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A LEVEL SET ANALYSIS OF THE WITTEN SPINOR WITH APPLICATIONS TO CURVATURE ESTIMATES

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Abstract. We analyze the level sets of the norm of the Witten spinor in an asymptotically flat Riemannian spin manifold of positive scalar curvature. Level sets of small area are constructed. We prove curvature estimates which quantify that, if the total mass becomes small, the manifold becomes flat with the exception of a set of small surface area. These estimates involve either a volume bound or a spectral bound for the Dirac operator on a conformal compactification, but they are independent of the isoperimetric constant.

1. Introduction and Statement of Results

Asymptotically flat manifolds of positive scalar curvature describe isolated time-symmetric gravitating systems in general relativity. The main point of mathematical interest is to explore the connections between the total mass and the local geometry of the manifold. A common general method for analyzing related questions is to consider a flow of hypersurfaces, under which a certain quasi-local mass functional is monotone. The most prominent example is the inverse mean curvature flow, under which the Hawking mass is monotone [6]. In order to conveniently parametrize the hypersurfaces, one often represents the hypersurfaces as level sets of a real-valued function ϕ, which is then a solution of a suitable partial differential equation on the manifold. In this paper, we follow this approach, taking for the function ϕ the norm $|\psi|$ of a Witten spinor in an asymptotically flat spin manifold M. This is particularly simple because the well-known existence of the Witten spinors ensures global existence of the corresponding flow. Nevertheless, this flow has nice and useful properties, above all that, in analogy to a monotonicity property, integrating $|D\phi|^2$ over the set $\{x \in M \mid \phi(x) < \tau\}$ gives a *convex* function in τ.

As an application we construct level sets $\{x \mid \phi(x) = t\}$ of small area. Combining this result with the curvature estimates [3, 4], we prove estimates of the following type: There is an exceptional set $\Omega \subset M$ of small surface area $|\partial\Omega|$ such that on $M \setminus \Omega$ the Riemann tensor is small in an L^2-sense. Here by "small" we mean that the upper bounds involve positive powers of the total mass m, and thus tend to zero as $m \searrow 0$. These curvature estimates are stated in two versions, either involving a volume bound, or, using methods and results from [5], involving a spectral bound for the Dirac operator on a conformal compactification of M.

We now introduce the mathematical framework and state our results. Let (M^n, g) be a complete Riemannian spin manifold of dimension $n \geq 3$. We assume that the

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scalar curvature of M is *non-negative* and integrable. Furthermore, we assume that M is *asymptotically flat*, for simplicity with one asymptotic end. Thus there is a compact set $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus B_\rho(0)$, $\rho > 0$, such that

$$(\Phi_* g)_{ij} \;=\; \delta_{ij} \;+\; \mathcal{O}(r^{2-n})\,, \quad \partial_k (\Phi_* g)_{ij} \;=\; \mathcal{O}(r^{1-n})\,, \quad \partial_{kl} (\Phi_* g)_{ij} \;=\; \mathcal{O}(r^{-n})\,.$$

Under these assumptions, the *total mass* of the manifold is defined by [1]

$$m \;=\; \frac{1}{c(n)} \lim_{\rho \to \infty} \int_{S_\rho} \left(\partial_j (\Phi_* g)_{ij} - \partial_i (\Phi_* g)_{jj} \right) d\Omega^i\,, \tag{1.1}$$

where $c(n) > 0$ is a normalization constant and $d\Omega^i$ denotes the product of the volume form on $S_\rho \subset \mathbb{R}^n$ by the i-th component of the normal vector on S_ρ.

Spinors are very useful for the analysis of asymptotically flat spin manifolds. The basic identity is the *Lichnerowicz-Weitzenböck formula*

$$\mathcal{D}^2 \;=\; -\nabla^2 + \frac{s}{4}\,, \tag{1.2}$$

where $\mathcal{D}$ is the Dirac operator, ∇ is the spin connection, and s denotes scalar curvature. Witten [12] considered solutions of the Dirac equation with constant boundary values ψ_0 in the asymptotic end,

$$\mathcal{D}\psi \;=\; 0\,, \qquad \lim_{|x| \to \infty} \psi(x) \;=\; \psi_0\,, \tag{1.3}$$

where ψ is a smooth section of the spinor bundle SM. In [9, 2] it is proven that for any ψ_0, this boundary value problem has a unique solution. We refer to ψ as the *Witten spinor* with boundary values ψ_0. For a Witten spinor, the Lichnerowicz-Weitzenböck formula implies that

$$\nabla_i \langle \psi, \nabla^i \psi \rangle \;=\; |\nabla \psi|^2 + \frac{s}{4}\, |\psi|^2\,. \tag{1.4}$$

Integrating over M, applying Gauss' theorem and relating the boundary values at infinity to the total mass (where we choose $c(n)$ in (1.1) appropriately), one obtains the identity [12, 9, 2]

$$\int_M \left(|\nabla \psi|^2 + \frac{s}{4}\, |\psi|^2 \right) d\mu_M \;=\; m\,. \tag{1.5}$$

This identity immediately implies the *positive mass theorem* for spin manifolds (for the positive mass theorem on non-spin manifolds see [11, 8]).

In this paper, we consider the *level sets* of the norm of a Witten spinor ψ. We introduce the function $\phi = |\psi|$ and set

$$\tau_0 \;=\; \inf_M \phi\,, \qquad \tau_1 \;=\; \sup_M \phi\,.$$

For any $\tau \in [\tau_0, \tau_1]$ we define the set

$$\Omega(\tau) \;=\; \{ x \in M \,|\, \phi(x) < \tau \}\;.$$

Clearly, the sets $\Omega(\tau)$ are open and form an increasing family in the sense that $\tau' \leq \tau$ implies $\Omega(\tau) \subset \Omega(\tau')$. Moreover, the boundary of $\Omega(\tau)$ is the level set $\{x \,|\, \phi(x) = \tau\}$.

We also introduce the two functions

$$m(\tau) \;=\; \int_{\Omega(\tau)} \Big(|\nabla\psi|^2 + \frac{s}{4}\,|\psi|^2 \Big)\, d\mu_M \tag{1.6}$$

$$F(\tau) \;=\; \int_{\Omega(\tau)} |D\phi|^2\, d\mu_M\,. \tag{1.7}$$

Since the integrands are non-negative, it is obvious that these functions are monotone increasing. Furthermore, $m(\tau_0) = 0 = F(\tau_0)$. Comparing (1.6) with (1.5), one sees that

$$m(\tau_1) \;=\; m\,;$$

this is why we refer to $m(\tau)$ as the *mass function*. Furthermore, the Kato inequality $|D|\psi|| \le |\nabla\psi|$ implies that

$$F(\tau) \;\le\; m(\tau)\,. \tag{1.8}$$

Our main result relates F' to m and makes a convexity statement.

Theorem 1.1. *The function $F(\tau) : [\tau_0, \tau_1] \to \mathbb{R}$ is convex and differentiable almost everywhere. It satisfies for almost all τ the identity*

$$\frac{dF(\tau)}{d\tau} \;=\; \frac{1}{\tau}\, m(\tau)\,. \tag{1.9}$$

As an immediate application, this theorem implies the inequalities

$$\frac{m(\tau')}{\tau'} \;\le\; \frac{F(\tau) - F(\tau')}{\tau - \tau'} \;\le\; \frac{m(\tau)}{\tau} \qquad \text{for almost all } \tau, \tau' \in [\tau_0, \tau_1] \text{ with } \tau' < \tau\,,$$

giving useful information on the behavior of the Witten spinor. For example, setting $\tau' = \tau_0$, the right inequality yields the following upper bound for τ_0,

$$\tau_0 \;\le\; \tau \left(1 - \frac{F(\tau)}{m(\tau)} \right).$$

Inequalities of this type seem surprising. However, one should keep in mind that in all interesting applications, the function F is difficult to compute, and therefore these inequalities are of limited practical value.

Here we focus on applications of the above theorem to *curvature estimates* in asymptotically flat manifolds in the spirit of [3, 4]. The main point is that we now obtain estimates which do not depend on the isoperimetric constant, and where the exceptional set has small surface area (instead of small volume as in [3, 4]). Here we state the results in the physically interesting case of dimension $n = 3$, but we also prove similar results in general dimension (see Theorems 3.3 and 4.5 below). We begin with a curvature estimate assuming a volume bound for some set $\Omega(t_1) \setminus \Omega(t_0)$.

Theorem 1.2. *Let (M^3, g) be a complete, asymptotically flat manifold whose scalar curvature is non-negative and integrable. For any Witten spinor ψ (1.3) and any interval $[t_0, t_1] \subset (0, 1]$, there is $t \in [t_0, t_1]$ with the following properties. The level set $|\psi| = t$ is a submanifold of M, whose 2-volume $A(t)$ is bounded by*

$$A(t) \;\le\; \sqrt{F(t_1) - F(t_0)}\; \frac{\sqrt{V(t_1) - V(t_0)}}{t_1 - t_0}\,, \tag{1.10}$$

where $V(t) := \mu_M(\Omega(t))$. On the set $M \setminus \Omega(t)$, the Riemann tensor satisfies the inequality

$$\int_{M\setminus\Omega(t)} |R|^2 \;\leq\; \frac{m\,c_1}{t^2}\,\sup_M |R| \;+\; \frac{\sqrt{m}\,c_2}{t^2}\,\|\nabla R\|_{L^2(M)}$$

with constants c_1, c_2 which are independent of the geometry of M.

Note that by (1.8), we can always bound F by the total mass. Furthermore, it is in most applications sufficient to drop the term $V(t_0)$ in (1.10) and to choose $t_0 = t_1/2$. This gives the following corollary.

Corollary 1.3. *Let (M^3, g) be a complete, asymptotically flat manifold whose scalar curvature is non-negative and integrable. For any Witten spinor ψ (1.3) and any $t_1 \in (0,1]$, there is $t \in [\frac{t_1}{2}, t_1]$ with the following properties. The level set $|\psi| = t$ is a submanifold of M whose 2-volume $A(t)$ is bounded by*

$$A(t) \;\leq\; \frac{2\sqrt{m\,V(t_1)}}{t_1}\,.$$

On the set $M \setminus \Omega(t)$, the Riemann tensor satisfies the inequality

$$\int_{M\setminus\Omega(t)} |R|^2 \;\leq\; \frac{m\,c_1}{t_1^2}\,\sup_M |R| \;+\; \frac{\sqrt{m}\,c_2}{t_1^2}\,\|\nabla R\|_{L^2(M)}\,.$$

The remaining question is how to control the volume $V(t_1) - V(t_0)$. We here propose the method to work with a *spectral bound* for the Dirac operator *on a conformal compactification* of M. As in [5] we assume for simplicity that M is *asymptotically Schwarzschild*, although the method should apply to more general asymptotically flat manifolds as well. Thus under the diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus B_\rho(0)$ the metric becomes the Schwarzschild metric,

$$(\Phi_* g)_{ij} \;=\; \Big(1 + \frac{m}{2r}\Big)^4\,\delta_{ij}\,. \tag{1.11}$$

We point compactify M by a conformal transformation

$$\tilde{g} \;=\; \lambda^2\, g\,,$$

in such a way that the geometry of K remains unchanged, the scalar curvature stays non-negative, and the compactification of the asymptotic end is isometric to a cap $C_R \subset S^n_\sigma$ of a sphere of radius σ (for details see [5] and Section 4). The Dirac operator on the compactification $(\bar{M}, \tilde{g})$ is denoted by $\tilde{\mathcal{D}}$.

Theorem 1.4. *Let (M^3, g) be a complete, manifold of non-negative scalar curvature such that $M \setminus K$ is isometric to the Schwarzschild geometry (1.11). Then there is $t \in (\frac{1}{4}, \frac{1}{2})$ such that the level set $|\psi| = t$ is a submanifold of M. Its 2-volume $A(t)$ is bounded by*

$$A(t) \;\leq\; c_0\,\sqrt{m}\,\frac{(\rho+m)^{\frac{3}{2}}}{\sigma\,\inf\mathrm{spec}|\tilde{\mathcal{D}}|}\,.$$

On the set $M \setminus \Omega(t)$, the Riemann tensor satisfies the inequality

$$\int_{M\setminus\Omega(t)} |R|^2 \;\leq\; m\,c_1\,\sup_M |R| \;+\; \sqrt{m}\,c_2\,\|\nabla R\|_{L^2(M)}$$

with constants c_0, c_1 and c_2, which are independent of the geometry of M.

2. Level Set Analysis

Let ψ be a Witten spinor (1.3). By linearity, it is no loss of generality to always normalize ψ_0 by one,

$$|\psi_0| \;=\; 1\,. \tag{2.1}$$

We introduce the level sets of ϕ by

$$L(\tau) \;=\; \{x \in M \mid \phi(x) = \tau\}\,.$$

We call $\tau \in [\tau_0, \tau_1]$ a *regular value* if $D\phi$ has no zeros on $L(\tau)$, otherwise it is called a *singular value*. According to Sard's lemma, the singular values form a set of Lebesgue measure zero in $[\tau_0, \tau_1]$. If τ is a regular value, the implicit function theorem yields that $L(\tau)$ is a smooth submanifold of M of codimension one. In this case, we denote the induced measure on $L(\tau)$ by $d\mu_{L(\tau)}$.

We first motivate our method, neglecting the subtle issue of the singular values. A promising idea for getting information on the level sets is to integrate a smooth function h on M (which may be a curvature expression or an expression involving the Witten spinor) over the level sets,

$$\int_{L(\tau)} h(x)\, d\mu_{L(\tau)}(x)\,,$$

and to analyze a "flow equation" for this expression. To derive the flow equation, we first apply Gauss' theorem to obtain

$$\int_{L(\tau)} h\, d\mu_{L(\tau)} \;=\; \int_{\Omega(\tau)} \nabla_i \left(h\, \nu^i\right) d\mu_M\,,$$

where $\nu^i = (D^i\phi)/|D\phi|$ denotes the outer normal. Now the co-area formula yields

$$\int_{L(\tau)} h\, d\mu_{L(\tau)} \;=\; \int_{\tau_0}^{\tau} d\sigma \int_{L(\sigma)} \frac{1}{|D\phi|}\, \nabla_i \left(\frac{h\, D^i\phi}{|D\phi|}\right) d\mu_{L(\sigma)}\,,$$

and differentiating with respect to τ gives for any regular value τ the differential equation

$$\frac{d}{d\tau} \int_{L(\tau)} h\, d\mu_{L(\tau)} \;=\; \int_{L(\tau)} \frac{1}{|D\phi|}\, \nabla_i \left(\frac{h\, D^i\phi}{|D\phi|}\right) d\mu_{L(\tau)}\,. \tag{2.2}$$

The basic problem is that the right hand side will in general involve new geometric quantities, which are difficult to control. For example, setting $h \equiv 1$, we obtain the area functional $A(\tau) := \mu_{L(\tau)}(L(\tau))$. Its flow equation is

$$\frac{d}{d\tau} A(\tau) \;=\; \int_{L(\tau)} \frac{1}{|D\phi|}\, \nabla_i \left(\frac{D^i\phi}{|D\phi|}\right) d\mu_{L(\tau)}\,.$$

Here in the integrand the well-known mean curvature operator appears. But the mean curvature of the level sets is not known, and it seems difficult to get information on mean curvature. Therefore, in order to make use of (2.2), we must look for a special function h for which the right side of (2.2) has nice properties. Choosing $h = |D\phi|$, we get the simple equation

$$\frac{d}{d\tau} \int_{L(\tau)} |D\phi|\, d\mu_{L(\tau)} \;=\; \int_{L(\tau)} \frac{\Delta\phi}{|D\phi|}\, d\mu_{L(\tau)}\,.$$

The integral on the left equals the function f which we shall define below. It is preferable to introduce it using volume integrals over $\Omega(\tau)$ instead of surface integrals, because such volume integrals make sense even if τ is a singular value. This motivates the following constructions.

A direct calculation using the Licherowicz-Weitzenböck formula (1.2) gives

$$\begin{aligned}
\Delta\phi^2 &= \Delta\langle\psi,\psi\rangle \;=\; 2\,\mathrm{Re}\langle\nabla^2\psi,\psi\rangle \;+\; 2\,|\nabla\psi|^2 \;=\; \frac{s}{2}\,|\psi|^2 + 2\,|\nabla\psi|^2 \\
\nabla\phi &= \frac{\nabla\phi^2}{2\phi} \\
\Delta\phi &= \frac{\Delta\phi^2}{2\phi} - \frac{(\nabla\phi^2)^2}{4\phi^2} \;=\; \frac{s}{4}\,\phi \;+\; \frac{|\nabla\psi|^2}{\phi} \;-\; \frac{|\mathrm{Re}\langle\nabla\psi,\psi\rangle|^2}{\phi^3}\,.
\end{aligned}$$

Applying the the Schwarz inequality $|\mathrm{Re}\langle\nabla\psi,\psi\rangle| \leq |\nabla\psi|\,\phi$, we obtain the inequality

$$\Delta\phi \;\geq\; \frac{s}{4}\,\phi\,. \tag{2.3}$$

It is an important observation that ϕ is subharmonic. In particular, we can apply the maximum principle to conclude that $|\phi| \leq |\psi_0|$. Comparing with (2.1), we find that

$$\tau_1 \;=\; 1\,. \tag{2.4}$$

We introduce the function $f(\tau)$ by

$$f(\tau) \;=\; \int_{\Omega(\tau)} \Delta\phi\, d\mu_M\,. \tag{2.5}$$

Lemma 2.1. *The function f is monotone increasing and left-sided continuous, i.e. for all $\tau \in (\tau_0, 1]$,*

$$\lim_{\tau' \nearrow \tau} f(\tau') \;=\; f(\tau)\,.$$

For almost all $\tau \in [\tau_0, 1]$,

$$f(\tau) \;=\; \int_{L(\tau)} |D\phi|\, d\mu_{L(\tau)}\,. \tag{2.6}$$

Proof. We write f in the form

$$f(\tau) \;=\; \int_M g\, d\mu_M \quad \text{with} \quad g(x) \;:=\; \Delta\phi(x)\,\chi_{\Omega(\tau)}(x)\,,$$

where χ denotes the characteristic function. According to (2.3), the integrand is non-negative. The monotonicity of f is obvious because the family $\Omega(\tau)$ is increasing. To prove left-sided continuity, we note that for all $\tau' < \tau$

$$f(\tau) - f(\tau') \;=\; \int_M \Delta\phi(x)\,\chi_{\Omega(\tau)\setminus\Omega(\tau')}(x)\, d\mu_M\,.$$

As $\tau' \nearrow \tau$, the characteristic function tends to zero pointwise. Hence in this limit, $f(\tau) - f(\tau')$ tends to zero due to Lebesgue's monotone convergence theorem.

To prove (2.6), we let τ be a regular value of ϕ. Then the outer normal on $L(\tau)$ is given by

$$\nu \;=\; \frac{D\phi}{|D\phi|}\,.$$

Thus applying Gauss' theorem in (2.5), we obtain

$$f(\tau) \;=\; \int_{\Omega(\tau)} \nabla_i \left(D^i\phi\right) \;=\; \int_{L(\tau)} (D^i\phi)\,\nu_i\,d\mu_{L(\tau)} \;=\; \int_{L(\tau)} |D\phi|\,d\mu_{L(\tau)}\,. \qquad \square$$

Proof of Theorem 1.1. The co-area formula yields

$$F(\tau) \;=\; \int_{\tau_0}^{\tau} \left[\int_{L(\sigma)} |D\phi|\,d\mu_{L(\sigma)}\right] d\sigma\,, \tag{2.7}$$

where the square bracket is defined almost everywhere according to Sard's lemma. Applying Lemma 2.1, the square bracket coincides with f and is thus monotone increasing and left-sided continuous. This implies that F is $C^0([\tau_0,1])$, and is differentiable almost everywhere with $F'(\tau) = f(\tau)$. Since f is monotone increasing, we conclude that F is convex.

It remains to show that for almost all $\tau \in [\tau_0, 1]$,

$$f(\tau) \;=\; \frac{1}{\tau}\,m(\tau)\,.$$

According to Sard's lemma, we may assume that τ is a regular value. Then, applying (1.4) and Gauss' theorem in (1.6), we obtain

$$m(\tau) \;=\; \int_{\Omega(\tau)} \mathrm{Re}\,\nabla_i\,\langle\psi, \nabla^i\psi\rangle\,d\mu_M \;=\; \int_{L(\tau)} \mathrm{Re}\,\langle\psi, \nabla^\nu\psi\rangle\,d\mu_{L(\tau)}\,.$$

Using furthermore that $D\phi = \mathrm{Re}\langle\psi, \nabla\psi\rangle/\phi$, and that $D\phi$ points in normal direction, we obtain

$$m(\tau) \;=\; \int_{L(\tau)} \phi\,D_\nu\phi\,d\mu_{L(\tau)} \;=\; \tau\int_{L(\tau)} |D\phi|\,d\mu_{L(\tau)} \;=\; \tau\,f(\tau)\,. \qquad \square$$

Next we want to derive an inequality involving the volume of the sets $\Omega(\tau)$ and the area of the level sets. We define the volume $V(\tau)$ by

$$V(\tau) \;=\; \int_{\Omega(\tau)} d\mu_M\,, \qquad \tau \in [\tau_0, 1]\,.$$

The area of the level sets is only defined if τ is a regular value, and we extend the area function by zero to the singular values,

$$A(\tau) \;=\; \begin{cases} \mu_{L(\tau)}(L(\tau)) & \text{if } \tau \text{ is a regular value} \\ 0 & \text{otherwise}\,. \end{cases}$$

Proposition 2.2. *For all $\tau, \tau' \in [\tau_0, \tau_1]$ with $\tau' < \tau$, the following inequality holds:*

$$\left(\int_{\tau'}^{\tau} A(\sigma)\,d\sigma\right)^2 \;\le\; (V(\tau) - V(\tau'))\,(F(\tau) - F(\tau'))\,.$$

Proof. The function $V(\tau)$ is clearly monotone increasing. Hence there is a unique Borel measure ν such that (see [10, Section I.4])

$$V(\tau) \;=\; \int_{\tau_0}^{\tau} d\nu\,. \tag{2.8}$$

The Lebesgue decomposition theorem [10, Section I.4] allows us to decompose this measure with respect to the Lebesgue measure,

$$d\nu \;=\; g(\tau)\, d\tau + d\nu_{\text{sing}} \quad \text{with} \quad g \in L^1([\tau_0, 1], d\tau)\,. \tag{2.9}$$

If τ is a regular value, we can compute g by differentiation,

$$g(\tau) \;=\; \frac{d}{d\tau}\, V(\tau) \;=\; \frac{d}{d\tau} \int^\tau d\sigma \int_{L(\sigma)} \frac{1}{|D\phi|}\, d\mu_{L(\sigma)} \;=\; \int_{L(\tau)} \frac{1}{|D\phi|}\, d\mu_{L(\tau)}\,.$$

Integrating (2.9) from τ' to τ and using (2.8), we find that

$$V(\tau) - V(\tau') \;\geq\; \int_{\tau'}^\tau g(\sigma)\, d\sigma\,. \tag{2.10}$$

We introduce the function Θ_{reg} by

$$\Theta_{\text{reg}}(\tau) \;=\; \begin{cases} 1 & \text{if } \tau \text{ is a regular value} \\ 0 & \text{otherwise}\,. \end{cases}$$

Then the Schwarz inequality yields

$$\begin{aligned} \int_{\tau'}^\tau A(\sigma)\, d\sigma \;&=\; \int_{\tau'}^\tau \Theta_{\text{reg}}(\sigma)\, d\sigma \int_{L(\sigma)} d\mu_{L(\sigma)} \\ &\leq\; \left(\int_{\tau'}^\tau \Theta_{\text{reg}}(\sigma)\, d\sigma \int_{L(\sigma)} |D\phi|\, d\mu_{L(\sigma)} \right)^{\frac{1}{2}} \left(\int_{\tau'}^\tau \Theta_{\text{reg}}(\sigma)\, d\sigma \int_{L(\sigma)} \frac{1}{|D\phi|}\, d\mu_{L(\sigma)} \right)^{\frac{1}{2}} \\ &=\; \left(\int_{\tau'}^\tau f(\sigma)\, d\sigma \right)^{\frac{1}{2}} \left(\int_{\tau'}^\tau g(\sigma)\, d\sigma \right)^{\frac{1}{2}}\,. \end{aligned}$$

Taking the square and using (2.7) as well as (2.10) gives the result. □

3. Applications to Curvature Estimates

For clarity, we begin with the simpler and physically interesting case of dimension $n = 3$, the extension to higher dimension will be given afterwards. Our starting point is the following integral estimate as derived in [3].

Lemma 3.1. *Suppose ψ is a Witten spinor (1.3) in a complete, asymptotically flat manifold (M^3, g) whose scalar curvature is non-negative and integrable. Then*

$$\int_M |R|^2\, |\psi|^2\, d\mu_M \;\leq\; m\, c_1 \sup_M |R| \;+\; \sqrt{m}\, c_2\, \|\nabla R\|_{L^2(M)}\,,$$

where the constants c_1 and c_2 are independent of the geometry of M.

Proof of Theorem 1.2. If $t_0 < \tau_0$, we set $t = t_0$. Then $L(t)$ is empty and $A(t) = 0$. If conversely $t_0 \geq \tau_0$, we apply Proposition 2.2 with $\tau' = t_0$ and $\tau = t_1$. According to the mean value theorem, there is a subset of $[t_0, t_1]$ of positive Lebesgue measure on which

$$(t_1 - t_0)\, A \;\leq\; \sqrt{(V(t_1) - V(t_0))\, (F(t_1) - F(t_0))}\,.$$

Out of this subset we choose a regular value t. This gives (1.10). On the set $M \setminus \Omega(t)$, the Witten spinor clearly satisfies the bound $|\psi| \geq t$. Using this in Lemma 3.1 gives the result. □

Let us now prove the extension of Theorem 1.2 to higher dimension, Theorem 3.3. If $n \geq 4$, the statement of Lemma 3.1 no longer holds. Instead, we must work with the spinor operator P_x as introduced in [4, Section 4]: We choose an orthonormal basis of constant spinors $(\psi_0^i)_{i=1,\dots,N}$, $N = 2^{[n/2]}$, $\langle \psi_0^i, \psi_0^j \rangle = \delta^{ij}$, and denote the corresponding solutions of the boundary problem (1.3) by $(\psi^i)_{i=1,\dots,N}$. We define the *spinor operator* P_x by

$$P_x \;:\; S_xM \longrightarrow S_xM \;:\; \psi \longmapsto \sum_{i=1}^{N} \langle \psi_x^i,\, \psi \rangle\, \psi_x^i \,. \tag{3.1}$$

Clearly, this operator is non-negative. We set

$$p(x) \;=\; \inf \Big\{ \langle \chi, P_x \chi \rangle \;\Big|\; \chi \in S_xM,\; |\chi| = 1 \Big\} \;\geq\; 0 \,. \tag{3.2}$$

Our starting point is the following curvature estimate, which is an improvement of the estimates in [4].

Lemma 3.2. *Suppose that (M^n, g), $n \geq 4$, is a complete, asymptotically flat manifold whose scalar curvature is non-negative and integrable. Then the Riemann tensor R and the infimum of the spinor operator (3.1, 3.2) satisfy the inequality*

$$\int_M |R|^2\, p(x)\, d\mu_M \;\leq\; m\, c_1(n) \sup_M |R| \;+\; \sqrt{m}\, c_2(n)\, \|\nabla R\|_{L^2(M)} \,,$$

where the constants c_1 and c_2 depend on the dimension, but are independent of the geometry of M.

Proof. In [4, Corollary 3.2, Lemma 5.1] it was proved that, choosing an orthonormal frame $(s_\alpha)_{\alpha=1,\dots,n}$,

$$\int_M \sum_{\alpha,\beta=1}^{n} \mathrm{Tr}\big(R^S(s_\alpha, s_\beta)^2\, P(x) \big)\, d\mu \;\leq\; m\, c_1(n) \sup_M |R| \;+\; \sqrt{m}\, c_2(n)\, \|\nabla R\|_{L^2(M)} \,,$$

where R^S is the curvature of the spin connection, which is related to the Riemann tensor by

$$R^S(X,Y)\, \psi = \frac{1}{4} \sum_{\alpha,\beta=1}^{n} R(X,Y,s_\alpha,s_\beta) s_\alpha \cdot s_\beta \cdot \psi \,.$$

Introducing the abbreviation

$$\mathcal{R}^2 \;=\; \sum_{\alpha,\beta=1}^{n} R^S(s_\alpha, s_\beta)^2 \,,$$

the operator $\mathcal{R}^2(x)$ acts on $S_x(M)$ as a positive operator. Its trace is a positive multiple of the norm squared of the Riemann tensor,

$$\mathrm{Tr}\big(\mathcal{R}^2\big) \;=\; c(n)\, |R|^2 \,.$$

Hence

$$\mathrm{Tr}\big(\mathcal{R}^2\, P(x)\big) \;=\; c(n)\, |R|^2\, p(x) \;+\; \mathrm{Tr}\big(\mathcal{R}^2\, (P(x) - p(x))\big) \;\geq\; c(n)\, |R|^2\, p(x) \,,$$

where in the last step we used that the trace of the product of two positive operators is positive. □

Theorem 3.3. *Let (M^n, g), $n \geq 4$, be a complete, asymptotically flat manifold whose scalar curvature is non-negative and integrable. Suppose that for an interval $[t_0, t_1] \subset (0,1]$ there is a constant C such that every Witten spinor (1.3) satisfies the volume bound*

$$V(t_1) - V(t_0) \;\leq\; C\,.$$

Then there is an open set $\Omega \subset M$ with the following properties. The $(n-1)$-dimensional Hausdorff measure μ_{n-1} of the boundary of Ω is bounded by

$$\mu_{n-1}(\partial\Omega) \;\leq\; \sqrt{m}\, c_0(n, t_0)\, \frac{\sqrt{C}}{t_1 - t_0}\,.$$

On the set $M \setminus \Omega$, the Riemann tensor satisfies the inequality

$$\int_{M \setminus \Omega} |R|^2 \;\leq\; m\, c_1(n, t_0) \sup_M |R| \;+\; \sqrt{m}\, c_2(n, t_0)\, \|\nabla R\|_{L^2(M)}\,.$$

Here the constants c_0, c_1 and c_2 depend on the dimension and on t_0, but they are independent of the geometry of M.

Proof. For given $x \in M$ we introduce the mapping

$$B \;:\; \mathbb{C}^N \to S_x M \;:\; \zeta \mapsto \sum_{i=1}^N \zeta_i\, \psi_i(x)\,.$$

The a-priori bound $|\psi(x)| \leq 1$ for all Witten spinors (see the argument before (2.4)) yields that $\|B\| \leq 1$. Furthermore, the spinor operator can be written as $P_x = BB^*$. Since the operators BB^* and B^*B are both Hermitian and have the same spectrum, we find

$$p(x) \;=\; \inf \operatorname{spec}(BB^*) \;=\; \inf \operatorname{spec}(B^*B) \;=\; \inf_{\zeta \text{ with } |\zeta|=1} |B\zeta|^2\,.$$

We choose a finite number of points $\zeta^1, \ldots, \zeta^L$ on the unit sphere S^N_1 in $\mathbb{C}^N$ such that the balls $B_{t_0/2}(\zeta^1), \ldots, B_{t_0/2}(\zeta^L)$ cover S^N_1 (with a constant $L = L(n, t_0)$). For every $a \in \{1, \ldots, L\}$, we let ψ^a be the Witten spinor (1.3) with boundary conditions $\psi_0 = \zeta^a$. Then $\psi^a = B\zeta^a$. Exactly as in the proof of Theorem 1.2, for every ψ^a we can choose a regular value $t \in [t_0, t_1]$ such that

$$A(t) \;\leq\; \frac{\sqrt{mC}}{t_1 - t_0}\,.$$

We also denote $\Omega(t)$ by Ω^a and set $\Omega = \cup_{a=1}^L \Omega^a$. Then $\partial\Omega$ is a subset of $\cup_{a=1}^L \partial\Omega^a$, and thus its Hausdorff measure is bounded by

$$\mu_{n-1}(\partial\Omega) \;\leq\; L\, \frac{\sqrt{mC}}{t_1 - t_0}\,.$$

In view of the estimates of Lemma 3.2, it remains to show that

$$p(x) \;\geq\; \frac{t_0^2}{4} \qquad \forall x \in M \setminus \Omega\,.$$

For any $\zeta \in S_1^N$ we can choose an index $a \in \{1, \dots, L\}$ such that $|\zeta - \zeta^a| < t_0/2$. Thus

$$\begin{aligned} |B\zeta| &\geq |B\zeta^a| - |B(\zeta - \zeta^a)| \;=\; |\psi^a| - |B(\zeta - \zeta^a)| \\ &\geq |\psi^a| - \|B\|\,|\zeta - \zeta^a| \;\geq\; t_0 - \|B\|\,|\zeta - \zeta^a| \;\geq\; \frac{t_0}{2}\,, \end{aligned}$$

where in the last step we used that $\|B\| \leq 1$. □

4. Estimates in a Conformal Compactification

This section is devoted to the proof of Theorem 1.4 and its generalization to higher dimension, Theorem 4.5. By rescaling, we can arrange that the total mass m equals two. We assume that (M^n, g), $n \geq 3$, is asymptotically Schwarzschild, i.e. there is a diffeomorphism

$$\varphi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_\rho(0)}$$

such that

$$(\varphi_* g)_{ij} \;=\; \left(1 + \frac{1}{|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}\,. \tag{4.1}$$

On $M \setminus K$ we introduce the function $r(x) = |\phi(x)|$. For the point compactification, we choose parameters σ and R in the range

$$\rho \;\leq\; \sigma \;\leq\; R \;<\; c(n)\,\rho \tag{4.2}$$

and consider a function λ with the following properties (for the construction of λ see [5]):

(i) $\lambda_{|K} \;\equiv\; 1$

(ii) $\lambda(x) \;=\; \left(\dfrac{2\sigma^2}{\sigma^2 + r(x)^2}\right) \cdot \left(1 + \dfrac{1}{r(x)^{n-2}}\right)^{-\frac{2}{n-2}}$ on $\phi^{-1}(\mathbb{R}^n \setminus B_R(0))$.

(iii) The scalar curvature corresponding to the conformally changed metric

$$\tilde{g} \;=\; \lambda^2\, g \tag{4.3}$$

is non-negative.

After the conformal change, the region $r > R$ is isometric to a neighborhood of the north pole $\mathfrak{n}$ of the sphere S^n_σ with the north pole removed. Adding the north pole, we obtain the conformal compactification $(\bar{M}, \tilde{g})$. For any $r \geq R$ we introduce the spherical cap

$$C_r \;=\; \overline{(\varphi^{-1}(\mathbb{R}^n \setminus B_r(0)), \tilde{g})} \subset \bar{M}\,.$$

Finally, we denote the geodesic radius of the spherical cap $C_R \subset S^n_\sigma$ by δ.

Our main task is to bound the function ϕ in the asymptotic end from below, see Proposition 4.3. As in [5] we work on the sphere S^n_σ with Sobolev norms which are scaling invariant in σ, namely

$$\|f\|^2_{H^{k,2}(S^n_\sigma)} \;:=\; \sum_{\kappa \text{ with } |\kappa| \leq k} \sigma^{2|\kappa| - n} \int_{S^n_\sigma} \|\nabla^\kappa f(x)\|^2\, dx\,.$$

We let η be a smooth function on $\bar{M}$ with

$$\operatorname{supp} \eta \subset C_R\,, \qquad \eta|_{C_{2R}} \;\equiv\; 1\,.$$

The next lemma is an elliptic estimate for the Dirac operator on S^n_σ, for the proof see [5, Lemma 5.1].

Lemma 4.1. *For any smooth section χ in $S(S^n_\sigma)$,*

$$\|\eta^{k+1}\chi\|^2_{H^{k,2}(S^n_\sigma)} \;\leq\; c(n)\sum_{l=0}^{k}\sigma^{2l-n}\;\|\eta^{l+1}\,\tilde{\mathcal{D}}^l\chi\|^2_{L^2(S^n_\sigma)}\,.$$

For a given Witten spinor ψ we introduce the spinors

$$\psi_{\text{asy}} \;=\; \eta\left(1+\frac{1}{r(x)^{n-2}}\right)^{-\frac{n-1}{n-2}}\psi_0 \qquad \text{on } M\setminus K \tag{4.4}$$

$$\delta\psi \;=\; \psi-\eta\,\psi_{\text{asy}} \qquad \text{on } M\,, \tag{4.5}$$

where ψ_0 is the boundary value of ψ at infinity. The spinor $\delta\psi$ is referred to as the *Witten deviation*. We shall also consider the above spinors on the conformal compactification $(\bar{M},\tilde{g})$. We then denote them with an additional tilde and rescale them as usual by

$$\tilde{\psi} \;=\; \lambda^{\frac{1-n}{2}}\;\psi\,. \tag{4.6}$$

Since the metric on $M\setminus K$ is Schwarzschild, the conformal invariance of the Dirac equation shows that $\mathcal{D}\psi_{\text{asy}}=0$, and therefore also $\tilde{\mathcal{D}}\tilde{\psi}_{\text{asy}}=0$. As a consequence,

$$\tilde{\mathcal{D}}(\widetilde{\delta\psi}) \;=\; -(\tilde{\mathcal{D}}\eta)\,\tilde{\psi}_{\text{asy}} \;=:\; h\,.$$

It is important that the function h appearing here can be given explicitly and is supported inside the spherical cap C_R.

Lemma 4.2. *There is a constant c depending only on n and the ratio δ/σ such that*

$$\sup_{C_{2R}}\left(\lambda^{\frac{1-n}{2}}\;|\delta\psi|\right) \;\leq\; \frac{c}{\sigma\;\inf\text{spec}|\tilde{\mathcal{D}}|}\,.$$

Proof. In view of (4.6), we must estimate the sup-norm of $\widetilde{\delta\psi}$ on C_{2R}. Obviously,

$$\sup_{C_{2R}}|\widetilde{\delta\psi}| \;\leq\; \sup_{C_R}|\eta^{k+1}\widetilde{\delta\psi}|\,.$$

Extending the last function by zero to S^n_σ, we can apply the Sobolev imbedding theorem on the sphere S^n_σ. Thus for sufficiently large k,

$$\sup_{C_R}|\eta^{k+1}\widetilde{\delta\psi}| \;=\; \sup_{S^n_\sigma}|\eta^{k+1}\widetilde{\delta\psi}| \;\leq\; c\,\|\eta^{k+1}\widetilde{\delta\psi}\|_{H^{k,2}(S^2_\sigma)}\,,$$

where $c=c(n)$ is the Sobolev constant on the unit sphere. Applying Lemma 4.1, we obtain

$$\begin{aligned}\sup_{C_{2R}}|\widetilde{\delta\psi}|^2 \;&\leq\; c^2\sum_{l=0}^{k}\sigma^{2l-n}\;\|\eta^{l+1}\,\tilde{\mathcal{D}}^l(\widetilde{\delta\psi})\|^2_{L^2(S^n_\sigma)}\\ &=\; c^2\sigma^{-n}\;\|\eta\,(\widetilde{\delta\psi})\|^2_{L^2(S^n_\sigma)} \;+\; c^2\sum_{l=1}^{k}\sigma^{2l-n}\;\|\eta^{l+1}\,\tilde{\mathcal{D}}^{l-1}h\|^2_{L^2(S^n_\sigma)}\,.\end{aligned}$$

The obtained terms can be estimated as follows,

$$\begin{aligned}
\|\eta\,(\widetilde{\delta\psi})\|^2_{L^2(S^n_\sigma)} &\le \|\widetilde{\delta\psi}\|^2_{L^2(M)} \le \frac{1}{\inf\operatorname{spec}(\tilde{\mathcal{D}}^2)}\,\|h\|^2_{L^2(S^n_\sigma)} \le \frac{c\,\sigma^{n-2}}{\inf\operatorname{spec}(\tilde{\mathcal{D}}^2)} \\
\|\eta^{l+1}\,\tilde{\mathcal{D}}^{l-1}h\|^2_{L^2(S^n_\sigma)} &\le \|\tilde{\mathcal{D}}^{l-1}h\|^2_{L^2(S^n_\sigma)} \le c\,\sigma^{n-2l}\,.
\end{aligned}$$

Putting these estimates together and taking the square root gives the estimate

$$\sup_{C_{2R}}\Big(\lambda^{\frac{1-n}{2}}\,|\delta\psi|\Big) \le c\left[1+\frac{1}{\sigma\,\inf\operatorname{spec}|\tilde{\mathcal{D}}|}\right]. \tag{4.7}$$

Finally, we can use the lower spectral bound (see [5, Proof of Theorem 7.5])

$$\inf\operatorname{spec}(\tilde{D}^2) \le \frac{c}{\sigma^2} \tag{4.8}$$

to drop the first term in the square brackets in (4.7). □

Proposition 4.3. *There is a constant c depending only on n and the quotient δ/σ such that for all $x \in M \setminus K$ with*

$$r(x) \ge r_1 := c\,\sigma\Big(\sigma\inf\operatorname{spec}|\tilde{D}|\Big)^{-\frac{1}{n-1}}\,,$$

the norm of the Witten spinor is bounded from below by

$$\phi(x) \ge \frac{1}{2}\,.$$

Proof. We estimate ϕ from below by

$$\phi(x) = |\psi(x)| \ge |\psi_{\mathrm{asy}}| - |\delta\psi|\,.$$

As is obvious from the definition of ψ_{asy}, (4.4), by choosing c sufficiently large we can arrange that $|\psi_{\mathrm{asy}}| > 3/4$. Hence we need to arrange that $|\delta\psi| < 1/4$. According to Lemma 4.2, this can be achieved by choosing

$$\lambda^{\frac{1-n}{2}} \ge \frac{4c}{\sigma\,\inf\operatorname{spec}|\tilde{\mathcal{D}}|}\,.$$

Using the explicit form of λ in **(ii)** gives the result. □

Next we derive the desired volume bound.

Lemma 4.4. *There is a constant c which depends only on n but is independent of the geometry of M such that*

$$V\Big(\frac{1}{2}\Big) - V\Big(\frac{1}{4}\Big) \le c\,\frac{(\rho+1)^n}{\sigma^2\,\inf\operatorname{spec}(\mathcal{D}^2)}\,.$$

Proof. Since $|\psi| \ge 1/4$ on $\Omega(1/2)\setminus\Omega(1/4)$, we clearly have

$$V\Big(\frac{1}{2}\Big) - V\Big(\frac{1}{4}\Big) \le 16\int_{\Omega(1/2)}|\psi|^2\,d\mu_M\,.$$

Furthermore, applying Proposition 4.3,

$$\int_{\Omega(1/2)}|\psi|^2\,d\mu_M \le \int_{M\setminus C_{r_1}}|\psi|^2\,d\mu_M \le \int_K|\psi|^2\,d\mu_M + \mu\Big(\{x\in M\setminus K \mid r(x)\le r_1\}\Big),$$

where in the last step we used that $|\psi| \leq 1$, (2.4). To the integral over K we apply the weighted L^2-estimates in [5, Corollary 7.6], whereas the additional measure can be estimated by the volume of a Euclidean ball in the asymptotic end,

$$\int_K |\psi|^2 \, d\mu_M \;\leq\; \int_K |\psi|^2 \, d\mu_M \;+\; \int_{M \setminus K} |\delta\psi|^2 \, d\mu_M \;\leq\; c \, \frac{(\rho+1)^n}{\sigma^2 \, \inf \operatorname{spec}(\tilde{D}^2)}$$
$$\mu\Big(\{x \in M \setminus K \mid r(x) \leq r_1\}\Big) \;\leq\; c\, r_1^n \;\leq\; c\,(\rho+1)^n \left(\sigma \inf \operatorname{spec}|\tilde{D}|\right)^{-\frac{n}{n-1}} \;,$$

where in the last step we used that, according to (4.2), σ and ρ have the same scaling. Combining these inequalities with (4.8), we obtain the result. □

Proof of Theorem 1.4. We apply Theorem 1.2 with $t_0 = 1/4$ and $t_1 = 1/2$, using the estimate $F(t_1) - F(t_0) \leq m$. We then put in the estimate of Lemma 4.4. □

Applying Lemma 4.4 in the same way to Theorem 3.3 gives the following result.

Theorem 4.5. *Let (M^n, g), $n \geq 4$, be a complete manifold of non-negative scalar curvature such that $M \setminus K$ is isometric to the Schwarzschild geometry. Then there is an open set $\Omega \subset M$ with the following properties. The $(n-1)$-dimensional Hausdorff measure μ_{n-1} of the boundary of Ω is bounded by*

$$\mu_{n-1}(\partial\Omega) \;\leq\; c_0(n) \sqrt{m} \, \frac{\left(\rho + m^{\frac{1}{n-2}}\right)^{\frac{n}{2}}}{\sigma \, \inf \operatorname{spec}|\tilde{\mathcal{D}}|} \;.$$

On the set $M \setminus \Omega$, the Riemann tensor satisfies the inequality

$$\int_{M \setminus \Omega} |R|^2 \;\leq\; m\, c_1(n) \sup_M |R| \;+\; \sqrt{m}\, c_2(n) \, \|\nabla R\|_{L^2(M)} \;.$$

Here the constants c_0, c_1 and c_2 depend on the dimension, but are independent of the geometry of M.

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