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# CONJECTURE OF TITS TYPE FOR COMPLEX VARIETIES AND THEOREM OF LIE-KOLCHIN TYPE FOR A CONE

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Abstract. First, we formulate and prove Theorem of Lie-Kolchin type for a cone and derive some algebro-geometric consequences. Next, inspired by a recent result of Dinh and Sibony we pose a conjecture of Tits type for a group of automorphisms of a complex variety and verify its weaker version. Finally, applying Theorem of Lie-Kolchin type for a cone, we confirm the conjecture of Tits type for complex tori, hyperkähler manifolds, surfaces, and minimal threefolds.

## 1. Introduction

We often study a group $G$ of automorphisms of a projective variety $X$ through its action on cohomological spaces such as on the Néron-Severi group $\mathrm{NS}(X)$. Set

$$G^* = \mathrm{Im}(G \longrightarrow \mathrm{GL}\,(\mathrm{NS}(X))).$$

Then the famous Tits Alternative Theorem [Ti] says:

(i) either $G^*$ contains a subgroup isomorphic to $\mathbf{Z} * \mathbf{Z}$ (highly noncommutative); or

(ii) $G^*$ contains a Z-connected solvable subgroup $G_1$ of finite index.

Here, $G_1$ is *Z-connected* if its Zariski closure in $\mathrm{GL}\,(\mathrm{NS}(X)_{\mathbf{C}})$ is connected with respect to the Zariski topology.

This *algebraic* property provides us with a useful tool in studying groups of automorphisms of projective varieties; see e.g. [Og2], [Zh1] for some concrete applications. There is a more *geometric* property of $G^*$: it preserves both the ample cone $\mathrm{Amp}\,(X)$ and the nef cone $\overline{\mathrm{Amp}}\,(X)$ (the closure of the ample cone), both of which encode lots of geometric information of $X$.

Keeping these two properties in mind, we shall prove first the following Theorem of Lie-Kolchin type (see also Theorem 2.3 and Corollary 2.5), and derive some direct consequences for groups of automorphisms of projective varieties or compact Kähler manifolds (Theorems in Section 3):

**Theorem 1.1. (Theorem of Lie-Kolchin type for a cone)**
*Let $V$ be a finite-dimensional real vector space, and $C \neq \{0\}$ a salient cone of $V$ (See Definition 2.2). Let $G$ be a Z-connected solvable subgroup of* $\mathrm{GL}(V)$ *such that $G(C) \subseteq C$. Then $G$ has a common eigenvector in the cone $C$.*

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This theorem is purely algebraic and does not *a priori* have anything to do with algebraic geometry or complex geometry. In our proof, we use Birkhoff-Perron-Frobenius' Theorem [Bir] and the proof of Dinh-Sibony [DS] for the case where $G$ is abelian.

In the same paper of theirs, Dinh and Sibony proved the following inspiring result:

**Theorem 1.2.** [DS] *Let $M$ be an $n$-dimensional compact Kähler manifold. Let $G$ be an abelian subgroup of* $\mathrm{Aut}\,(M)$ *such that each element of* $G \setminus \{\mathrm{id}\}$ *is of positive entropy. Then $G$ is a free abelian group of rank at most $n-1$. Furthermore, the rank estimate is optimal.*

See for instance [ibid] or 4.2 for the notion of entropy of an automorphism.

In view of the Tits Alternative Theorem and Dinh-Sibony's Theorem, it is natural to pose the following conjecture:

**Conjecture 1.3. (Conjecture of Tits type)**
*Let $X$ be an $n$-dimensional compact Kähler manifold or an $n$-dimensional complex projective variety with at most rational $\mathbf{Q}$-factorial singularities. Let $G$ be a subgroup of* $\mathrm{Aut}\,(X)$. *Then, one of the following two assertions holds:*

(1) *$G$ contains a subgroup isomorphic to $\mathbf{Z} * \mathbf{Z}$.*
(2) *There is a finite-index subgroup $G_1$ of $G$ such that the subset*

$$N(G_1) = \{g \in G_1 \,|\, g \ \text{is of null entropy}\}$$

*of $G_1$ is a normal subgroup of $G_1$ and the quotient group $G_1/N(G_1)$ is a free abelian group of rank at most $n-1$.*

It turns out that the crucial part of the conjecture is the rank estimate in the statement (2), as a weaker version of the conjecture (Theorem 4.3) can be easily verified. We shall also verify the conjecture for some basic cases, say, for surfaces (Theorem 4.4), hyperkähler manifolds (Theorem 4.6), complex tori (Theorem 4.7), and minimal threefolds (Theorem 5.1). Except for the case of complex tori, Theorem of Lie-Kolchin type (Theorem 1.1) plays a crucial role in our proof.

It would be interesting to confirm/disprove the conjecture for higher dimensional Calabi-Yau manifolds and manifolds with Kodaira dimension $-\infty$.

## 2. Theorem of Lie-Kolchin type for a cone

**Definition 2.1.** A group is *virtually solvable* if it has a solvable subgroup of finite index. A solvable subgroup $G$ of $\mathrm{GL}(n, \mathbf{C})$ is *Z-connected* if its Zariski closure $\overline{G}$ in $\mathrm{GL}(n, \mathbf{C})$ is connected with respect to the Zariski topology.

Given a virtually solvable subgroup $G$ of $\mathrm{GL}(n, \mathbf{C})$, one can always find a Z-connected solvable, finite-index subgroup $H$ of $G$. Indeed, choose a solvable finite-index subgroup $G_1$ of $G$ and take the identity component $\overline{G_1}^0$ of the Zariski closure $\overline{G_1}$. Then the group $H := \overline{G_1}^0 \cap G_1$ is a Z-connected solvable finite-index subgroup of $G$.

**Definition 2.2.** Let $V$ be an $r$-dimensional real vector space. We regard $V$ as an $r$-dimensional euclidean space with the natural topology. A subset $C$ of $V$ is a *salient cone of $V$* if $C$ is closed in $V$, closed under addition and multiplication by a non-negative scalar, and contains no 1-dimensional linear space.

In this section, we shall prove the following theorem:

**Theorem 2.3. (Theorem of Lie-Kolchin type for a cone)**
*Let $V$ be an $r$-dimensional real vector space, and $C \neq \{0\}$ a salient cone of $V$. Let $G$ be a virtually solvable subgroup of* $\mathrm{GL}(V)$ *such that $G(C) \subseteq C$. Then there are a finite-index subgroup $H$ of $G$ and a real vector $v \in C \setminus \{0\}$ such that $H(\mathbf{R}_{\geq 0}v) = \mathbf{R}_{\geq 0}v$. In other words, a suitable finite-index subgroup of $G$ has a common eigenvector in the cone $C$. Moreover, if $G$ is Z-connected and solvable, then one may choose $H = G$.*

Before starting the proof, we recall several relevant results and state one special but useful version.

*Remark* 2.4. (1) The Tits Alternative Theorem [Ti] states that any subgroup $G$ of $\mathrm{GL}(V_{\mathbf{C}})$ is either *virtually solvable*, or contains a free subgroup isomorphic to $\mathbf{Z} * \mathbf{Z}$. In the latter case, the conclusion in Theorem 2.3 is not true in general even in the case $V = \mathbf{Z}^r \otimes \mathbf{R}$ and $G \subset \mathrm{GL}(\mathbf{Z}^r)$. Therefore, our condition in the theorem on the group $G$ is quite natural.

(2) The original version of Lie-Kolchin Theorem (see e.g. [Hu]) states that any Z-connected solvable subgroup $G$ of $\mathrm{GL}(V_{\mathbf{C}})$ has a common eigenvector. Note that it is, in general, far from being real, even in the case $V = \mathbf{Z}^r \otimes \mathbf{R}$ and $G \subset \mathrm{GL}(\mathbf{Z}^r)$.

(3) Birkhoff-Perron-Frobenius' Theorem [Bir] for a cone $C$ states that any $g \in \mathrm{GL}(V)$ with $g(C) \subset C$ has an eigenvector $v_g$ in $C$. Furthermore, since $g$ acts on the real vector space $W = \langle C \rangle$, one can choose $v_g$ so that $g(v_g) = \rho(g|W)\ v_g$. Here $\rho(g|W)$ is the spectral radius of $g|W$ (see [ibid]).

(4) Dinh-Sibony ([DS, Proposition 4.1]) proved Theorem 2.3 when $G$ is abelian. So, our theorem is not only a refinement of Lie-Kolchin Theorem but also an optimal generalization of Birkhoff-Perron-Frobenius' Theorem [Bir] and Dinh-Sibony's version [DS] in view of the Tits Alternative Theorem.

For algebro-geometric applications, the following arithmetical version is most useful.

**Corollary 2.5.** *Let $L$ be a free $\mathbf{Z}$-module of finite rank $r \geq 1$, and $C \neq \{0\}$ a salient cone of $L_{\mathbf{R}} := L \otimes \mathbf{R}$. Let $H$ be a Z-connected solvable subgroup of* $\mathrm{Aut}\, L = \mathrm{GL}(L)$ *such that $H(C) \subseteq C$. Then there is a real vector $v \in C \setminus \{0\}$ such that $H(\mathbf{R}_{\geq 0}v) = \mathbf{R}_{\geq 0}v$.*

This version together with the next lemma or its proof will be frequently used in the sequel. In this lemma, the very crucial condition is: "the action of $H$ is defined over $\mathbf{Z}$".

**Lemma 2.6.** *Let $L$ be a free $\mathbf{Z}$-module of finite rank $r \geq 1$ and let $H$ be a Z-connected solvable subgroup of* $\mathrm{GL}(L)$. *Then, the following set*

$$N(H) := \{h \in H \mid \rho(h) = 1\}$$

*is a normal subgroup of $H$ and $H/N(H)$ is a free abelian group of rank at most $r - 1$. Here, $\rho(h)$ is the spectral radius of $h$ on $L_{\mathbf{C}}$.*

*Proof.* By Lie-Kolchin Theorem, $H$ is a subgroup of the group $T(r)$ of upper triangle matrices of $\mathrm{GL}(r, \mathbf{C})$ under the identification $\mathrm{GL}(L_{\mathbf{C}}) = \mathrm{GL}(r, \mathbf{C})$ with respect to a suitable basis of $L_{\mathbf{C}}$.

Regarding $H \subset T(r)$, we write the $(i,i)$-th entry of the matrix $h \in H$ by $\chi_i(h)$ ($1 \le i \le r$). Note that $\{\chi_i(h)\}_{i=1}^r$ is the set of eigenvalues of $h$ (listed with multiplicity) and

$$\chi_i(h_1h_2) = \chi_i(h_1)\chi_i(h_2),$$
$$|\chi_1(h)\chi_2(h)\cdots\chi_r(h)| = 1.$$

The last equality holds because $h \in \mathrm{GL}(L)$ and $L$ is a free $\mathbf{Z}$-module. Thus, we have a group homomorphism defined as follows:

$$\chi : H \longrightarrow \mathbf{R}^{r-1} = \{(x_i)_{i=1}^r \in \mathbf{R}^r \mid \sum_{i=1}^r x_i = 0\} \subset \mathbf{R}^r,$$
$$\chi(h) := (\log|\chi_1(h)|,\ \log|\chi_2(h)|,\ \cdots,\ \log|\chi_r(h)|)\ .$$

By the definition of $N(H)$, we have

$$N(H) = \mathrm{Ker}\,\chi.$$

Thus $N(H)$ is a normal subgroup and its quotient group $H/N(H) \simeq \chi(H)$ is a subgroup of $\mathbf{R}^{r-1}$.

It now suffices to show that $\chi(H)$ is discrete in the additive group $\mathbf{R}^{r-1}$ with respect to the usual topology. Or equivalently, it suffices to show that the identity element $(0)_{i=1}^r$ is an isolated point of $\chi(H)$. Here we will use the crucial fact: "$H$ is defined over $\mathbf{Z}$".

Let $\delta$ be any positive real number. Consider the set of all elements $h \in H$ satisfying the inequalities

$$\log|\chi_i(h)| < \delta\ ,\quad i = 1, \cdots, r\ .$$

For such an $h$, $\chi_i(h)$ are all bounded, and hence the coefficients of the characteristic polynomial $\Phi_h(x)$ are all bounded. Since $\Phi_h(x) \in \mathbf{Z}[x]$, there are only finitely many characteristic polynomials for all such $h$. Thus the set of vectors $(\log|\chi_i(h)|)_{i=1}^r$ for all such $h$ is also finite. Hence $(0)_{i=1}^r$ is an isolated point of $\chi(H)$. □

In the rest of this section, we shall prove Theorem 2.3. First, we observe the following:

**Lemma 2.7.** *Let $H$ be a Z-connected solvable (possibly discrete) subgroup of* $\mathrm{GL}(r, \mathbf{C})$. *Denote by $[H, H]$ the commutator subgroup of $H$. Then the eigenvalues of every element of $[H, H]$ are all equal to* 1.

*Proof.* By Lie-Kolchin Theorem, $H$ is a subgroup of the group $T(r)$ of upper triangle matrices of $\mathrm{GL}(r, \mathbf{C})$. From this, it is easy to see that the diagonal entries of any commutator $g_1g_2g_1^{-1}g_2^{-1}$ are all equal to 1. So are the diagonal entries of any product of commutators. □

2.8. **Proof of Theorem 2.3.**

Replacing $G$ by a Z-connected solvable subgroup of finite index, we may assume that $G$ itself is Z-connected and solvable.

We consider the derived series of $G$:

$$G = G^{(0)} > G^{(1)} > \cdots > G^{(k-1)} > A := G^{(k)} > G^{(k+1)} = \{\ \mathrm{id}\}\,.$$

Here, the series has length $k \ge 0$, $G^{(l)}$ are normal subgroups of $G$ and $[G^{(l)}, G^{(l)}] = G^{(l+1)}$ for $l = 0, 1, \cdots, k$. Note that $A := G^{(k)}$ is an abelian group.

We prove Theorem 2.3 by induction on the length $k$. If $k = 0$, then $G = A$, i.e. $G$ itself is abelian, and the result follows from the result of Dinh-Sibony (Remark 2.4 (4)).

Suppose $k \geq 1$. Assuming that Theorem 2.3 is true when the length $\leq k - 1$, we shall show that Theorem 2.3 is also true when the length is $k$. First, note that $A \subseteq [G, G] = G^{(1)}$. By Lemma 2.7, the eigenvalues on $V_{\mathbf{C}}$ of every element of $A$ are all equal to 1. For each $a \in A$, we write the associated real eigenspace by $V(a, 1)$:

$$V(a, 1) := \{\, x \in V \mid a(x) = x \,\} \subset V .$$

Let us consider the following linear subspace $W$ of $V$

$$W = \cap_{a \in A} V(a, 1) .$$

**Claim 2.9.**

(1) $C_W := C \cap W \neq \{0\}$ .
(2) For each $g \in G$, one has $g(W) = W$. In particular, $G(C_W) \subseteq C_W$.

*Proof.* (1) Since Theorem 2.3 is true for the abelian group $A$, there is a common eigenvector in $C$ for $A$. So (1) follows.

(2) Let $0 \neq w \in W$. Since $A$ is a normal subgroup of $G$, we can write $g^{-1}ag = a_g$ for some $a_g \in A$. Thus, regardless of the choice of $w$, we have

$$(g^{-1}ag)\ (w) = a_g(w) = w .$$

Hence

$$a(g(w)) = g(w).$$

So

$$g(W) \subseteq \cap_{a \in A} V(a, 1) = W.$$

Since $g$ is invertible, this implies $g(W) = W$. □

From now on, let us consider the action of $G$ on $W$ and on $C_W$. Set

$$G_W^{(l)} := \mathrm{Im}\,(G^{(l)} \longrightarrow \mathrm{GL}(W)) .$$

Then, $G_W := G_W^{(0)}$ is Z-connected solvable, and the new series

$$G_W = G_W^{(0)} \geq G_W^{(1)} \geq \cdots \geq G_W^{(k-1)} \geq A_W := G_W^{(k)} \geq G_W^{(k+1)} = \{\, \mathrm{id}_W \}$$

gives the derived series, possibly redundant, of $G_W$.

Since $W = \cap_{a \in A} V(a, 1)$ and hence $A_W = \{\, \mathrm{id}_W \}$, *the new series is in fact redundant*. Now we can apply the induction hypothesis to the series of length $\leq k - 1$, and obtain the result. This completes the proof of Theorem 2.3.

## 3. Algebro-geometric consequences of Lie-Kolchin for a cone

In this section, we shall derive some direct consequences of Theorem of Lie-Kolchin type for a cone (Theorem 2.3 or Corollary 2.5) for groups of automorphisms or birational automorphisms of algebraic varieties and compact Kähler manifolds.

We now have some setups. Let $f : X \longrightarrow S$ be a projective surjective morphism of normal varieties defined over a field $k$, with connected fibers of positive dimension. Assume that $S$ is quasi-projective. Important cases are the case where $S = \mathrm{Spec}\, k$ (abstract case) and the case where $f := \Phi_{|mK_X|} : X \longrightarrow S$ is the relatively minimal

Kodaira-Iitaka fibration of $X$. We note that the relative canonical class $K_{X/S}$ is trivial in $\mathrm{NS}(X/S)$ (cf. 3.2) in the second case.

**Definition 3.1.** By $\mathrm{Bir}\,(X)$ and by $\mathrm{Aut}\,(X)$, we denote the group of birational transformations of $X$ and the group of biregular transformations of $X$ respectively. Set

$$\mathrm{Bir}(X/S) := \{\varphi \in \mathrm{Bir}\,(X) \,|\, f = f \circ \varphi\}\,,$$
$$\mathrm{Bir}^s(X/S) := \{\varphi \in \mathrm{Bir}\,(X/S) \,|\, \varphi \ \text{ is isomorphic in codimension one}\}\,,$$
$$\mathrm{Aut}(X/S) := \{\varphi \in \mathrm{Aut}\,(X) \,|\, f = f \circ \varphi\}\ .$$

Note that $\mathrm{Bir}^s\,(X) = \mathrm{Bir}\,(X)$ when $X$ is a (terminal) minimal model.

**Definition 3.2.** (1) Denote by $\mathrm{NS}(X/S)$ the set of numerical equivalence classes of Cartier divisors on $X$ over $S$. Note that $\mathrm{NS}(X/S)$ is the free part of the "Néron-Severi" group. It is well known that $\mathrm{NS}(X/S)$ is a free $\mathbf{Z}$-module of finite rank ([Kl]).

(2) We define the *relative nef cone* $\overline{\mathrm{Amp}}(X/S)$ to be the closure of the relative ample cone $\mathrm{Amp}(X/S)$ in $\mathrm{NS}(X/S)_{\mathbf{R}}$ (under the usual topology). Similarly, the *relative pseudo-effective cone* $\overline{\mathrm{Big}}(X/S)$ is defined to be the closure of the relative big cone $\mathrm{Big}(X/S)$ in $\mathrm{NS}(X/S)_{\mathbf{R}}$.

Both cones are closed convex cones of $\mathrm{NS}(X/S)$. Moreover, since the dimension of fibers of $f$ is assumed to be positive, both cones are salient. For more details on these cones and their roles in birational algebraic geometry, we refer the readers to [Ka2].

**Definition 3.3.** A subgroup of $\mathrm{Aut}\,(X/S)$ is said to be *numerically virtually solvable* if its image in $\mathrm{GL}\,(\mathrm{NS}(X/S))$ is virtually solvable.

By the Tits Alternative Theorem (Remark 2.4 (1)), every subgroup $G$ of $\mathrm{Aut}\,(X/S)$ satisfies either one of the following two, which are not in general mutually exclusive:

(i) $G$ is numerically virtually solvable, or
(ii) $G$ contains a subgroup isomorphic to $\mathbf{Z} * \mathbf{Z}$ .

Applying Corollary 2.5 to the action of $G$ on $(\mathrm{NS}(X/S), \overline{\mathrm{Amp}}(X/S))$, we have:

**Theorem 3.4.** *Let $G$ be a numerically virtually solvable subgroup of* $\mathrm{Aut}\,(X/S)$*. Then there are a finite-index subgroup $G_1$ of $G$ and a nonzero real vector $v \in \overline{\mathrm{Amp}}(X/S)$ such that $G_1(\mathbf{R}_{\geq 0}v) = \mathbf{R}_{\geq 0}v$. In particular, if $G$ is a numerically virtually solvable subgroup of* $\mathrm{Aut}\,(X)$*, then a suitable finite-index subgroup of $G$ admits a common real nef eigenvector.*

We know that the relative canonical divisor class $K_{X/S}$ is always fixed by the group $\mathrm{Aut}\,(X/S)$. However, the class $K_{X/S}$ is relatively trivial over $S$, if $f$ is a relatively minimal Kodaira-Iitaka fibration or if $X$ has a numerically trivial canonical class (i.e., $X$ is a Calabi-Yau variety in a broad sense) and $S = \mathrm{Spec}\,k$.

The next theorem will be applied in Section 5.

**Theorem 3.5.** *Let $X$ be an $n$-dimensional projective manifold whose $(n-1)$-th Chern class $c_{n-1}(X)$ is in the boundary of the cone of effective* 1*-cycles $\overline{\mathrm{NE}}(X)$. Let $G$ be a numerically virtually solvable subgroup of* $\mathrm{Aut}\,(X)$*. Then there are a finite-index subgroup $G_1$ of $G$ and a nonzero real nef vector $v$ such that $G_1(\mathbf{R}_{\geq 0}v) = \mathbf{R}_{\geq 0}v$ and $(v.c_{n-1}(X)) = 0$. The same is true for a minimal projective complex threefold* (*see* §5) *whose second Chern class $c_2(X)$ is not in the interior of $\overline{\mathrm{NE}}(X)$.*

*Proof.* Since $c_{n-1}(X)$ is stable under the action of $\mathrm{Aut}\,(X)$, the free submodule $M$ of $\mathrm{NS}(X)$ defined by

$$M := \{x \in \mathrm{NS}(X) \mid (x, c_{n-1}(X)) = 0\}$$

is also $\mathrm{Aut}\,(X)$-stable. Set

$$C := \overline{\mathrm{Amp}}\,(X) \cap M_{\mathbf{R}}.$$

Since $c_{n-1}(X)$ is in the boundary of $\overline{\mathrm{NE}}(X)$, it follows that $C \neq \{0\}$. So, we obtain the first assertion by applying Corollary 2.5 for the action of $G$ on $(M, C)$. The last statement follows from a result of Miyaoka [Mi], which says that $c_2(X)$ is well-defined and is always in $\overline{\mathrm{NE}}(X)$ for a minimal projective complex threefold. □

*Remark* 3.6. For several reasons, even $\mathrm{Bir}^s(X/S)$ does not act on $\mathrm{NS}(X/S)$ in general.

We quote the following result of Kawamata [Ka2, the proof of Lemma 1.1].

**Theorem 3.7.** *Assume that* $\mathrm{char}\, k = 0$ *and* $X$ *admits at most* $\mathbf{Q}$*-factorial rational singularities. Let* $\mathrm{N}^1(X/S)$ *be the group of numerical equivalence classes of Weil divisors on* $X$ *over* $S$*. Then* $\mathrm{N}^1(X/S)$ *is a free* $\mathbf{Z}$*-module of finite rank and satisfies* $[\mathrm{N}^1(X/S) : \mathrm{NS}(X/S)] < \infty$*. Moreover,* $\mathrm{Bir}^s(X/S)$ *acts naturally and linearly on* $\mathrm{N}^1(X/S)$ *and preserves the cone* $\overline{\mathrm{Big}}(X/S)$ $(\subset \mathrm{N}^1(X/S)_{\mathbf{R}})$.

**Definition 3.8.** A subgroup $G$ of $\mathrm{Bir}^s\,(X/S)$ is *numerically virtually solvable* if its image in $\mathrm{GL}(\mathrm{N}^1(X/S))$ is virtually solvable. Since $[\mathrm{N}^1(X/S) : \mathrm{NS}(X/S)] < \infty$, the definitions here and in 3.3 are consistent if $G \subset \mathrm{Aut}\,(X/S)$.

Applying Corollary 2.5 to the action $G$ on $(\mathrm{N}^1(X/S), \overline{\mathrm{Big}}(X/S))$, we obtain:

**Theorem 3.9.** *Assume that* $\mathrm{char}\, k = 0$ *and* $X$ *admits at most* $\mathbf{Q}$*-factorial rational singularities. Let* $G$ *be a numerically virtually solvable subgroup of* $\mathrm{Bir}^s\,(X/S)$*. Then there are a finite-index subgroup* $G_1$ *of* $G$ *and a nonzero real vector* $v \in \overline{\mathrm{Big}}(X/S)$ *such that* $G_1(\mathbf{R}_{\ge 0} v) = \mathbf{R}_{\ge 0} v$*. In particular, if* $X$ *is minimal and if* $G$ *is a numerically virtually solvable subgroup of* $\mathrm{Bir}\,(X)$*, then a suitable finite-index subgroup of* $G$ *admits a common real pseudo-effective eigenvector.*

In the above Theorem, we do not need the rationality of singularities, because $\mathbf{Q}$-factoriality already implies that $\mathrm{NS}(X/S)_{\mathbf{Q}} = \mathrm{N}^1(X/S)_{\mathbf{Q}}$. However, in the sequel, in order to obtain an extra discreteness as in Lemma 2.6, one needs as well an extra condition that the action of $G$ is defined over a $\mathbf{Z}$-module.

**Definition 3.10.** Let $X$ be a compact Kähler manifold and let $\overline{\mathcal{K}}(X)$ be the closure of the Kähler cone $\mathcal{K}(X)$ in $H^2(X, \mathbf{R})$. A subgroup $G$ of $\mathrm{Aut}\,(X)$ is *cohomologically virtually solvable* if its image in $\mathrm{GL}\,(H^2(X, \mathbf{Z})/\mathrm{torsion})$ is virtually solvable. When $X$ is projective, $G$ is numerically virtually solvable if $G$ is cohomologically virtually solvable.

Applying Corollary 2.5 to the action of $G$ on $(H^2(X, \mathbf{Z})/\mathrm{torsion}, \overline{\mathcal{K}}(X))$, we obtain the following:

**Theorem 3.11.** *If* $G$ *is a cohomologically virtually solvable subgroup of* $\mathrm{Aut}\,(X)$*, then there are a finite-index subgroup* $G_1$ *of* $G$ *and a nonzero real vector* $v \in \overline{\mathcal{K}}(X)$ *such that* $G_1(\mathbf{R}_{\ge 0} v) = \mathbf{R}_{\ge 0} v$*.*

## 4. Entropy and Conjecture of Tits type

In this section, we verify Conjecture of Tits type (Conjecture 1.3) posed in Introduction for some basic varieties, i.e., for surfaces, complex tori and hyperkähler manifolds. In the next section, we shall also verify the conjecture for minimal threefolds. In our proof, Theorem 2.3 or Corollary 2.5 also plays an important role.

We recall the definition of *entropy* of an automorphism of a compact Kähler manifold from [DS]. We will generalize it to a complex normal projective variety; see 4.2.

For the references to the following important results, see Dinh [Di, Proposition 5.7], Dinh-Sibony [DS] and Guedj [Gu, (1.2), (1.6)].

**Theorem 4.1.** *Let $M$ be a compact Kähler manifold of dimension $n$ and let $g$ be an automorphism of $M$. By $\rho(g^*|W)$ we denote the spectral radius of the action of $g^*$ on a $g^*$-stable subspace $W$ of the total cohomology group $H^*(M, \mathbf{C})$. Then we have:*

(1) *$\rho(g^*|H^*(M, \mathbf{C})) \geq 1$, and $\rho(g^*|H^* \ (M, \mathbf{C})) = 1$ (resp. $> 1$) if and only if $\rho(g^*|H^2(M, \mathbf{C})) = 1$ (resp. $> 1$).*
*Further, $\rho(g^*|H^*(M, \mathbf{C})) = 1$ (resp. $> 1$) if and only if so is for $g^{-1}$.*
(2) *$\rho(g^*|H^2(M, \mathbf{C})) = \rho(g^*|H^{1,1}(M))$.*
*Further, if $M$ is projective, then this value is also equal to $\rho(g^*|\mathrm{NS}(M))$.*
(3) *The spectral radius $\rho(g^*|H^{i,i}(X, \mathbf{C}))$ equals the $i$-th dynamical degree $d_i(g)$. Further, there are integers $m \leq m'$ such that*
$$1 = d_0(g) < \cdots < d_m(g) = \cdots = d_{m'}(g) > \cdots > d_n(g) = 1.$$
(4) *$\rho(g^*|H^*(X, \mathbf{C})) = \max_{0 \leq i \leq n} d_i(g) = e^{h(g)}$, with $h(g)$ the topological entropy of $g$.*

**Definition 4.2.** (1) Let $M$ be a compact Kähler manifold. An automorphism $g$ of $M$ is *of null entropy* (resp. *of positive entropy*) if the spectral radius $\rho(g^*|H^2(M, \mathbf{C})) = 1$ (resp. $> 1$). By Theorem 4.1, $g$ is of null entropy (resp. of positive entropy) if and only if $\rho(g^*|\mathrm{NS}(M)) = 1$ (resp. $> 1$) when $X$ is projective.

(2) Let $V$ be a complex normal projective variety. Taking account of the last characterization in (1), an automorphism $g$ of $V$ is *of null entropy* (resp. *of positive entropy*) if the spectral radius $\rho(g^*|\mathrm{NS}(V))$ on the free part $\mathrm{NS}(V)$ of the Néron-Severi group is 1 (resp. $> 1$). If $V$ is $\mathbf{Q}$-factorial, then $g \in \mathrm{Aut}(V)$ is of positive entropy if and only if so is the lifting of $g$ to an automorphism on an equivariant resolution of $V$; see the proof of [Zh2, Lemma 2.6].

In Introduction, inspired by the Tits Alternative Theorem and Dinh-Sibony's Theorem (Theorem 1.2) we posed Conjecture of Tits type (Conjecture 1.3). The easy result below will imply a weaker version of the conjecture.

**Theorem 4.3.** *Let $L$ be a free $\mathbf{Z}$-module of rank $r \geq 1$ and let $H$ be a subgroup of $\mathrm{GL}(L)$. Assume that $H$ does not contain a subgroup isomorphic to $\mathbf{Z} * \mathbf{Z}$. Then, there is a Z-connected solvable finite-index subgroup $H_1$ of $H$ such that the subset*
$$N(H_1) := \{h \in H_1 \mid \rho(h) = 1\}$$
*of $H_1$ is a normal subgroup of $H_1$ and $H_1/N(H_1)$ is a free abelian group of rank at most $r - 1$.*

*Proof.* By the Tits Alternative Theorem, $H$ is virtually solvable. Thus $H$ has a finite-index subgroup $H_1$ which is Z-connected solvable. So the result follows from Lemma 2.6. □

Applying this theorem to the action of $G$ on $H^2(X, \mathbf{Z})$ or on $\mathrm{NS}(X)$, we see that Conjecture 1.3 is true *except the rank estimate*. In other words, *the rank estimate is the most essential point of the conjecture*.

In the rest of this section, we verify Conjecture of Tits type for surfaces (Theorem 4.4), hyperkähler manifolds (Theorem 4.6), and complex tori (Theorem 4.7). We also study Conjecture of Tits type for coverings and fibrations (Propositions 4.8, 4.9).

**Theorem 4.4.** *Conjecture of Tits type is true if* $\dim X \le 2$.

*Proof.* There is nothing to prove when $\dim X = 0$ or 1.

Assume $\dim X = 2$. If $X$ is smooth and projective, the conjecture has been proved by [Zh1]. Therefore, when $X$ is projective, it is possible to reduce to the smooth projective case by considering a minimal resolution $\tilde{X}$ of $X$ and the induced action on $\tilde{X}$. It is also possible to argue by using classification of surfaces (see e.g. [BHPV]) when $X$ is Kähler. Instead of such a case-by-case proof, here we give a unified, classification-free proof.

Define

$$L := H^2(X, \mathbf{Z}) \text{ (resp. } = \mathrm{NS}(X)) \,,$$
$$C := \overline{\mathcal{K}}(X) \text{ (resp. } = \overline{\mathrm{Amp}}(X))$$

when $X$ is a compact Kähler surface (resp. $X$ is a normal projective surface).

By the Hodge index theorem, the real linear subspace

$$W := \langle C \rangle$$

of $L_{\mathbf{R}}$ is hyperbolic, i.e. of signature $(1, *)$ with respect to the intersection form. This is because

$$W = H^{1,1}(M, \mathbf{R}) \text{ or } W = \mathrm{NS}(X)_{\mathbf{R}}$$

according to the two cases above.

Consider the image

$$G^* := \mathrm{Im}\,(G \longrightarrow \mathrm{GL}(L)) \ .$$

We may assume that $G^*$ is Z-connected solvable. Let $v \in C$ be a common eigenvector of $G^*$ guaranteed by Corollary 2.5. Then, we can write $g^*(v) = \chi(g)v$.

**Claim 4.5.** (1) $\rho(g^*|W) = \rho(g^*|L)$. (2) $\chi(g) = \rho(g^*|L)$ or $\frac{1}{\rho(g^*|L)}$ .

*Proof of Claim 4.5.* (1) In the Kähler surface case, the intersection form on the subspace $(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbf{R})$ is positive definite, so the eigenvalues of $g^*$ on this subspace all have absolute value 1.

(2) By Birkhoff-Perron-Frobenius' Theorem (Remark 2.4 (3)) applied to the action of $g^*$ on $(W, C)$, there is a nonzero vector $v_g \in C$ such that $g^*(v_g) = \rho(g^*|W)\ v_g$. Note that

$$(v.v_g) = (g^*(v).g^*(v_g)) = \chi(g)\rho(g^*|W)(v.v_g) \ .$$

If $\chi(g) \ne \frac{1}{\rho(g^*|W)}$ , then we must have $(v, v_g) = 0$. Since $v$, $v_g \in C$ and since $W$ is hyperbolic by the Hodge index theorem, it follows that $v$ is parallel to $v_g$, so $\chi(g) = \rho(g^*|W)$. Now, (2) follows from (1). This proves the claim.

Resuming the proof of the theorem, we consider the group homomorphism

$$\chi : G \longrightarrow \mathbf{R} \; ; \; g \mapsto \log|\chi(g)| = \log \chi(g) \; .$$

By Claim 4.5, we see that $\chi(g) = \rho(g^*|L) = 1$ if and only if $g$ is of null entropy. This means that

$$\mathrm{Ker}\, \chi = N(G),$$

and hence

$$G/N(G) \simeq \chi(G).$$

Furthermore, by Claim 4.5 again, we have

$$\chi(G) \subset \{\pm \log \rho(g^*|L) \mid g \in G\} \; .$$

Note here that if $\rho(g^*|L)$ is bounded, then all eigenvalues of $g^*|L$ are also bounded. We also note that the action $G^*$ is defined over the $\mathbf{Z}$-module $L$. Thus, $\chi(G)$ is discrete in $\mathbf{R}$ as we have seen in the proof of Lemma 2.6. Hence $\chi(G) \simeq \mathbf{Z}^s$ with $s \le 1$. □

By a *hyperkähler* manifold $M$, we mean a simply-connected compact Kähler manifold admitting an everywhere non-degenerate holomorphic 2-form $\sigma_M$ such that $H^0(M, \Omega^2_M) = \mathbf{C}\sigma_M$. According to the Bogomolov decomposition Theorem [Be1], hyperkähler manifolds form one of the three building blocks of compact Kähler manifolds with vanishing first Chern class. The other two building blocks are complex tori and Calabi-Yau manifolds (in the narrow sense).

**Theorem 4.6.** *Conjecture of Tits type is true for hyperkähler manifolds.*

*Proof.* By Beauville [Be1], $H^2(M, \mathbf{Z})$ of a hyperkähler manifold $M$ admits an integral symmetric bilinear form $(*, **)$ of signature $(3, b_2(M) - 3)$. This form is invariant under $\mathrm{Aut}\,(M)$ and is of signature $(1, h^{1,1}(M) - 1)$ on $H^{1,1}(M, \mathbf{R})$. A bit more precisely, $(\eta, \eta') > 0$ for any Kähler classes $\eta$ and $\eta'$. Thus, the same proof as in Theorem 4.4 shows the result. Furthermore, in the statement (2) of the conjecture, we have $G_1/N(G_1) \simeq \mathbf{Z}^s$ with $s \le 1$. □

**Theorem 4.7.** *Conjecture of Tits type is true for complex tori.*

*Proof.* Let $X$ be a complex torus of dimension $n$ and let $G$ be a subgroup of $\mathrm{Aut}\,(X)$. Note that

$$H^1(X, \mathbf{Z}) \otimes \mathbf{C} = H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)} \; ,$$
$$H^*(X, \mathbf{Z}) = \oplus_{k=0}^{2n} \wedge^k H^1(X, \mathbf{Z}) \; .$$

Thus $g \in G$ is of null entropy if and only if the spectral radius $\rho(g^*|H^0(X, \Omega^1_X)) = 1$. Let us consider the action of $G$ on $H^0(X, \Omega^1_X) \simeq \mathbf{C}^n$, and define

$$G^* := \mathrm{Im}\,(G \longrightarrow \mathrm{GL}\,(H^0(X, \Omega^1_X)) \; .$$

As in the proof of Theorem 4.4, we may assume that $G^*$ is Z-connected solvable. Then, as in Lemma 2.6, choosing a suitable basis of $H^0(X, \Omega^1_X)$, one can embed $G^*$ into the subgroup $T(n)$ of upper triangle matrices of $\mathrm{GL}\,(n, \mathbf{C}) = \mathrm{GL}\,(H^0(X, \Omega^1_X))$, and we have a group homomorphism

$$\chi : G \longrightarrow \mathbf{R}^{n-1} = \{(x_i)_{i=1}^n \in \mathbf{R}^n \mid \sum_{i=1}^n x_i = 0\} \subset \mathbf{R}^n$$

which is defined by

$$\chi(g) = (\log|\chi_1(g)|,\ \log|\chi_2(g)|,\ \cdots,\ \log|\chi_n(g)|)\ .$$

Here $\chi_i(g)$ is the $(i,i)$-th entry of the matrix $g$ (regarded as an element of $T(n)$). Thus

$$N(G) = \operatorname{Ker}\chi\,.$$

Note that the eigenvalues of $g^*|H^1(X,\mathbf{Z})$ are $\chi_i(g)$ and $\overline{\chi_i(g)}$ $(1 \le i \le n)$. So one can apply the same argument as in Lemma 2.6 to get

$$G/N(G) \simeq \operatorname{Im}\chi \simeq \mathbf{Z}^s$$

for some $s \le n-1$. □

It is highly interesting to verify the conjecture for Calabi-Yau manifolds (in the narrow sense). Unfortunately, we could not yet do it. All we can say now is that the conjecture is true for Calabi-Yau threefold, as a special case of Theorem 5.1 in Section 5. See also Proposition 4.10 for a relevant result.

The next two propositions will be used in Section 5.

**Proposition 4.8.** *Let $G$ be a subgroup of $\operatorname{Aut}X$ and let $\pi : \tilde{X} \longrightarrow X$ be a generically finite surjective morphism between compact Kähler manifolds or between complex normal projective varieties. Assume that $G$ lifts to a subgroup $\tilde{G} \simeq G$ of $\operatorname{Aut}(\tilde{X})$ equivariantly. Then Conjecture of Tits type is true for $(X,G)$ if so is for $(\tilde{X},\tilde{G})$.*

*Proof.* Let $\dim X = \dim\tilde{X} = n$. If $G$ contains a subgroup isomorphic to $\mathbf{Z} * \mathbf{Z}$, then we are done. Therefore, we may assume that $\tilde{G} \simeq G$ does not contain a subgroup isomorphic to $\mathbf{Z}*\mathbf{Z}$. Then, by the Tits Alternative Theorem, any linear action of $\tilde{G}$ on a free $\mathbf{Z}$-module of finite rank is virtually solvable. Furthermore, replacing $G \simeq \tilde{G}$ by a suitable finite-index subgroup, we may assume that $\tilde{G}|H^2(\tilde{X},\mathbf{Z})$ and $G|H^2(X,\mathbf{Z})$ or $\tilde{G}|\mathrm{NS}(\tilde{X})$ and $G|\mathrm{NS}(X)$ are Z-connected solvable. Since $\pi^* : H^2(X,\mathbf{Z}) \longrightarrow H^2(\tilde{X},\mathbf{Z})$ or $\pi^* : \mathrm{NS}(X) \longrightarrow \mathrm{NS}(\tilde{X})$ is injective, there is then a surjective homomorphism $\phi : \tilde{G}/N(\tilde{G}) \longrightarrow G/N(G)$. Since $G/N(G)$ is free by Lemma 2.6, the result follows. □

If both $X$ and $\tilde{X}$ are $\mathbf{Q}$-factorial, then the group homomorphism $\phi$ is indeed an isomorphism (see [NZ, Appendix, Lemma A.8] and also [Zh2, Lemma 2.6]), so the converse of the conclusion is also true.

Thanks to Proposition 4.8, we may freely replace the variety $X$ by its $G$-equivariant resolution, which always exists.

**Proposition 4.9.** *Let $f : X \longrightarrow Y$ be a surjective morphism with connected fibres. Assume that $X$ is smooth. Let $F$ be a general fiber. Let $G$ be a subgroup of $\operatorname{Aut}(X/Y)$ (so that $G$ acts faithfully on $F$). Then, Conjecture of Tits type is true for $G$ with the rank bound $\dim F - 1$ if and only if so is for the action of $G$ on $F$.*

*Proof.* We write by $G_F$ the action of $G$ on $F$. As in the previous proposition, we may assume that $G$ does not contain $\mathbf{Z} * \mathbf{Z}$. Replacing $G$ by a finite-index subgroup, we may assume that both

$$G^* := \operatorname{Im}(\varphi_X : G \longrightarrow \mathrm{GL}\,(\mathrm{NS}(X)) \text{ and } G_F^* := \operatorname{Im}(\varphi_F : G_F \longrightarrow \mathrm{GL}\,(\mathrm{NS}(F))$$

are Z-connected solvable. The groups $N(G)$ and $N(G_F)$ are then well-defined normal subgroups of $G = G_F$ (cf. Lemma 2.6). Then, by [Zh2, 2.1 (11)] or [NZ, Appendix, Theorem D], we have $N(G_F) = N(G)$ as subgroups of $G$. Thus we have the identification:

$$G_F/N(G_F) = G/N(G) .$$

This implies the result. □

As an application, we have the following:

**Proposition 4.10.** *Conjecture of Tits type is true for compact Kähler manifolds $M$ with $c_1(M) = 0$ in $H^2(M, \mathbf{R})$, provided that Conjecture of Tits type is true for Calabi-Yau manifolds (in the narrow sense).*

*Proof.* Let $\pi : \tilde{M} \longrightarrow M$ be the minimal splitting cover of $M$ in the sense of [Be2]. $\tilde{M}$ is the product of complex torus $T$, finitely many hyperkähler manifolds $V_i$ and finitely many Calabi-Yau manifolds $W_j$. The subgroup $G$ of $\mathrm{Aut}\,(M)$ lifts equivariantly to the subgroup $\tilde{G} \simeq G$ of $\mathrm{Aut}\,(\tilde{M})$ and $\tilde{G}$ preserves each factor $T$, $V_i$, $W_j$ [ibid]. By the Künneth formula, $H^2(\tilde{M}, \mathbf{Z})$ is naturally isomorphic to the direct sum of $H^2(T, \mathbf{Z})$, $H^2(V_i, \mathbf{Z})$ and $H^2(W_j, \mathbf{Z})$. Thus, Conjecture of Tits type is true for $\tilde{G}$ by Theorems 4.6, 4.7 and by our assumption for Calabi-Yau manifolds. Now the result follows from Proposition 4.8. □

*Remark* 4.11. Furthermore, the proof of Proposition 4.10 together with Theorem 5.1 below implies that Conjecture of Tits type is true for compact Kähler manifolds $M$ with $c_1(M) = 0$ in $H^2(M, \mathbf{R})$ if its minimal splitting cover does not have a Calabi-Yau manifold of dimension $\geq 4$ as a factor.

## 5. Conjecture of Tits type for minimal threefolds

A complex normal projective variety $X$ is *minimal* if $X$ admits at most $\mathbf{Q}$-factorial terminal singularities and the canonical class $K_X$ is nef.

In this section, we show the following:

**Theorem 5.1.** *Conjecture of Tits type is true for minimal threefolds $X$.*

Our proof is mostly classification free, but unfortunately, we could not avoid using some classification result of Calabi-Yau threefolds of very special type at the final step.

*Proof.* By the abundance theorem for threefolds [Ka3], the linear system $|mK_X|$ for some $m > 0$ defines a surjective morphism $\varphi : X \longrightarrow W$ from $X$ to the canonical model $W$ of $X$ (Iitaka-Kodaira fibration). The morphism $\varphi$ is $\mathrm{Aut}\,(X)$-equivariant and the induced action of $\mathrm{Aut}\,X$ on $W$ is finite by Deligne-Nakamura-Ueno [Ue, Theorem 14.10]. Thus, replacing $G$ by a finite-index subgroup, we may assume that $G \subset \mathrm{Aut}\,(X/W)$. Since the conjecture of Tits type is true when $\dim \leq 2$ by Theorem 4.4, the conjecture is also true if $\dim W \geq 1$, i.e. if $K_X$ is not numerically trivial. This can be seen by applying Proposition 4.8 to a $G$-equivariant resolution $\pi : \tilde{X} \longrightarrow X$, and then Proposition 4.9 to the composition $\varphi \circ \pi : \tilde{X} \longrightarrow W$.

Consider the case where $K_X$ is numerically trivial. By Proposition 4.8 applied to the global index-one cover of $X$, we may assume that $K_X$ is linearly equivalent to 0.

If $q(X) := h^1(\mathcal{O}_X) > 0$, then the albanese morphism $a : X \longrightarrow \mathrm{Alb}(X)$ is an étale fiber bundle by [Ka1]. Thus, $X$ is smooth and its minimal splitting cover is either the product of an elliptic curve and a K3 surface or an abelian threefold. In this case, the result follows from the proof of Proposition 4.10.

If $c_2(X) = 0$ as a linear form on $\mathrm{NS}(X)$, then $X$ is an étale quotient of an abelian threefold by [SW]. Hence $X$ is smooth and its minimal splitting cover is an abelian threefold. So the result also follows from the proof of Proposition 4.10.

Thus, we may assume further that $q(X) = 0$ and $c_2(X) \neq 0$.

**Lemma 5.2.** *Let $X$ be a minimal threefold such that $K_X \sim 0$, $q(X) = 0$ and $c_2(X) \neq 0$ (or more generally $K_X \sim_{\mathbf{Q}} 0$ and $c_2(X) \neq 0$). Let $G \subset \mathrm{Aut}\,(X)$. Then, either one of the following holds:*

(1) *Conjecture of Tits type is true for $(X, G)$.*
(2) *There is a nef and big Cartier divisor $D$ such that $(c_2(X).D) = 0$.*

*Proof.* Set $G^* := \mathrm{Im}\,(G \longrightarrow \mathrm{GL}\,(\mathrm{NS}(X)))$. If $\mathbf{Z} * \mathbf{Z} \subset G^*$, then (1) holds. So, by the Tits Alternative Theorem, we may assume that $G^*$ is Z-connected solvable (after replacing $G$ by a finite-index subgroup).

Set

$$M := \{x \in \mathrm{NS}(X) \mid (c_2(X).x) = 0\,\}\ , \quad C := \overline{\mathrm{Amp}}\,(X) \cap M_{\mathbf{R}}\ .$$

For $g \in G$, we denote by $\rho(g)$ the spectral radius of $g^*|\mathrm{NS}(X)$, i.e.

$$\rho(g) := \rho(g^*|\mathrm{NS}(X))\,.$$

If each element of $G$ is of null entropy, then we are done. So we may assume that $\rho(g_1) > 1$ for some $g_1 \in G$. Then, by Birkhoff-Perron-Frobenius' Theorem, there is a real vector $v_{g_1} \in \overline{\mathrm{Amp}}\,(X) \setminus \{0\}$ such that $g_1^* v_{g_1} = \rho(g_1) v_{g_1}$. Thus $c_2(X).v_{g_1} = 0$ as shown below. Hence $C \neq \{0\}$ and we can apply Theorem 3.5 to obtain a real vector $v_0 \in C \setminus \{0\}$ such that $G(\mathbf{R}_{\geq 0} v_0) = \mathbf{R}_{\geq 0} v_0$. Thus, for each $g \in G$ we can write $g^* v_0 = \chi(g) v_0$, and hence we have the group homomorphism:

$$\chi : G \longrightarrow \mathbf{R}\ ;\ g \mapsto \log \chi(g)\ .$$

Note that $\rho(g) > 1$ if and only if $\rho(g^{-1}) > 1$ (for $\det g^*|\mathrm{NS}(X) = \pm 1$). Further, if $\rho(g) = 1$ then $\rho(g) = \chi(g) = \rho(g^{-1}) = 1$. So we have either one of the following two cases:

Case(i) There is an element $g \in G$ such that $\rho(g) > 1$ and the three positive real numbers $\chi(g),\ \rho(g),\ 1/\rho(g^{-1})$ are mutually distinct.

Case(ii) For each $g \in G$, either $\rho(g) = 1$, or $\chi(g) = \rho(g) > 1$, or $\chi(g) = 1/\rho(g^{-1}) < 1$.

We deal with Case (i) first. By Birkhoff-Perron-Frobenius' Theorem, there is a real vector $v_g \in \overline{\mathrm{Amp}}\,(X) \setminus \{0\}$ such that

$$g^* v_g = \rho(g)\ v_g\ .$$

Since $g^* c_2(X) = c_2(X)$, we have

$$(c_2(X).v_g) = (g^* c_2(X).g^* v_g) = \rho(g)(c_2(X).v_g)\ .$$

Thus

$$(c_2(X).v_g) = 0\,.$$

Similarly, since $\rho(g^{-1}) > 1$, there is a real vector $v_{g^{-1}} \in \overline{\mathrm{Amp}}\,(X) \setminus \{0\}$ such that

$$(g^{-1})^* v_{g^{-1}} = \rho(g^{-1}) v_{g^{-1}} \quad \text{and} \quad (c_2(X).v_{g^{-1}}) = 0\,.$$

Note that $c_2(X)$ gives rise to an integer-valued linear form on $\mathrm{NS}(X)$, so $v_g, v_{g^{-1}} \in M_{\mathbf{R}}$. Hence

$$v_g,\, v_{g^{-1}} \in C \setminus \{0\}\,.$$

Let

$$D' := v_0 + v_g + v_{g^{-1}}\,.$$

Then

$$(D'.c_2(X)) = 0\,.$$

Since $v_0$, $v_g$ and $v_{g^{-1}}$ are eigenvectors of $g^*$ with distinct eigenvalues $\chi(g)$, $\rho(g)$ and $1/\rho(g^{-1})$, we have

$$(v_0.v_g.v_{g^{-1}}) \neq 0$$

by [DS, Lemma 4.4]. Since $v_0$, $v_g$ and $v_{g^{-1}}$ are all nef, we have then

$$((D')^3) > 0$$

and hence $D'$ is nef and big. Since $M_{\mathbf{R}}$ is a rational hypersurface in $\mathrm{NS}(X)_{\mathbf{R}}$ and the nef cone is locally rational polyhedral away from the cubic cone by [Ka2] or [Wi], we can then find a rational nef and big divisor $D$ such that $(D.c_2(X)) = 0$. Therefore, the assertion (2) in the lemma holds.

Next we deal with Case (ii). First note that the image $\mathrm{Im}\,\chi$ of the group homomorphism $\chi$ defined above is discrete in $\mathbf{R}$. This follows from the same argument as in the last part of the proof of Theorem 4.4. Thus

$$\mathrm{Im}\,\chi \simeq \mathbf{Z}^s \text{ with } s \leq 1\,.$$

Also, by the case assumption it is easy to see that Ker $\chi = N(G)$. Therefore, $G/N(G) \simeq \mathrm{Im}\,\chi$ and the assertion (1) in the lemma holds. This proves the lemma. □

Now the following lemma completes the proof of Theorem 5.1.

**Lemma 5.3.** *Let $X$ be a minimal threefold such that $K_X \sim 0$, $q(X) = 0$ and $c_2(X) \neq 0$. Assume that there is a nef and big Cartier divisor $D$ such that $(c_2(X).D) = 0$. Then, Conjecture of Tits type is true for $X$.*

*Proof.* Write $\zeta_n := e^{2\pi i/n}$. Let $l = 3$ or $7$. By [Og1] or [OS], the universal cover of $X$ is either one of $X_l$ ($l = 3, 7$). Here $X_3$ is the unique crepant resolution of the quotient threefold $\overline{X}_3 := A_3/\langle g_3\rangle$, where $A_3$ is the 3-times self product of the elliptic curve $E_3$ of period $\zeta_3$ and $g_3 := \mathrm{diag}\,(\zeta_3, \zeta_3, \zeta_3)$; $X_7$ is the unique crepant resolution of the quotient threefold $\overline{X}_7 := A_7/\langle g_7\rangle$, where $A_7$ is the Jacobian threefold of the Klein quartic curve and $g_7 := \mathrm{diag}\,(\zeta_7, \zeta_7^2, \zeta_7^4)$. Note that the singular locus of $\overline{X}_l$ consists of finitely many points.

By Proposition 4.8, we may assume that $X$ is $X_l$. In each case, we denote by $\nu_l : X_l \longrightarrow \overline{X}_l$ the unique crepant resolution. By [OS], this $\nu_l$ is also the unique birational contraction which contracts the class $c_2(X_l)$ in the cone of effective 1-cycles $\overline{\mathrm{NE}}(X_l)$. Thus $\mathrm{Aut}\,(X_l)$ acts equivariantly on $\overline{X}_l$.

Let $G \subset \mathrm{Aut}\,(X_l)$. Then $G$ acts on $\overline{X}_l$ equivariantly. Replacing $G$ by a finite-index subgroup, we may assume that each $\nu_l$-exceptional divisor is $G$-stable. Thus, it suffices to show the assertion for the action of $G$ on $\overline{X}_l$. Let $\overline{X}_l^0$ be the smooth locus of $\overline{X}_l$. Clearly $\overline{X}_l^0$ is $G$-stable. Moreover, by the shape of $g_l$, we see that $\pi_1(\overline{X}_l^0) = \pi_1(A_l).\langle g_l\rangle$ (a semi-direct product). We regard $\pi_1(\overline{X}_l^0)$ as a deck transformation group of the universal cover $U_l = \mathbf{C}^3 \setminus B_l$ of $\overline{X}_l^0$. Here $B_l$ is a discrete set of points ('lying over' the singular points of $\overline{X}_l$). Then $\pi_1(\overline{X}_l^0)$ is an affine transformation subgroup of $\mathbf{C}^3$, in which the factor $\pi_1(A_l)$ forms the group of parallel translations and the factor $\langle g_l \rangle$ forms the group of linear transformations defined by the matrix $g_l$. From this, one may see that, in $\pi_1(\overline{X}_l^0)$,

$$\pi_1(A_l) = \{\sigma \in \pi_1(\overline{X}_l^0) \mid \operatorname{ord} \sigma = \infty \} \cup \{\mathrm{id}\} .$$

Thus the (equivariant) action of $G$ on the universal cover $U_l$ normalizes not only $\pi_1(\overline{X}_l^0)$, but also $\pi_1(A_l)$. Hence the action of $G$ on $\mathbf{C}^3$ descends to an action on $A_l$ equivariantly. Now the result follows from Theorem 4.7 and Proposition 4.8. □

This completes the proof of the theorem. □

Remark. Using the result of this paper, Conjecture 1.3 has been proved in [Zh3].

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