Math. Res. Lett. **15** (2008), *no.* 6, 1211–1221

ON THE UNIVERSAL GRÖBNER BASES OF VARIETIES OF MINIMAL DEGREE

Sonja Petrović

Abstract. A universal Gröbner basis of an ideal is the union of all its reduced Gröbner bases. It is contained in the Graver basis, the set of all primitive elements. Obtaining an explicit description of either of these sets, or even a sharp degree bound for their elements, is a nontrivial task.
In their '95 paper, Graham, Diaconis and Sturmfels give a nice combinatorial description of the Graver basis for any rational normal curve in terms of primitive partition identities. Their result is extended here to rational normal scrolls. The description of the Graver bases of scrolls is given in terms of *colored* partition identities. This leads to a sharp bound on the degree of Graver basis elements, which is always attained by a circuit.
Finally, for any variety obtained from a scroll by a sequence of projections to some of the coordinate hyperplanes, the degree of any element in any reduced Gröbner basis is bounded by the degree of the variety.

1. Introduction

Fix a subset $\mathcal{A} = \{a_1, \ldots, a_n\}$ of $\mathbb{Z}^d$. The set $\mathcal{A}$ determines a toric ideal in the following way. Every vector $u \in \mathbb{Z}^n$ can be written uniquely as $u = u^+ - u^-$ where u^+ and u^- are nonnegative and have disjoint support. Considering $\mathcal{A}$ as a $d \times n$ matrix induces a parametrization of a variety $X := X_\mathcal{A}$ whose defining ideal is the toric ideal $I_\mathcal{A} := (x^{u^+} - x^{u^-} : Au = 0)$ in the polynomial ring $k[\mathbf{x}] := k[x_1, \ldots, x_n]$. We may write I_X for $I_\mathcal{A}$. A standard reference on toric ideals is [10].

Given any term order $\prec$ on the monomials of $k[\mathbf{x}]$, the initial ideal of $I_\mathcal{A}$ is defined to be $in_\prec(I_\mathcal{A}) := (in_\prec(f) : f \in I_\mathcal{A})$. Any generating set $\mathcal{G}_\prec$ of the ideal such that $in_\prec(I_\mathcal{A}) = (in_\prec(g) : g \in \mathcal{G}_\prec)$ is called a *Gröbner basis*. If $\mathcal{G}_\prec$ is *reduced*, that is, no term of g is divisible by $in_\prec(f)$ for any $f, g \in \mathcal{G}_\prec$, then $\mathcal{G}_\prec$ is uniquely determined by the term order $\prec$. For applications of Gröbner bases, see [9] and [10].

The union of the (finitely many) reduced Gröbner bases of $I_\mathcal{A}$ is called the *universal Gröbner basis* and denoted $\mathcal{U}_\mathcal{A}$. It is contained in the set of primitive binomials called the *Graver basis* $\mathcal{G}r_\mathcal{A}$ of $I_\mathcal{A}$; a binomial $x^{u^+} - x^{u^-} \in I_\mathcal{A}$ is called primitive if there is no $x^{v^+} - x^{v^-} \in I_\mathcal{A}$ such that $x^{v^+} | x^{u^+}$ and $x^{v^-} | x^{u^-}$. A set of primitive binomials with minimal support is the set $\mathcal{C}_\mathcal{A}$ of *circuits* of the ideal. It is known that $\mathcal{C}_\mathcal{A} \subset \mathcal{U}_\mathcal{A} \subset \mathcal{G}r_\mathcal{A}$. In general, both containments are proper.

There exists a general bound on the degrees of the elements of the universal Gröbner basis ([10]), however it is far too large for many specific examples. One might expect the sharp bound to be smaller for varieties that are special in some sense.

Rational normal scrolls are examples of varieties of *minimal degree*, that is, the varieties which attain the general lower bound $\deg(X) \geq \operatorname{codim}(X) + 1$. They have been classified ([5]) as quadratic hypersurfaces, rational normal scrolls, the Veronese

Received by the editors November 20, 2007.

surface in $\mathbb{P}^5$, and cones over these. The scrolls are the only infinite family among these. Their defining ideals have the following special property:

Theorem 1.1 (Corollary 4.4)**.** *The degree of any binomial in the Graver basis (and the universal Gröbner basis) of any rational normal scroll is bounded above by the degree of the scroll.*

We also derive a sharp bound (Theorem 4.1), which is usually much smaller than the degree of the scroll.

Another remarkable property of the defining ideals of rational normal scrolls is that their Graver bases admit a particularly nice combinatorial description. Namely, each primitive element in the ideal of a scroll corresponds to a suitable *primitive colored partition identity*. Such a characterization of primitive elements imposes a restriction on the *structure* of any binomial in the universal Gröbner bases of scrolls.

The paper is organized as follows. Section 2 contains the necessary information about the defining ideals of rational normal scrolls. In Section 3 we introduce colored partition identities and use them to characterize the Graver bases of the scrolls (Proposition 3.8), generalizing the result for rational normal curves in [7]. Section 4 contains the degree bounds. An important consequence of the sharp bound in Theorem 4.1 is that if X is any variety that can be obtained from a scroll by a sequence of projections to some of the coordinate hyperplanes, then the degree of the variety gives an upper bound on the degrees of elements in the universal Gröbner basis of its defining ideal I_X. In the final section, we conjecture that the universal Gröbner basis equals the Graver basis for any scroll, and discuss its consequences. We also derive the dimension of the state polytopes of scrolls.

2. Parametrization of Scrolls

Let $S := S(n_1 - 1, \dots, n_c - 1)$ be the rational normal scroll in $\mathbb{P}^{n_1+\dots+n_c-1}$. Its defining ideal is given by I_2M, where

$$M := [M_{n_1}|M_{n_2}|\dots|M_{n_c}], \text{ and } M_{n_j} := \begin{bmatrix} x_{j,1} & \dots & x_{j,n_j-1} \\ x_{j,2} & \dots & x_{j,n_j} \end{bmatrix}.$$

If $c = 1$, then the 2-minors of the matrix above give the defining ideal of a rational normal curve $S(n-1)$ in $\mathbb{P}^{n-1}$ ([5]).

Lemma 2.1. *$I_S = \ker\varphi$, where $\varphi(x_{j,i}) = [v_1^1, \dots, v_j^1, v_{j+1}^0, \dots, v_c^0, t^i]^T$ for $1 \le j \le c$. That is, the matrix $\mathcal{A}$ that encodes the parametrization of the scroll S is*

$$\mathcal{A} = \begin{bmatrix} 1 & \dots & 1 & 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ \vdots & & & & & & & & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 1 & \dots & 1 \\ 1 & 2\dots & n_1 & 1 & \dots & n_2 & \dots & 1 & \dots & n_c \end{bmatrix}.$$

Proof. Indeed, let the generators of I_S be the minors

$$m_{i,j,k,l} := x_{i,k}x_{j,l+1} - x_{j,l}x_{i,k+1}$$

for $1 \le i, j \le c$, $1 \le k \le n_i - 1$, and $1 \le l \le n_j - 1$. (Note that we allow $i = j$ and $k = l$.) Then the exponent vector of $m_{i,j,k,l}$ is

$$v_{i,j,k,l} = [0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots, 0]$$

where the positive entries are in columns $n_1 + \cdots + n_{i-1} + k$ and $n_1 + \cdots + n_{j-1} + l + 1$, while the negative entries are in columns $n_1 + \cdots + n_{j-1} + l$ and $n_1 + \cdots + n_{i-1} + k + 1$. (If $i = j$ and $k = l$, then the two locations for the negative entries coincide; in that case, the negative entry is -2.) Denote by $\mathcal{A}_c$ the c^{th} column of $\mathcal{A}$. Then clearly

$$\mathcal{A}_{n_1+\cdots+n_{i-1}+k} + \mathcal{A}_{n_1+\cdots+n_{j-1}+l+1} = \mathcal{A}_{n_1+\cdots+n_{j-1}+l} + \mathcal{A}_{n_1+\cdots+n_{i-1}+k+1}$$

since

$$\begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ k \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ l+1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ k+1 \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ l \end{bmatrix}.$$

Thus $m_{i,j,k,l} \in I_{\mathcal{A}}$ for each generator $m_{i,j,k,l}$ of I_S.

In addition, the matrix $\mathcal{A}$ has full rank; thus the dimension of the variety it parametrizes is $\operatorname{rank}\mathcal{A} - 1 = c$. But this is precisely the dimension of the scroll S. □

Example 2.2. The ideal of the scroll $S(3, 2)$ is $I_{\mathcal{A}_{S(3,2)}}$ where

$$\mathcal{A}_{S(3,2)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 \end{bmatrix}.$$

3. Colored partition identities and Graver bases

Let us begin by generalizing the definitions from Chapter 6 of [10].

Definition 3.1. A **colored partition identity** (or a **cpi**) in the colors $(1),\dots,(c)$ is an identity of the form

$$(*)\qquad \begin{aligned} & a_{1,1} + \cdots + a_{1,k_1} + a_{2,1} + \cdots + a_{2,k_2} + \cdots + a_{c,1} + \cdots + a_{c,k_c} = \\ & b_{1,1} + \cdots + b_{1,s_1} + b_{2,1} + \cdots + b_{2,s_2} + \cdots + b_{c,1} + \cdots + b_{c,s_c}, \end{aligned}$$

where $1 \le a_{p,j}, b_{p,j} \le n_p$ are positive integers for all j, $1 \le p \le c$ and some positive integers $n_1, \dots, n_c$.

If $c = 1$ then this is precisely the definition of the usual partition identity with $n = n_1$.

Remark 3.2. A cpi in c colors with $n_1, \dots, n_c$ as above is a partition identity (in one color) with largest part $n = \max\{n_1, \dots, n_c\}$.

Example 3.3. Denote by i_r the number i colored red, and by i_b the number i colored blue. Then

$$1_r + 4_r + 3_b = 5_b + 1_b + 2_r$$

is a colored partition identity with two colors, with $n_1 = 4$ and $n_2 = 5$. Erasing the coloring gives $1 + 4 + 3 = 5 + 1 + 2$, a (usual) partition identity with largest part $n = 5$.

Definition 3.4. A colored partition identity (*) is a **primitive** cpi (or a **pcpi**) if there is no proper sub-identity $a_{-,i_1} + \cdots + a_{-,i_l} = b_{-,j_1} + \cdots + b_{-,j_t}$, with $1 \leq l+t < k_1 + \cdots + k_c + s_1 + \cdots + s_c$, which is a cpi.

A cpi is called **homogeneous** if $k_1 + \cdots + k_c = s_1 + \cdots + s_c$. If $k_j = s_j$ for $1 \leq j \leq c$, then it is called **color-homogeneous**. The **degree** of a pcpi is the number of summands $k_1 + \cdots + k_c + s_1 + \cdots + s_c$.

Note that color-homogeneity implies homogeneity, and that a homogeneous pcpi need not be primitive in the inhomogeneous sense.

Example 3.5. Here is a list of all primitive color-homogeneous partition identities with $c = 2$ colors and $n_1 = n_2 = 3$.

$$\begin{aligned}
1_1 + 3_1 &= 2_1 + 2_1\\
1_1 + 2_2 &= 2_1 + 1_2\\
1_1 + 1_1 + 3_2 &= 2_1 + 2_1 + 1_2\\
1_1 + 3_2 &= 2_1 + 2_2\\
2_1 + 3_2 &= 3_1 + 2_2\\
2_1 + 2_2 &= 3_1 + 1_2\\
1_1 + 3_2 &= 3_1 + 1_2\\
1_2 + 3_2 &= 2_2 + 2_2\\
1_1 + 3_2 + 3_2 &= 3_1 + 2_2 + 2_2\\
1_1 + 2_2 + 2_2 &= 3_1 + 1_2 + 1_2\\
2_1 + 2_1 + 3_2 &= 3_1 + 3_1 + 1_2
\end{aligned}$$

We are now ready to relate the ideals of scrolls and the colored partition identities.

Lemma 3.6. *A binomial* $x_{1,a_{1,1}} \dots x_{1,a_{1,k_1}} \dots x_{c,a_{c,1}} \dots x_{c,a_{c,k_c}} - x_{1,b_{1,1}} \dots x_{c,b_{c,s_c}}$ *is in the ideal* $I_{\mathcal{A}_{S(n_1-1,\dots,n_c-1)}}$ *if and only if (*) is a color-homogeneous cpi.*

Proof. This follows easily from the definitions and Lemma 2.1. □

Example 3.7. Let $c = 2$. Then

$$A := \mathcal{A}_{S(n_1-1,n_2-1)} = \begin{bmatrix} 1 & \dots & 1 & 1 & \dots & 1\\ 0 & \dots & 0 & 1 & \dots & 1\\ 1 & \dots & n_1 & 1 & \dots & n_2 \end{bmatrix}$$

and

$$I_A = I_2 \begin{bmatrix} x_{1,1} & \dots & x_{1,n_1-1} & x_{2,1} & \dots & x_{2,n_2-1}\\ x_{1,2} & \dots & x_{1,n_1} & x_{2,2} & \dots & x_{2,n_2} \end{bmatrix}.$$

Then $x_{1,a_{1,1}} \dots x_{1,a_{1,k_1}} x_{2,a_{2,1}} \dots x_{2,a_{2,k_2}} - x_{1,b_{1,1}} \dots x_{1,b_{1,s_1}} x_{2,b_{2,1}} \dots x_{2,b_{2,s_2}} \in I_A$ if and only if

$$\begin{bmatrix} v_1^{k_1+k_2} \\ v_2^{0+k_2} \\ t^{a_{1,1}+\dots+a_{2,k_2}} \end{bmatrix} = \begin{bmatrix} v_1^{s_1+s_2} \\ v_2^{0+s_2} \\ t^{b_{1,1}+\dots+b_{2,s_2}} \end{bmatrix}$$

if and only if $k_1 + k_2 = s_1 + s_2$, $k_2 = s_2$, and

$$a_{1,1} + \dots + a_{1,k_1} + a_{2,1} + \dots + a_{2,k_2} = b_{1,1} + \dots + b_{1,s_1} + b_{2,1} + \dots + b_{2,s_2}.$$

The last equality clearly describes a color-homogeneous pcpi.

The Lemmas above imply the following characterization of the Graver bases of rational normal scrolls.

Proposition 3.8. *The Graver basis elements for the scroll $S(n_1 - 1, \dots, n_c - 1)$ are precisely the color-homogeneous primitive colored partition identities of the form (*).*

Proof. With all the tools in hand, it is not difficult to check that the binomial in the ideal of the scroll is primitive if and only if the corresponding colored partition identity is primitive. □

If $c = 1$, this is just the observation in Chapter 6 of [10].

4. Degree bounds

Now we can generalize the degree bound given in [10] for the rational normal curves. The degree bound is sharp, and it is remarkable that it is always attained by a circuit. By a *subscroll* of $S(n_1 - 1, \dots, n_c - 1)$ we mean a scroll $S' := S(n'_1 - 1, \dots, n'_c - 1)$ such that $n'_i \le n_i$ for each i. Clearly, $I_{S'}$ can be obtained from I_S by eliminating variables.

Theorem 4.1. *Let $S := S(n_1 - 1, \dots, n_c - 1)$ for $c \ge 2$. Let P and Q be the indices such that*

$$n_P = \max\{n_i : 1 \le i \le c\}$$

and

$$n_Q = \max\{n_j : 1 \le j \le c, j \ne P\}.$$

Then the degree of any primitive binomial in I_S is bounded above by

$$n_P + n_Q - 2.$$

This bound is sharp exactly when $n_P - 1$ and $n_Q - 1$ are relatively prime.

More precisely, the primitive binomials in I_S have degree at most

$$u + v - 2,$$

where u and v are maximal integers such that $S(n'_1 - 1, \dots, n'_c - 1)$ is a subscroll of S with $n'_i = u$ and $n'_j = v$ for some $1 \le i, j \le c$, and subject to $(u - 1, v - 1) = 1$.

This degree bound is sharp; there is always a circuit having this degree. For any number of colors c, such a maximal degree circuit is two-colored.

Before proving the Theorem, let us look at an example.

Example 4.2. Consider the scroll $S(5,6)$. Here $n_P - 1 = 6$ and $n_Q - 1 = 5$, and since they are relatively prime, the sharp degree bound is $5+6 = 11$. On the other hand, if $S := S(4,4,2,2)$, then $n_P - 1 = n_Q - 1 = 4$ so we look for a subscroll $S' := S(4,3,2,2)$. Then $u-1 = 4$ and $v-1 = 3$, and the degree of any primitive element is at most 7. Finally, if $S := S(5,5,5)$, then $n_P - 1 = n_Q - 1 = 5$. The desired subscroll is $S' := S(5,4,4)$ so that the degree bound is $u-1+v-1 = 5+4 = 9$.

Proof of Theorem 4.1. Let $x_{1,a_{1,1}} \dots x_{c,a_{c,k_c}} - x_{1,b_{1,1}} \dots x_{c,b_{c,k_c}} \in I_S$. Consider the corresponding color-homogeneous pcpi:

$$\begin{aligned} &a_{1,1} + \dots + a_{1,k_1} + a_{2,1} + \dots + a_{2,k_2} + \dots + a_{c,1} + \dots + a_{c,k_c} = \\ (**)\qquad &b_{1,1} + \dots + b_{1,k_1} + b_{2,1} + \dots + b_{2,k_2} + \dots + b_{c,1} + \dots + b_{c,k_c}. \end{aligned}$$

Note that the number of terms on either side of (**) equals the degree of the binomial. We shall first show that $k_1 + \dots + k_c \leq n_P + n_Q - 2$ holds for (**).

Let $d_{i,j} = a_{i,j} - b_{i,j}$ be the differences in the i^{th}-color entries for $1 \leq j \leq k_i$, $1 \leq i \leq c$. Then

$$\sum_{\substack{1 \leq i \leq c \\ 1 \leq j \leq k_i}} d_{i,j} = 0.$$

Separating positive and negative terms gives an inhomogeneous pcpi $\sum d_{i,j}^+ = \sum d_{i,j}^-$. Indeed, if it is not primitive then there would be a subidentity in (**). Note that an inhomogeneous pcpi is defined to be a ppi with arbitrary coloring. Therefore, the sum-difference algorithm from the proof of Theorem 6.1. in [10] can be applied. For completeness, let us recall the algorithm.

`Set` $x := 0$, $\mathcal{P} := \{d_{i,j}^+\}$, $\mathcal{N} := \{d_{i,j}^-\}$.
`While` $\mathcal{P} \cup \mathcal{N}$ `is non-empty do`
  `if` $x \geq 0$
    `then select an element` $\nu \in \mathcal{N}$, `set` $x := x - \nu$ `and` $\mathcal{N} := \mathcal{N} \backslash \{\nu\}$
    `else select an element` $\pi \in \mathcal{P}$, `set` $x := x + \pi$ `and` $\mathcal{P} := \mathcal{P} \backslash \{\pi\}$.

The number of terms in the pcpi is bounded above by the number of values x can obtain during the run of the algorithm. Primitivity ensures no value is reached twice. Let

$$D_{i,+} := \max_j \{d_{i,j} : d_{i,j} > 0\}$$

and

$$D_{i,-} := \max_j \{-d_{i,j} : d_{i,j} < 0\}.$$

Then $k_1 + \dots + k_c \leq \max_i\{D_{i,+}\} + \max_i\{D_{i,-}\} =: D_+ + D_-$ (Corollary 6.2 in [10]). Let D_+ and D_- occur in colors P and Q, respectively, so that $D_+ = a_P - b_P$, and $D_- = b_Q - a_Q$. Then the sequence of inequalities

$$1 + D_+ + 1 \leq 1 + D_+ + b_P = 1 + a_P \leq 1 + n_P \leq a_Q + n_P = b_Q - D_- + n_P \leq n_Q - D_- + n_P$$

implies that

$$D_+ + D_- \leq n_P + n_Q - 2,$$

and the degree bound follows.

The maximum degree occurs when there is equality in the above sequence of inequalities, and x reaches every possible value during the run of the algorithm. Following the argument of Sturmfels from the proof of Theorem 6.1. in [10], this means that the inhomogeneous pcpi $\sum d_{i,j} = 0$ is of the form

$$\underbrace{D_+ + \cdots + D_+}_{D_- \text{ terms}} = \underbrace{D_- + \dots D_-}_{D_+ \text{ terms}}.$$

In addition,

$$1+D_+ + 1 = 1+D_+ + b_P = 1+a_P = 1+n_P = a_Q+n_P = b_Q - D_- + n_P = n_Q - D_- + n_P$$

implies that $b_P = 1$, $a_P = n_P$, $a_Q = 1$, and $b_Q = n_Q$. Therefore, the maximal degree identity $\sum d_{i,j} = 0$ provides that (**) is of the following form:

$$\underbrace{1_P + \cdots + 1_P}_{n_Q - 1 \text{ terms}} + \underbrace{n_Q + \cdots + n_Q}_{n_P - 1 \text{ terms}} = \underbrace{1_Q + \cdots + 1_Q}_{n_P - 1 \text{ terms}} + \underbrace{n_P + \cdots + n_P}_{n_Q - 1 \text{ terms}},$$

where 1_P denotes the number 1 colored using the color P. This colored partition identity is primitive if and only if there does not exist a proper subidentity if and only if $n_P - 1$ and $n_Q - 1$ are relatively prime. Indeed, if $n_P - 1 = zy$ and $n_Q - 1 = zw$ for some $z, y, w \in \mathbb{N}$, then there is a subidentity of the form

$$\underbrace{1_P + \cdots + 1_P}_{w \text{ terms}} + \underbrace{n_Q + \cdots + n_Q}_{y \text{ terms}} = \underbrace{1_Q + \cdots + 1_Q}_{y \text{ terms}} + \underbrace{n_P + \cdots + n_P}_{w \text{ terms}}.$$

Furthermore, assume that $n_P - 1$ and $n_Q - 1$ are relatively prime. Then the exponent vector of the binomial corresponding to the maximal degree identity has support of cardinality four. It is thus a circuit for any $c \geq 2$. Clearly, it is a two-colored circuit, regardless of the number of colors c in our scroll S.

Finally, if $n_P - 1$ and $n_Q - 1$ are not relatively prime, the degree $n_Q + n_P - 2$ cannot be attained by a primitive binomial. In that case, we may simply eliminate one of the variables to obtain a smaller scroll, say $S' := S(n_1 - 1, \dots, n_P - 2, \dots, n_c - 1)$, whose defining ideal is embedded in that of S (that is, $u := n_P - 1$ and $v := n_Q$). Clearly, primitive binomials from $I_{S'}$ lie in I_S. If $u - 1$ and $v - 1$ are relatively prime, then we have the smaller bound for the degree: $n_P + n_Q - 3$. If not, we continue eliminating variables until the condition is satisfied.

This completes the proof. □

Remark 4.3. In view of the comment on p.36 of [10], it is interesting to note that in the case of varieties of minimal degree, the maximum degree of any Graver basis element is attained by a circuit. This is not true in general.

Now the following is trivial.

Corollary 4.4. *The degree of any binomial in the Graver basis (and the universal Gröbner basis) of any rational normal scroll is bounded above by the degree of the scroll.*

In addition, this also gives the upper bound for the degrees of any element in the universal Gröbner basis of any variety whose parametrization can be embedded into that of a scroll, generalizing Corollary (6.5) from [10].

Corollary 4.5. *Let X be any toric variety that can be obtained from a scroll by a sequence of projections to some of the coordinate hyperplanes. Then the degree of an element of any reduced Gröbner basis of I_X is at most the degree of the toric variety X.*

Proof. The claim follows from degree-preserving coordinate projections and the elimination property of the universal Gröbner basis. The variety $X = X_{\mathcal{A}}$ is parametrized by

$$\mathcal{A} = \begin{bmatrix} 1 & \dots & 1 & 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ \vdots & & & & & & & & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 1 & \dots & 1 \\ i_{1,1} & i_{1,2}\dots & i_{1,r_1} & i_{2,1} & \dots & i_{2,r_2} & \dots & i_{c,1} & \dots & i_{c,r_c} \end{bmatrix}$$

In what follows, we may assume that $1 = i_{k,1} < \cdots < i_{k,r_k} = n_k$ for $1 \le k \le c$. Then X can be obtained by coordinate projections from the scroll $S := S(n_1-1, \dots, n_c-1)$, parametrized by $\mathcal{A}_S$ as before. The degree of the toric variety $X_{\mathcal{A}}$ is the normalized volume of the polytope formed by taking the convex hull of the columns of $\mathcal{A}$. But $\mathrm{vol}(\mathrm{conv}(\mathcal{A})) = \mathrm{vol}(\mathrm{conv}(\mathcal{A}_S))$ implies that the two varieties have the same degree.

Suppose $x^u - x^v$ is in some reduced Gröbner basis of I_X. Then Proposition 4.13. and Lemma 4.6. in [10] provide that $x^u - x^v \in \mathcal{U}_{\mathcal{A}} \subset \mathcal{U}_{\mathcal{A}_S} \subset \mathcal{G}r_{\mathcal{A}_S}$. Applying Corollary 4.4 completes the proof. □

Remark 4.6. In particular, note that this degree bound (which equals the degree of the scroll, $n_1 + \cdots + n_c - c$) is always better then the general one given for toric ideals in [10], Corollary 4.15, which equals $1/2(c+2)(n_1 + \cdots + n_c - c - 1)D(\mathcal{A})$ where $D(\mathcal{A})$ is the maximum over all $(c+1)$-minors of $\mathcal{A}$.

Let us conclude this section by listing the number of all elements in the Graver basis of some small scrolls, sorted by degree of the binomial. The entries in this table have been obtained using the software 4ti2 [1], which was essential in this project.

	Degrees									
Scroll	2	3	4	5	6	7	8	9	10	11
S(2,2)	7	4								
S(2,2,2)	18	24								
S(4)	7	7	2							
S(3,2)	12	16	4	1						
S(3,2,2)	26	58	22	4						
S(3,3)	20	40	18	4						
S(3,3,2,2)	59	242	208	36						
S(4,2)	19	39	20	4						
S(4,3)	30	86	58	15	2	1				
S(4,4)	44	166	146	52	12	4				
S(4,3,2,2)	75	391	524	176	6	1				
S(5,2)	28	83	72	32	4	1				
S(6,2)	40	157	182	95	28	4				
S(5,3)	42	166	174	78	16	6	1			
S(6,3)	57	290	412	210	62	14	2			
S(7,2)	55	280	432	294	130	46	4	1		
S(5,5,5)	204	2526	10002	10404	5088	1764	444	78		
S(6,5)	105	813	1678	1136	454	149	42	12	2	1

5. Universal Gröbner bases

The Graver basis is a good approximation to the universal Gröbner basis, but they are not equal in general. However, extensive computations using the software 4ti2 ([1]) show evidence supporting the following conjecture:

Conjecture 5.1. *$\mathcal{U}_\mathcal{A} = \mathcal{G}r_\mathcal{A}$ for the defining matrix $\mathcal{A}$ of any rational normal scroll.*

Note that the defining ideal of $S := S(n_1-1, \ldots, n_c-1)$ is contained in the defining ideal of the scroll

$$S(n_1 - 1, \ldots, n_c - 1, \underbrace{1, \ldots, 1}_{l \text{ terms}})$$

for any l. Define S' to be any such scroll, where l is chosen so that the inequality

$$c + l + 3 > 2(n_P + n_Q - 2 - j_0)$$

is satisfied, where $n_P + n_Q - 2 - j_0$ is the degree bound for the scroll S' from Theorem 4.1. This puts a restriction on the size of the support of any primitive binomial. Let $f \in \mathcal{G}r_\mathcal{A}$. Then $f \in I_{\mathcal{A}'}$ where $\mathcal{A}' := \mathcal{A}_{S'}$. The primitivity of f implies $f \in \mathcal{G}r_{\mathcal{A}'}$. If the conjecture is true for the scroll S', then f lies in the universal Gröbner basis of the ideal $I_{\mathcal{A}'}$, and hence in the universal Gröbner basis of $I_\mathcal{A}$.
Therefore, to prove this conjecture, it suffices to prove a weaker one:

Conjecture 5.2. *$\mathcal{U}_\mathcal{A} = \mathcal{G}r_\mathcal{A}$ for rational normal scrolls of sufficiently high dimension.*

Recently, Hemmecke and Nairn in [6] stated that if the universal Gröbner basis and Graver basis of $I_\mathcal{A}$ coincide, then the Gröbner and Graver complexities of $\mathcal{A}$ are equal. We plan to study the higher Lawrence configurations of the rational normal scrolls.

Next, we consider state polytopes of rational normal scrolls. Knowing a universal Gröbner basis of $I_\mathcal{A}$ is equivalent to knowing its *state polytope* ([10]). It is defined to be any polytope whose normal fan coincides with the *Gröbner fan* of the ideal. The cones of the Gröbner fan correspond to the reduced Gröbner bases $\mathcal{G}_\prec$ of $I_\mathcal{A}$. In addition, the Gröbner fan is a refinement of the secondary fan $\mathcal{N}(\Sigma(\mathcal{A}))$, which classifies equivalence classes of lifting functions giving a particular regular triangulation of the point configuration $\mathcal{A}$.

Theorem 5.3. *The dimension of the state polytope of a rational normal scroll is one less then the degree of the scroll:*

$$\dim \mathit{State}(I_{S(n_1-1,\ldots,n_c-1)}) = n_1 + \cdots + n_c - c - 1.$$

Proof. Eliminating variables results in taking faces of the state polytope. Thus the state polytope for the scroll $S(n_1 - 1)$ is a face of that of $S(n_1 - 1, 1)$, which in turn is a face of the state polytope of $S(n_1 - 1, 2)$, etc. so that each time we add a column to the parametrization matrix $\mathcal{A}_S$, the dimension of the state polytope grows by at least one. The ideal of the scroll $S(1, \ldots, 1)$ is just the ideal of 2-minors of a generic $2 \times c$ matrix. The minors form a universal Gröbner basis for the ideal which is a reduced Gröbner basis of the ideal with respect to every term order. Hence, the state polytope is a Minkowski sum of the Newton polytopes of the minors (Cor. 2.9. in [10]), a permutohedron $\Pi_{2,c}$ ([2],[12]). Its dimension is $c - 1$.

By induction,

$$\dim \operatorname{State}(S(n_1-1,\dots,n_c-1)) \geq n_1-2+n_2-1+\cdots+n_c-1=\sum n_i-c-1.$$

On the other hand, the ideal of the scroll is homogeneous with respect to the grading given by all the rows of $\mathcal{A}_S$. There are $c+1$ independent rows, thus the vertices of the state polytope lie in $c+1$ hyperplanes, and the claim follows. □

Let us conclude with an example.

Example 5.4. Let S be the scroll $S(5,6)$. Its defining ideal I_S is the ideal of 2-minors of the matrix

$$M := \begin{bmatrix} x_1 & \dots & x_5 & y_1 & \dots & y_6 \\ x_2 & \dots & x_6 & y_2 & \dots & y_7 \end{bmatrix}.$$

The matrix $\mathcal{A}$ providing the parametrization of the scroll is

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & 2 & \dots & 6 & 1 & \dots & 7 \end{bmatrix}.$$

The number and degrees of elements in the universal Gröbner basis of the ideal $I_{\mathcal{A}}$ can be found in the Table of degrees. The primitive colored partition identity of maximal degree is

$$\begin{aligned} &1_1+1_1+1_1+1_1+1_1+1_1+7_2+7_2+7_2+7_2+7_2 \\ &=1_2+1_2+1_2+1_2+1_2+6_1+6_1+6_1+6_1+6_1+6_1. \end{aligned}$$

The corresponding binomial in the ideal $I_{\mathcal{A}}$ is

$${x_1}^6{y_7}^5={y_1}^5{x_6}^6.$$

The state polytope of the ideal $I_{\mathcal{A}}$ is 10-dimensional.

There exist primitive elements that are not circuits. In fact, using [1], we can see that there is a circuit in every degree from 2 to 11 except degree 10, but the number of circuits in each degree is considerably smaller then the number of primitive binomials.

Acknowledgments

The author would like to thank Bernd Sturmfels for suggesting a generalization of primitive partition identities, and my advisor, Uwe Nagel, for his continuous support and guidance.

References

[1] 4ti2 team, *4ti2—A software package for algebraic, geometric and combinatorial problems on linear spaces*, Available at www.4ti2.de.

[2] D. Bernstein and A. Zelevinsky, *Combinatorics of Maximal Minors*, J. Algebraic Comb. **2** (1993), no. 2, 111–121.

[3] J. L. David Cox and D. O'Shea, Using Algebraic Geometry, Vol. 185 of *Graduate Texts in Mathematics*, Springer-Verlag, Berlin (1995).

[4] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Vol. 150 of *Graduate Texts in Mathematics*, Springer-Verlag, Berlin (1995).

[5] D. Eisenbud and J. Harris, *On varieties of minimal degree (a centennial account)*, Proc. Sympos. Pure Math. **46** (1987), no. Part 1, 3–13.

[6] R. Hemmecke and K. A. Nairn, *On the Gröbner complexity of matrices* (2007).

[7] P. D. Ron Graham and B. Sturmfels, *Primitive partition identities*, in T. S. D.Miklós, V.T. Sós, editor, Combinatorics, Paul Erdos is Eighty, 173–192, Janos Bolyai Mathematical Society, Budapest, Hungary (1996).

[8] F. Santos and B. Sturmfels, *Higher Lawrence configurations*, J. Comb. Theory Ser. A **103** (2003), no. 1, 151–164.

[9] J. L. Serkan Hoşten and R. Thomas, editors, Trends in Optimization, Vol. 61 of *Proceedings of Symposia in Applied Mathematics*, Amer. Math. Soc. (2004).

[10] B. Sturmfels, Gröbner bases and convex polytopes, Vol. 8 of *University Lecture Series*, Amer. Math. Soc. (1996).

[11] B. Sturmfels and R. R. Thomas, *Variation of cost functions in integer programming*, Math. Program. **77** (1997), no. 3, 357–387.

[12] B. Sturmfels and A. Zelevinsky, *Maximal minors and their leading terms*, Adv. Math. **98** (1993), no. 1, 65–112.

Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, IL, 60607 USA

E-mail address: `petrovic@math.uic.edu`

URL: `www.math.uic.edu/ petrovic`