlet forms with coefficients of Lipschitz regularity. These bounds are new for the range $6 < p < \infty$. A key step in the proof is bounding the rate at which energy spreads for solutions to hyperbolic equations with Lipschitz coefficients.

1. Introduction

The purpose of this paper is to establish L^p bounds on eigenfunctions, or more generally spectrally localized functions, associated to Dirichlet forms on a compact manifold. The question of interest is the dependance of the bounds on the Hölder regularity of the coefficients of the form. We consider here the case of Dirichlet forms with Lipschitz coefficients for simplicity, but the proofs can be adapted to the case of C^s coefficients, where $0 < s < 2$. Our work is restricted to the case of two-dimensional manifolds, however.

Consider the eigenvalue problem for a Dirichlet form, where we work on a compact manifold M without boundary,

$$d^*(a\,d\phi) + \lambda^2 \rho\,\phi = 0\,.$$

Here, a is a section of real, symmetric quadratic forms on $T^*(M)$, with associated linear transforms $a_x : T^*_x(M) \to T_x(M)$, and ρ is a real valued function on M. Here, d^* denotes the adjoint of d relative to a fixed volume form dx. We assume both a and ρ are strictly positive, with uniform bounds above and below. We note that this setting includes the Laplace-Beltrami operator on a Riemannian manifold. The parameter $\lambda \geq 0$ is referred to as the frequency of the eigenfunction ϕ.

A *spectral cluster* of frequency λ is a linear combination of eigenfunctions with frequencies in the range $[\lambda-1,\lambda]$. In the case that a and ρ are smooth, Sogge [9] established the following best possible L^p bounds on spectral clusters,

$$\|f\|_{L^p(M)} \lesssim \begin{cases} \lambda^{\frac{n-1}{2}(\frac12-\frac1p)}\,\|f\|_{L^2(M)}\,, & 2 \leq p \leq p_n \\ \lambda^{n(\frac12-\frac1p)-\frac12}\,\|f\|_{L^2(M)}\,, & p_n \leq p \leq \infty \end{cases} \tag{1}$$

The critical index is $p_n = \frac{2(n+1)}{n-1}$. Semiclassical generalizations were obtained by Koch-Tataru-Zworksi [4]. The bounds (1) hold in case a and ρ are of regularity $C^{1,1}$ by [5], but based on an observation of Grieser [2], and examples of Smith-Sogge [7] and the authors [3], they fail for coefficients of C^s regularity if $s < 2$.

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For metrics of regularity C^s with $s < 2$ (or Lipschitz in case $s = 1$) best possible L^p bounds on spectral clusters have been established on the range $2 \le p \le p_n$, as well as for $p = \infty$; see [6] for the case $1 \le s < 2$, and [3] for the case $s < 1$. This leaves open the *subcritical* case $p_n < p < \infty$, where the upper bounds on the exponent of λ that can be obtained from [3] and [6] by interpolation do not match the lower bounds that follow from the examples of [3] and [7].

In this paper we obtain bounds for $p_n < p < \infty$, for Lipschitz coefficients and $n = 2$, which improve upon the results of [6]. They do not match the exponent displayed by the Rayleigh whispering mode example noted in [2], but the difference is exponentially small as $p \to \infty$. Our results are restricted to $n = 2$, but all steps adapt to C^s coefficients for $0 < s < 2$, and improve upon the results of [3] and [6] for this range of p.

Thus, consider a Dirichlet form on a two-dimensional compact manifold without boundary, with a and ρ of Lipschitz regularity. Let

$$\gamma(p) = 2\Big(\tfrac{1}{2} - \tfrac{1}{p}\Big) - \tfrac{1}{2}$$

be the exponent occuring in the subcritical estimates (1). By Theorem 2 of [6], in this case the following *no-loss estimates* hold on cubes Q of sidelength $\lambda^{-\frac{1}{3}}$,

$$\|f\|_{L^p(Q)} \lesssim \lambda^{\gamma(p)} \|f\|_{L^2(M)}\,, \qquad 6 \le p \le \infty\,. \tag{2}$$

This estimate was in fact proved under a weaker *quasimode* condition (6) on f, which is preserved under smooth cutoffs in x, as well as dyadic and conic localization in frequency.

The Rayleigh whispering mode examples show that if $p = 6$ the size of Q cannot be increased without increasing the exponent in (2). On the other hand, for $p \ge 8$, the Rayleigh mode examples satisfy no-loss estimates on cubes Q of sidelength 1.

The main result of this paper is to establish *log-loss estimates* on cubes of larger sidelength for larger p. Precisely, for cubes Q of sidelength $\lambda^{-\frac{1}{3}2^{(6-p)/2}}$, we establish

$$\|f\|_{L^p(Q)} \lesssim (\log \lambda)^{p-6} \lambda^{\gamma(p)} \|f\|_{L^2(M)}\,, \qquad p = 6, 8, 10, 12, \ldots \tag{3}$$

If f is conically microlocalized in frequency, then (see [6, (14)-(15)]) Q can be replaced by a thin slab of size $1 \times \lambda^{-\frac{1}{3}2^{(6-p)/2}}$. Summing over such slabs in local coordinate charts, one obtains the following, for spectral clusters f of frequency λ.

Theorem 1. *Suppose that a and ρ are of Lipschitz regularity, on a two-dimensional compact manifold without boundary. Then, for $p = 6, 8, 10, 12, \ldots$*

$$\|f\|_{L^p(M)} \lesssim (\log \lambda)^{p-6} \lambda^{\sigma(p)} \|f\|_{L^2(M)}\,, \qquad \sigma(p) = \gamma(p) + \tfrac{1}{3p}\, 2^{\frac{6-p}{2}}\,. \tag{4}$$

To place this result in context, the Rayleigh whispering mode eigenfunctions on the disc, together with a reflection argument, show that, for Lipschitz coefficients and $n = 2$, one cannot establish better estimates than the following

$$\|f\|_{L^p(M)} \lesssim \begin{cases} \lambda^{\frac{2}{3}(\frac{1}{2}-\frac{1}{p})}\, \|f\|_{L^2(M)}\,, & 2 \le p \le 8 \\ \lambda^{\gamma(p)}\, \|f\|_{L^2(M)}\,, & 8 \le p \le \infty \end{cases} \tag{5}$$

The estimate (5) for $2 \le p \le 6$ and $p = \infty$ was established in [6]; interpolation then establishes (4) with $\sigma(p) = \gamma(p) + \frac{1}{3p}$ for $6 < p < \infty$. Thus, Theorem 1 improves upon [6] in the subcritical range, but misses the conjectured result (5) for $6 < p < \infty$ by a factor which decays exponentially as $p \to \infty$.

In a related direction, the bounds (5) were established in [8] for smooth Dirichlet forms on two-dimensional manifolds with boundary, with either Dirichlet or Neumann conditions at the boundary. A smooth Dirichlet form on a manifold with boundary can be thought of as a special case of a form with Lipschitz coefficients on a manifold without boundary, by extending coefficients evenly across the boundary in the geodesic normal coordinates determined by a. The examples of [7] for Lipschitz metrics and $n = 2$ are generated by reflecting a Rayleigh whispering mode from the unit disc, and currently no examples are known which exhibit larger growth in L^p norm for $p > 6$ than these ones.

The proof of (3) is inductive. The estimate for $p + 2$ is derived from the estimate for p, together with an almost orthogonal decomposition of f into tubular pieces. Essentially, one can localize f in frequency to a cone of angle δ, and in space to a characteristic tube of diameter δ^2, and control the energy flow over distance δ. For this reason, the diameter of the log-loss cubes for $p + 2$ is the square root of the diameter of the log-loss cubes for p. The argument that allows summation over different tubes with a $(\log \lambda)^2$ loss works only for $n = 2$, however. Improving Theorem 1 appears then to hinge on controlling energy flow over longer distances, and improving the summation argument to allow $n \geq 3$.

The bounds we obtain are proved for functions satisfying a quasimode condition

$$d^*(a\, df) + \lambda^2 \rho\, f = d^* g_1 + g_2\,. \tag{6}$$

Here, g_2 and the components of g_1 are L^2 functions, and the norm of f as a quasimode is taken as $\|f\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2}$. If f is a spectral cluster, then (6) holds on M with $g_1 = 0$ and $\|g_2\|_{L^2} \lesssim \lambda \|f\|_{L^2}$. Allowing the term g_1 makes localization arguments simpler. In particular, (6) holds, with $\|g_1\|_{L^2} \lesssim \|f\|_{L^2}$ and $\|g_2\|_{L^2} \lesssim \lambda\|f\|_{L^2}$, for the product of a spectral cluster f with a unit size bump function, so we may smoothly localize the function and the equation to a coordinate patch. After a linear change of coordinates and extending the coefficients, we may assume that a and ρ are defined on $\mathbb{R}^2$ and globally close to the flat metric, in that for a constant c_0 which can be taken sufficiently small as needed

$$\|a^{ij} - \delta^{ij}\|_{Lip(\mathbb{R}^2)} + \|\rho - 1\|_{Lip(\mathbb{R}^2)} \leq c_0\,. \tag{7}$$

We establish the estimate (3) by an induction argument, for which the starting point is Corollary 7 of [6], which states that Hypotheses 2 holds for $p = 6$ and $\ell(Q) = \lambda^{-\frac{1}{3}}$. At each step of the induction p increases by 2, and we establish estimates on cubes of square-root the sidelength of the previous step. A loss of $(\log \lambda)^2$ is incurred at each step, however. The hypothesis and induction step follow, and apply generally to functions satisfying equation (6) on $\mathbb{R}^2$; futhermore, constants are uniform over a and ρ given (7).

Hypothesis 2. *Suppose that f satisfies* (6) *on Q^*, where Q^* is the double of the cube Q. Then the following inequality holds, where $\ell(Q)$ denotes the sidelength of Q*

$$\|f\|_{L^p(Q)} \leq C_p\, (\log\lambda)^{p-6} \lambda^{\gamma(p)} \Big(\ell(Q)^{-\frac12}\|f\|_{L^2(Q^*)} + \lambda^{-1}\ell(Q)^{-\frac12}\|df\|_{L^2(Q^*)} \\ + \ell(Q)^{\frac12}\,\|g_1\|_{L^2(Q^*)} + \lambda^{-1}\ell(Q)^{\frac12}\|g_2\|_{L^2(Q^*)}\Big)\,.$$

Theorem 3. *Assume that Hypothesis 2 holds for a given $p \in [6, \infty)$, uniformly over cubes Q of a given sidelength $\ell(Q) = \delta^2$, where $1 \geq \delta \geq \lambda^{-\frac{1}{6}}$. Then Hypothesis 2 holds with p replaced by $p + 2$, uniformly over cubes Q of sidelength $\ell(Q) = \delta$.*

We remark that the norm on the right hand side in Hypothesis 2 should be thought of as the L^2-energy of f on Q^*. If the functions involved are localized to frequencies ξ of magnitude λ, and ξ in a small cone about the ξ_1 axis, then the right hand side is a replacement for $\|f\|_{L^\infty_{x_1} L^2_{x_2}} + \|Pf\|_{L^1_{x_1} L^\infty_{x_2}}$, where $Pf = d^*(adf) + \lambda^2 \rho f$.

The outline of this paper is as follows. In Section 2, we establish the key decomposition of f as a sum of terms, each supported in a thin tube of dimensions $\delta \times \delta^2$. This decomposition at multiple scales is inspired by work of Geba-Tataru [1]. In Section 3 we establish ℓ^q bounds on the overlaps of collections of such tubes, which is applied in Section 4 to complete the proof.

For the remainder of this section we carry out some simple initial reductions. Let the function f satisfy (6) on $\mathbb{R}^2$. We first observe that, for Theorem 3, it suffices to establish Hypotheses 2 for cubes with $\ell(Q) = \delta$ with the norm on the right hand side taken over $\mathbb{R}^2$ instead of Q^*. This is because the conclusion is unchanged if we replace f by ψf, where ψ is a scaled bump function, supported in Q^* and equal to 1 on Q. In subsequent steps we do not assume that f is compactly supported, however.

Next, we split f into components $f = f_{<\lambda} + f_\lambda + f_{>\lambda}$, by localizing respectively to frequencies smaller than $c^2\lambda$, comparable to λ, and larger than $c^{-2}\lambda$, where c is a fixed small constant (assuming only that c_0 in (7) is sufficiently small.) By the arguments of [6, Corollary 5],

$$\lambda \|f_{<\lambda}\|_{L^2} + \|df_{>\lambda}\|_{L^2} \lesssim \|f\|_{L^2} + \lambda^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \lambda^{-1}\|g_2\|_{L^2} \,.$$

Since $\lambda^{2(\frac{1}{2}-\frac{1}{p})-1} \leq \lambda^{\gamma(p)}\ell(Q_0)^{\frac{1}{2}}$, where we use that $\ell(Q) \geq \lambda^{-\frac{1}{6}}$, Sobolev embedding yields that Hypothesis 2 holds with $\|f_{<\lambda}\|_{L^p}$ and $\|f_{>\lambda}\|_{L^p}$ on the left hand side. Thus we restrict attention to the case that f is frequency localized to $|\xi| \approx \lambda$. By further decomposing f as a finite sum of terms, we may also assume that f is frequency localized to $|\xi_2| \leq c\lambda$.

Define the operator

$$P_{\lambda\delta} = d^* a_{\lambda\delta}\, d + \lambda^2 \rho_{\lambda\delta} \,,$$

where the coefficients $a_{\lambda\delta}$ and $\rho_{\lambda\delta}$ are smoothly truncated in frequency to $|\xi| \leq c\lambda\delta$.

Provided $\ell(Q) \leq \delta$, then Hypothesis 2 is unchanged if we replace the defining equation by $P_{\lambda\delta} f = d^* g_1 + g_2$, since the difference $(P - P_{\lambda\delta})f$ can be absorbed into g_1 and g_2, leaving the right hand side of the inequality unchanged up to a constant. To see this one uses the bound

$$\|a - a_{\lambda\delta}\|_{L^\infty} + \|\rho - \rho_{\lambda\delta}\|_{L^\infty} \lesssim (\lambda\delta)^{-1} \,.$$

Given a cube Q and parameters λ, δ, we now set

$$|||f|||_{\lambda,\delta,Q} = \delta^{-\frac{1}{2}}\|f\|_{L^2(Q)} + \lambda^{-1}\delta^{-\frac{1}{2}}\|df\|_{L^2(Q)} + \lambda^{-1}\delta^{\frac{1}{2}}\|P_{\lambda\delta} f\|_{L^2(Q)} \,.$$

We use $|||f|||_{\lambda,\delta}$ to denote the norm where Q is replaced by $\mathbb{R}^2$.

Since f is frequency localized to $|\xi| \approx \lambda$, as is $P_{\lambda\delta} f$, we may absorb the term $d^* g_1$ into g_2. Thus, by the preceeding comments, we are reduced to the following.

Theorem 4. *Suppose that $f \in L^2(\mathbb{R}^2)$ is frequency localized to $|\xi| \approx \lambda$ and $|\xi_2| \le c\lambda$. Then the following holds, uniformly on cubes Q of sidelength δ,*

$$\|f\|_{L^{p+2}(Q)} \lesssim (\log \lambda)^{p-4} \lambda^{\gamma(p+2)} |||f|||_{\lambda,\delta}$$

under the assumption that the following holds, uniformly on cubes Q of sidelength δ^2,

$$\|f\|_{L^p(Q)} \lesssim (\log \lambda)^{p-6} \lambda^{\gamma(p)} |||f|||_{\lambda,\delta^2,Q^*}$$

2. The tube decomposition

Let f be as in Theorem 4, and fix a cube Q_0 of sidelength δ and center x_0. As above, let $\psi = 1$ on Q_0 and vanish outside Q_0^*. In this section we produce a decomposition

$$\psi f = \sum_{T \in \mathcal{T}} f_T + f_0 \,, \tag{8}$$

where f_0 is an error term whose L^p norms can be appropriately bounded by Sobolev embedding. Each f_T will be compactly supported in a tube T. The index T varies over a collection $\mathcal{T}$ of tubes of diameter δ^2 and length δ, each oriented along one of a set of bicharacteristic directions of $P_{\lambda\delta}$ with angular separation δ. Each $\widehat{f_T}$ will be concentrated (in a weighted L^2 sense) in a ball of diameter $\lambda\delta$. We further have the bounds,

$$\left(\sum_T |||f_T|||_{\lambda,\delta}^2 \right)^{\frac{1}{2}} \le C\, |||f|||_{\lambda,\delta} \,. \tag{9}$$

Let Γ be the characteristic set of $P_{\lambda\delta}$ which lies near $Q_0^* \times \mathrm{support}(\hat{f})$,

$$\Gamma = \big\{ (x,\xi) : \langle a_{\lambda\delta}(x)\,\xi, \xi\rangle = \lambda^2 \rho_{\lambda\delta}(x) \big\} \cap Q_0^* \times \{ |\xi_2| \le 2c\lambda \} \,.$$

Since $a_{\lambda\delta}$ and $\rho_{\lambda\delta}$ are pointwise close to the flat metric, the set Γ can be realized as the union of two graphs $\xi_1 = \gamma_\pm(x,\xi_2)$. Since $a_{\lambda\delta}$ and $\rho_{\lambda\delta}$ are Lipschitz and $|x - x_0| \le \delta$, the characteristic set Γ is contained in a $\lambda\delta$ neighborhood of $\xi_1 = \gamma_\pm(x_0,\xi_2)$. Let $q_j(\xi)$ be a finite-overlap collection of $\approx \delta^{-1}$ smooth bump functions, each supported in a ball of diameter $\approx \lambda\delta$ centered on $\xi_1 = \gamma_\pm(x_0,\xi_2)$, so that $\phi(\xi) = 1 - \sum_j q_j(\xi)$ vanishes on a $\lambda\delta$ size neighborhood of the ξ-projection of Γ. Thus, $\phi(\xi) P_{\lambda\delta}(x,\xi)^{-1} \lesssim (\lambda^2\delta)^{-1}$ near $Q_0^* \times \mathrm{support}(\hat{f})$. Set

$$\psi f = \sum_j \psi(x) q_j(D) f + \psi(x)\phi(D) f \,.$$

Let $q(x,\xi) = \psi(x)\phi(\xi)P_{\lambda\delta}(x,\xi)^{-1}$ smoothly cutoff in ξ to $\{|\xi_1| \approx \lambda\,, |\xi_2| \le 2c\lambda\}$. Then

$$\psi(x)\phi(D) f = q(x,D) P_{\lambda\delta} f + r(x,D) f \,,$$

where r is of size $\lambda^{-1}\delta^{-2}$. Precisely, q and r are supported where $|\xi| \approx \lambda$, and

$$\lambda^2 \delta\, q(\lambda^{-1}\delta^{-1}x, \lambda\,\delta\,\xi)\,, \quad \lambda\,\delta^2 r(\lambda^{-1}\delta^{-1}x, \lambda\,\delta\,\xi) \in S^0_{0,0} \,.$$

It follows that

$$\|\psi(x)\phi(D) f\|_{H^1} \lesssim \delta^{-\frac{3}{2}} |||f|||_{\lambda,\delta} \lesssim \lambda^{\frac{1}{4}} |||f|||_{\lambda,\delta}\,, \qquad \delta \ge \lambda^{-\frac{1}{6}} \,.$$

Since $p \ge 6$ we have $\gamma(p+2) \ge \frac{1}{4}$, and Sobolev embedding yields

$$\|\psi(x)\phi(D) f\|_{L^{p+2}} \lesssim \lambda^{\gamma(p+2)} |||f|||_{\lambda,\delta} \,,$$

hence we may take $\psi(x)\phi(D)f$ as the term f_0.

For each fixed j, now consider the term $\psi(x)q_j(D)f$, and let ξ_j be the center of the support of $q_j(\xi)$. We can assume that the angular separation of the ξ_j satisfies $\angle(\xi_i,\xi_j) \gtrsim \delta\,|i-j|$. Let V_j denote the vector

$$V_j = a_{\lambda\delta}(x_0)\,\xi_j\,.$$

Observe that V_j lies within a small angle of the x_1 axis.

We take a partition of unity $\{h\}_{h\in\mathcal{F}}$ on $\mathbb{R}^2$, such that for each h we have $V_j\cdot dh = 0$, and the intersection of $\mathrm{supp}(h)$ with the x_2 axis is contained in an interval of length δ^2. Multiplying by $\psi(x)$, we obtain a decomposition $\psi = \sum_{T\in\mathcal{T}_j}\psi_T$, where ψ_T is supported in a tube T of dimension $\delta\times\delta^2$, and T varies over a collection of approximately δ^{-1} tubes pointing in direction V_j. We let $\mathcal{T}=\cup_j\mathcal{T}_j$, so that there are δ^{-2} tubes in the collection $\mathcal{T}$. With $f_T=\psi_T(x)q_j(D)f$, we have the decomposition expressed in (8).

We also have derivative bounds on ψ_T, for $T\in\mathcal{T}_j$:

$$|(V_j\cdot d)^k\psi_T| \lesssim \lambda^k\delta^{-k}\,, \qquad |\partial_x^\alpha\psi_T| \lesssim \delta^{-2|\alpha|} \le \delta^{-2}(\lambda\delta)^{|\alpha|-1} \quad \text{if} \quad |\alpha|\ge 1\,, \tag{10}$$

where we use $\delta \ge \lambda^{-\frac{1}{6}}$. We then expand $P_{\lambda\delta}(\psi_T q_j(D)f)$ as

$$(d^*a_{\lambda\delta}d\psi_T)\,q_j(D)f + 2\langle a_{\lambda\delta}d\psi_T, d(q_j(D)f)\rangle + \psi_T[P_{\lambda\delta},q_j(D)]f + \psi_T q_j(D)P_{\lambda\delta}f\,, \tag{11}$$

and seek to show that

$$\sum_j\sum_{T\in\mathcal{T}_j}\lambda^{-2}\delta\,\|P_{\lambda\delta}(\psi_T q_j(D)f)\|_{L^2}^2 \lesssim |||f|||_{\lambda,\delta}^2\,.$$

For the first term in (11), this follows by the finite overlap of the ψ_T for $T\in\mathcal{T}_j$ and the finite overlap of the $q_j(\xi)$, together with the pointwise bounds (10). The fourth term is similary handled by the finite overlap properties.

For the third term in (11), we have the simple commutator bounds

$$\|[P_{\lambda\delta},q_j(D)]\|_{L^2\to L^2} \le \lambda^2(\lambda\delta)^{-1} = \lambda\delta^{-1}\,.$$

Additionally, by the frequency localization of $a_{\lambda\delta}$ and $\rho_{\lambda\delta}$, the commutators have finite overlap as j varies, yielding square summability over j.

We expand the brackets in the second term in (11) as

$$\begin{aligned}\langle\,(a_{\lambda\delta}(x)-a_{\lambda\delta}(x_0))d\psi_T, dq_j(D)f\rangle + i\langle V_j, d\psi_T\rangle\,q_j(D)f \\ + \langle a_{\lambda\delta}(x_0)d\psi_T,(d-i\xi_j)(q_j(D)f)\rangle\,.\end{aligned}$$

Each term has L^2 norm bounded by $\lambda\delta^{-1}\|f\|_{L^2}$, and finite overlap properties yield square summability as above.

The finite overlap properties similarly yield that

$$\sum_{T\in\mathcal{T}}\|f_T\|_{L^2}^2 + \lambda^{-2}\|df_T\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 + \lambda^{-2}\|df\|_{L^2}^2\,,$$

completing the verification of (9). □

We also need a stronger inequality. For $T\in\mathcal{T}$, let ξ_T equal ξ_j if $T\in\mathcal{T}_j$, so $|\xi_T|\approx\lambda$ and $f_T=\psi_T q_j(D)f$, with $q_j(\xi)$ centered on ξ_T. The following lemma expresses the fact that the frequencies of f_T are concentrated in the $\lambda\delta$-ball about ξ_T.

Lemma 5. *The following bounds hold, for each α, β,*

$$\left(\sum_T \lambda^{-2|\alpha|}(\lambda\delta)^{-2|\beta|}\|D^\alpha(D-\xi_T)^\beta f_T\|^2_{L^\infty_{x_1}L^2_{x_2}}\right)^{\frac{1}{2}} \le C_{\alpha,\beta}|||f|||_{\delta,\lambda}\,. \tag{12}$$

Proof. Observe that we can write

$$\lambda^{-|\alpha|}(\lambda\delta)^{-|\beta|}D^\alpha(D-\xi_T)^\beta\psi_T(x)\,q_j(D) = \tilde{\psi}_T(x)\,\tilde{q}_j(D)$$

where $\tilde{\psi}_T(x)$ and $\tilde{q}_j(\xi)$ satisfy similar support and derivative bounds as ψ_T and q_j. Hence, the proof we present for the case $\alpha=\beta=0$ applies to the general case.

Let $q_j'(\xi)$ be a smooth cutoff to the $\delta\lambda$-neighborhood of support(q_j). Then

$$\|(1-q_j'(D))\psi_T q_j(D)f\|_{L^2} \lesssim \lambda^{-N}\|f\|_{L^2}\,,\quad \forall N\,.$$

Since the number of tubes is bounded by $\delta^{-2}\ll\lambda$, Sobolev embedding establishes the desired bounds on these terms. We set $f_T' = q_j'(D)f_T$. By commutator arguments as above we have $|||f_T'|||_{\delta,\lambda}\lesssim |||f_T|||_{\delta,\lambda}$. The proof will then follow from (9) by showing that

$$\|f_T'\|_{L^\infty_{x_1}L^2_{x_2}} \lesssim |||f_T'|||_{\lambda,\delta}\,. \tag{13}$$

We establish (13) by energy inequality arguments. Let V denote the vector field

$$V = 2(\partial_1 f_T')\,a_{\lambda\delta}\,df_T' + \left(\lambda^2\rho_{\lambda\delta}\,{f_T'}^2 - \langle a_{\lambda\delta}\,df_T', df_T'\rangle\right)\overrightarrow{e_1}$$

Then

$$d^*V = 2(\partial_1 f_T')\,P_{\lambda\delta}f_T' + \lambda^2(\partial_1\rho_{\lambda\delta}){f_T'}^2 - \langle(\partial_1 a_{\lambda\delta})df_T', df_T'\rangle$$

Applying the divergence theorem on the set $x_1\le r$ yields

$$\int_{x_1=r} V_1\,dx' \lesssim \lambda^2\|f_T'\|^2_{L^2} + \delta^{-1}\|df_T'\|^2_{L^2} + \delta\,\|P_{\lambda\delta}f_T'\|^2_{L^2} \le \lambda^2|||f_T'|||^2_{\lambda,\delta}$$

Since $a_{\lambda\delta}$ and $\rho_{\lambda\delta}$ are pointwise close to the flat metric, we have pointwise that

$$V_1 \ge \tfrac{3}{4}|\partial_1 f_T'|^2 + \tfrac{3}{4}\lambda^2|f_T'|^2 - \tfrac{3}{2}|\partial_2 f_T'|^2$$

The frequency localization of $\widehat{f_T'}$ to $|\xi_2|\le c\lambda$ yields

$$2\int_{x_1=r} V_1\,dx' \ge \int_{x_1=r}|df_T'|^2 + \lambda^2|f_T'|^2\,dx'\,.\quad\square$$

3. Overlap estimates

In this section we establish simple bounds on the overlap of tubes, and resulting ℓ^q bounds on the overlap-counting function.

Lemma 6. *Let x and y be two points in Q_0. Then the number of distinct tubes $T\in\mathcal{T}$ which pass within distance $4\delta^2$ of both x and y is bounded by $C\min\left(\delta^{-1},\frac{\delta}{|x_1-y_1|}\right)$.*

Proof. For each j, there is a fixed bound on the number of tubes $T\in\mathcal{T}_j$ which pass within distance $4\delta^2$ of x. It thus suffices to bound the number of distinct j such that the line through x in direction V_j passes within distant $\sim\delta^2$ of y. The above bound is then a simple consequence of the fact that $\angle(V_i,V_j)\gtrsim\delta|\,i-j|$. $\square$

Now consider a collection $\mathcal{N} \subset \mathcal{T}$ containing N distinct tubes. We make a decomposition of the cube Q_0 into a $\delta^{-1} \times \delta^{-1}$ grid $\mathcal{Q}$ of cubes Q of sidelength δ^2. Let n_Q denote the number of tubes in $\mathcal{N}$ which intersect Q^*,

$$n_Q = \#\{T \in \mathcal{N} : T \cap Q^* \neq \emptyset\}\,.$$

Let $\|n_Q\|_{\ell^q(\mathcal{Q})} = \left(\sum_{Q\in\mathcal{Q}} |n_Q|^q\right)^{1/q}$ denote the ℓ^q norm of the counting function n_Q. By $\|n_Q\|_{\ell^p\ell^q}$ we understand the mixed $\ell^p_{x_1}\ell^q_{x_2}(\mathcal{Q})$ norm of n_Q, taken over the grid $\mathcal{Q}$.

Corollary 7. *The following bounds hold,*

$$\|n_Q\|_{\ell^\infty\ell^1} \lesssim N\,, \qquad\qquad \|n_Q\|_{\ell^2\ell^\infty} \lesssim |\log\delta|^{\frac12}\delta^{-\frac12}N^{\frac12}\,.$$

Furthermore, for $q \geq 3$

$$\|n_Q\|_{\ell^q(\mathcal{Q})} \lesssim |\log\delta|^{\frac1q}\delta^{-\frac1q}N^{1-\frac1q}\,. \tag{14}$$

Proof. The first bound is an immediate consequence of the fact that, for each T and r, there is a fixed upper bound on the number of cubes Q centered on the line $x_1 = r$ such that Q^* intersect T. For the second bound, we consider the map

$$W\{c_T\} = \sum_T c_T\chi_T(Q) \quad \text{where} \quad \begin{cases}\chi_T(Q) = 1\,, & T\cap Q^* \neq \emptyset\\ \chi_T(Q) = 0\,, & T\cap Q^* = \emptyset\end{cases}$$

It suffices to show that $W : \ell^2(\mathcal{T}) \to \ell^2\ell^\infty(\mathcal{Q})$ with bound $|\log\delta|^{\frac12}\delta^{-\frac12}$. The map WW^* takes the form

$$WW^*\{c_Q\}(Q') = \sum_Q n(Q',Q)c_Q\,, \qquad n(Q',Q) = \#\{\,T : T\cap Q^*\neq\emptyset\,, T\cap Q'^* \neq \emptyset\,\}\,.$$

We need to show that $WW^* : \ell^2\ell^1(\mathcal{Q}) \to \ell^2\ell^\infty(\mathcal{Q})$ with norm $|\log\delta|\,\delta^{-1}$. This is an easy consequence of the bound from Lemma 6,

$$n(Q',Q) \lesssim \min\Big(\delta^{-1},\, \delta\,|x_1(Q') - x_1(Q)|^{-1}\Big)\,.$$

Applying interpolation now yields the bounds

$$\|n_Q\|_{\ell^q\ell^r} \lesssim |\log\delta|^{\frac1q}\delta^{-\frac1q}N^{1-\frac1q}\,, \qquad \tfrac2q + \tfrac1r = 1\,.$$

Note that if $q \geq 3$ then $r \leq q$, yielding (14). ☐

4. Proof of Theorem 4

Given f and the cube Q_0, we decompose $\psi f = \sum_{T\in\mathcal{T}} f_T + f_0$ as in Section 2, and control $\|f_0\|_{L^{p+2}}$ by Sobolev embedding. We make a further decomposition by collecting together tubes for which f_T is of comparable energy. Precisely, decompose $\mathcal{T} = \cup_{k\geq -k_0}\mathcal{N}_k$ where $T \in \mathcal{N}_k$ if

$$2^{-k-1}|||f|||_{\lambda,\delta} \;<\; |||f_T|||_{\lambda,\delta} + \sum_{|\alpha|+|\beta|\leq 3}\lambda^{-|\alpha|}(\lambda\delta)^{-|\beta|}\|D^\alpha(D-\xi_T)^\beta f_T\|_{L^\infty_{x_1}L^2_{x_2}}$$
$$\leq 2^{-k}|||f|||_{\lambda,\delta}\,.$$

We handle the tubes for $k \geq 2\log_2\lambda$ by the Sobolev bound $\|f_T\|_{L^\infty} \leq \|Df_T\|_{L^\infty_{x_1}L^2_{x_2}}$, since there are at most $\delta^{-2} \leq \lambda^{\frac13}$ tubes in all. This leaves at most $\approx \log\lambda$ values of k, which we handle individually. We thus fix some $\mathcal{N} = \mathcal{N}_k$, and let N be the number of tubes in $\mathcal{N}$. We multiply f by a constant so that $|||f|||_{\lambda,\delta} = 2^k$, which by (9) and (12) implies $N^{\frac{1}{2}} \lesssim |||f|||_{\lambda,\delta}$. We then need to establish the following.

Theorem 8. *Suppose that $f = \sum_{T\in\mathcal{N}} f_T$, where each f_T is supported in Q_0, and*

$$|||f_T|||_{\delta,\lambda} + \sum_{|\alpha|+|\beta|\le 3} \lambda^{-|\alpha|}(\lambda\delta)^{-|\beta|}\|D^\alpha(D-\xi_T)^\beta f_T\|_{L^\infty_{x_1}L^2_{x_2}} \le 1\,.$$

Let N denote the cardinality of $\mathcal{N}$. Then, under the conditions of Hypothesis 2,

$$\|f\|_{L^{p+2}} \lesssim (\log\lambda)^{p-5}\lambda^{\gamma(p+2)}N^{\frac{1}{2}}\,.$$

Proof. As above we decompose Q_0 into cubes Q of size δ^2, $Q_0 = \cup_{\mathcal{Q}} Q$. By hypothesis, for each Q we have

$$\|f\|_{L^p(Q)} \lesssim (\log\lambda)^{p-6}\lambda^{\gamma(p)}\Big|\Big|\Big|\sum_{T\cap Q^*\neq\emptyset} f_T\,\Big|\Big|\Big|_{\delta^2,\lambda,Q^*}\,.$$

We first show that

$$\Big|\Big|\Big|\sum_{T\cap Q^*\neq\emptyset} f_T\,\Big|\Big|\Big|_{\delta^2,\lambda,Q^*} \lesssim n_Q^{\frac{1}{2}}\,. \tag{15}$$

For this, note that $|a_{\lambda\delta} - a_{\lambda\delta^2}| \le (\lambda\delta^2)^{-1}$, hence

$$\Big|\Big|\Big|\sum_{T\cap Q^*\neq\emptyset} f_T\,\Big|\Big|\Big|_{\delta^2,\lambda,Q^*} \le \sum_{|\alpha|\le 2}\lambda^{-|\alpha|}\Big\|\sum_{T\cap Q^*\neq\emptyset} D^\alpha f_T\,\Big\|_{L^\infty_{x_1}L^2_{x_2}} + \lambda^{-1}\delta\,\Big\|\sum_{T\cap Q^*\neq\emptyset} P_{\lambda\delta}f_T\,\Big\|_{L^2}\,.$$

For each j, there are a bounded number of tubes T with $\xi_T = \xi_j$ for which $T\cap Q^* \neq \emptyset$, hence we can assume the different ξ_T in the above sum are spaced by distance $\lambda\delta$ in the ξ_2 variable. Thus,

$$\begin{aligned}\lambda^{-|\alpha|}\Big\|\sum_{T\cap Q^*\neq\emptyset} D^\alpha f_T\Big\|_{L^\infty_{x_1}L^2_{x_2}} &\lesssim \left(\sum_{|\beta|\le 1}\sum_{T\cap Q^*\neq\emptyset}\lambda^{-2|\alpha|}(\lambda\delta)^{-2|\beta|}\|(D-\xi_T)^\beta D^\alpha f_T\|^2_{L^\infty_{x_1}L^2_{x_2}}\right)^{\frac{1}{2}}\\ &\lesssim\ n_Q^{\frac{1}{2}}\,.\end{aligned}$$

To complete the proof of (15), we use that $n_Q \le \delta^{-1}$ to bound

$$\lambda^{-1}\delta\sum_{T\cap Q^*\neq\emptyset}\|P_{\lambda\delta}f_T\|_{L^2} \le \delta^{\frac{1}{2}}\sum_{T\cap Q^*\neq\emptyset}|||f_T|||_{\delta,\lambda} \le \delta^{\frac{1}{2}}n_Q \le n_Q^{\frac{1}{2}}\,.$$

By (15) and Hypothesis 2, we thus have

$$||f||_{L^p(Q)} \lesssim (\log\lambda)^{p-6}\lambda^{\gamma(p)}n_Q^{\frac{1}{2}}\,. \tag{16}$$

We next note the bound

$$\begin{aligned}\|f_T\|_{L^\infty} &\le 2\|(D-\xi_T)f_T\|^{\frac{1}{2}}_{L^\infty_{x_1}L^2_{x_2}}\|f_T\|^{\frac{1}{2}}_{L^\infty_{x_1}L^2_{x_2}}\\ &\le \lambda^{\frac{1}{2}}\delta^{\frac{1}{2}}\sum_{|\beta|\le 1}(\lambda\delta)^{-|\beta|}\|(D-\xi_T)^\beta f_T\|_{L^\infty_{x_1}L^2_{x_2}}\\ &\le \lambda^{\frac{1}{2}}\delta^{\frac{1}{2}}\,.\end{aligned}$$

Consequently,

$$\|f\|_{L^\infty(Q)} \le \lambda^{\frac{1}{2}}\delta^{\frac{1}{2}}n_Q\,. \tag{17}$$

Combining (16)–(17) with (14) for $q = \frac{1}{2}p$ and $q = \infty$ respectively, we obtain

$$\|f\|_{L^p} \lesssim (\log\lambda)^{p-6+\frac{1}{p}}\lambda^{\gamma(p)}\delta^{-\frac{1}{p}}N^{\frac{1}{2}-\frac{1}{p}}$$

$$\|f\|_{L^\infty} \lesssim \lambda^{\frac{1}{2}}\delta^{\frac{1}{2}}N$$

Interpolation yields

$$\|f\|_{L^{p+2}} \lesssim (\log\lambda)^{\frac{p(p-6)+1}{p+2}}\lambda^{\gamma(p+2)}N^{\frac{1}{2}}$$

Observing that if $p \geq 6$ we have $p(p-6)+1 \leq (p+2)(p-5)$ concludes the proof. $\square$

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