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A_1 BOUNDS FOR CALDERÓN-ZYGMUND OPERATORS RELATED TO A PROBLEM OF MUCKENHOUPT AND WHEEDEN

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Abstract. We obtain an $L^p(w)$ bound for Calderón-Zygmund operators T when $w \in A_1$. This bound is sharp both with respect to $\|w\|_{A_1}$ and with respect to p. As a result, we get a new $L^{1,\infty}(w)$ estimate for T related to a problem of Muckenhoupt and Wheeden.

1. Introduction

Let T be a Calderón-Zygmund singular integral operator. It was conjectured by B. Muckenhoupt and R. Wheeden [9] many years ago that T satisfies

$$\|Tf\|_{L^{1,\infty}(w)} \le c\|f\|_{L^1(Mw)}, \tag{1.1}$$

where w is a weight (i.e., $w \ge 0$ and $w \in L^1_{\text{loc}}(\mathbb{R}^n)$) and M is the Hardy-Littlewood maximal operator. Observe that (1.1) with T replaced by M is well-known; it was proved by C. Fefferman and E.M. Stein [6] in 1971.

Recall that $w \in A_1$ if there exists $c > 0$ such that $Mw(x) \le cw(x)$ a.e.; the smallest possible c here is denoted by $\|w\|_{A_1}$. Clearly, (1.1) implies

$$\|Tf\|_{L^{1,\infty}(w)} \le c\|w\|_{A_1}\|f\|_{L^1(w)}. \tag{1.2}$$

We call (1.2) the weak Muckenhoupt-Wheeden conjecture.

Both conjectures (1.1) and (1.2) are known to be true for $w_\delta(x) = |x|^{-n(1-\delta)}, 0 < \delta < 1$, see [1]. However, to our best knowledge, they are still open, in general, even for the Hilbert transform.

In a recent paper [8], the following results towards (1.2) have been obtained: if $\nu_p = \frac{p^2}{p-1}\log\left(e + \frac{1}{p-1}\right)$ and $\varphi(t) = t(1+\log^+ t)(1+\log^+\log^+ t)$, then

$$\|Tf\|_{L^p(w)} \le c\nu_p\|w\|_{A_1}\|f\|_{L^p(w)} \quad (1 < p < \infty) \tag{1.3}$$

and

$$\|Tf\|_{L^{1,\infty}(w)} \le c\varphi(\|w\|_{A_1})\|f\|_{L^1(w)}. \tag{1.4}$$

Inequality (1.3) in the case $p = 2$ for classical convolution singular integrals was proved previously by R. Fefferman and J. Pipher [7] by means of different ideas. A

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general observation from [7] shows that (1.3) is sharp with respect to $\|w\|_{A_1}$ for any $p>1$. On the other hand, it was not clear whether (1.3) is sharp with respect to p for p close to 1, in general. For example, it is well-known that in the unweighted case (i.e., when $w\equiv 1$) $\|T\|_{L^p}\le c\,pp'$, where, as usual, $1/p'+1/p=1$, and this estimate is sharp. We also remark that the behavior of ν_p in (1.3) for p close to 1 was used in deducing (1.4).

In this paper we obtain the best possible behavior of ν_p in (1.3) and, as a consequence, an improvement of φ in (1.4). Our main result is the following.

Theorem 1.1. *Let T be a Calderón-Zygmund operator. Then*

$$\|Tf\|_{L^p(w)}\le c\,pp'\,\|w\|_{A_1}\|f\|_{L^p(w)}\quad(1<p<\infty) \tag{1.5}$$

and

$$\|Tf\|_{L^{1,\infty}(w)}\le c\|w\|_{A_1}(1+\log\|w\|_{A_1})\|f\|_{L^1(w)}, \tag{1.6}$$

where $c=c(n,T)$.

The proof of Theorem 1.1 is based on several ingredients. Some of them are exactly the same as in the proof of (1.3) and (1.4). Here we mention the key new ingredient leading to Theorem 1.1. This is the following lemma.

Lemma 1.2. *Let T be a Calderón-Zygmund operator. There exists a constant $c=c(n,T)$ such that for any weight w and for any $p,r\ge 1$,*

$$\left\|\frac{Tf}{M_rw}\right\|_{L^p(M_rw)}\le cp\left\|\frac{Mf}{M_rw}\right\|_{L^p(M_rw)}, \tag{1.7}$$

where $M_rw=M(w^r)^{1/r}$.

It is well-known that the weight $(M_rw)^{1-p}$ belongs to the A_∞ class with the corresponding constants independent of w. Hence, (1.7) is a particular case of the Coifman-type estimate (see [2, 3]). The standard proofs applied to this concrete weight yield constants of exponential type $C(p)\sim\ 2^p$. In [8], the growth of $C(p)$ at infinity was improved to $C(p)\sim p\log p$. Lemma 1.2 represents the subsequent improvement to the best possible growth $C(p)\sim p$. This can be seen by taking $w\equiv 1$ and recalling that $\|M\|_{L^p}\approx c_n$ as $p\to\infty$.

An extrapolation argument yields an interesting consequence for the A_p class of weights, $1<p<\infty$, that follows from (1.6). Recall that a weight w is said to belong to the class $A_p, 1<p<\infty$, if

$$\|w\|_{A_p}\equiv\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty.$$

Corollary 1.3. *Let $1<p<\infty$ and let T be a Calderón-Zygmund operator. Also let $w\in A_p$, then*

$$\|Tf\|_{L^{p,\infty}(w)}\le c\|w\|_{A_p}(1+\log\|w\|_{A_p})\|f\|_{L^p(w)}, \tag{1.8}$$

where $c=c(n,p,T)$.

It is a difficult open problem whether a Calderón-Zygmund operator T satisfies the following sharp inequality with respect to $\|w\|_{A_p}$:

$$\|Tf\|_{L^p(w)} \le c\|w\|_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}} \|f\|_{L^p(w)} \quad (1 < p < \infty). \tag{1.9}$$

Observe that it is enough to prove (1.9) only for $p = 2$; then it follows for any $p > 1$ by the extrapolation theorem of Rubio de Francia with sharp constant as can be found in [5]. In recent works by S. Petermichl and A. Volberg [11, 12, 13] inequality (1.9) has been proved for Beurling, Hilbert or any one of the Riesz transforms. It is clear that for these operators (1.9) is stronger than (1.8) for $p \ge 2$. However, we emphasize that (1.8) holds for *any* Calderón-Zygmund operator. Also, to our best knowledge, (1.8) for $1 \le p < 2$ is new even for the Hilbert transform.

By a duality argument, Corollary 1.3 implies the following.

Corollary 1.4. *Let $1 < p < \infty$ and let T be a Calderón-Zygmund operator. If $w \in A_p$, then for any measurable set E,*

$$\|T(\sigma\chi_E)\|_{L^p(w)} \le c\|w\|_{A_p}^{\frac{1}{p-1}} (1 + \log \|w\|_{A_p})\sigma(E)^{1/p}, \tag{1.10}$$

where $\sigma = w^{-1/(p-1)}$.

Inequality (1.10) can be regarded as a Sawyer-type condition (cf. [14]). Although (1.9) is sharp with respect to $\|w\|_{A_p}$, (1.10) shows however that for test functions of the form $f = \sigma\chi_E$ a much better dependence in terms of $\|w\|_{A_p}$ can be obtained for $p > 2$.

The paper is organized as follows. In the next section, we give a detailed proof of Lemma 1.2 along with some auxiliary statements. In the third section we outline briefly the main steps from [8] showing how this lemma leads to Theorem 1.1. In Section 4 we prove Corollaries 1.3 and 1.4.

2. Proof of Lemma 1.2

By a Calderón-Zygmund operator we mean a continuous linear operator $T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ that extends to a bounded operator on $L^2(\mathbb{R}^n)$, and whose distributional kernel K coincides away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ with a function K satisfying the size estimate

$$|K(x, y)| \le \frac{c}{|x - y|^n}$$

and the regularity condition: for some $\varepsilon > 0$,

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \le c\frac{|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}},$$

whenever $2|x - z| < |x - y|$, and so that

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy,$$

whenever $f \in C_0^\infty(\mathbb{R}^n)$ and $x \notin \mathrm{supp}(f)$.

Set $A_\infty = \cup_{p\geq 1} A_p$. The class A_∞ can be defined in several equivalent ways, see [3]. In particular, $w \in A_\infty$ if and only if there exist constants $0 < \alpha, \beta < 1$ such that for any cube Q and any measurable subset $E \subset Q$,

$$\frac{|E|}{|Q|} < \alpha \Rightarrow \frac{w(E)}{w(Q)} < \beta.$$

We shall use several well-known facts about the A_p weights. First, it follows from Hölder's inequality that if $w_1, w_2 \in A_1$, then $w = w_1 w_2^{1-p} \in A_p$, and

$$\|w\|_{A_p} \leq \|w_1\|_{A_1} \|w_2\|_{A_1}^{p-1} \tag{2.1}$$

Second, if $0 < \delta < 1$, then $(Mf)^\delta \in A_1$ (see [4]), and

$$\|(Mf)^\delta\|_{A_1} \leq \frac{c_n}{1-\delta}. \tag{2.2}$$

The proof of Lemma 1.2 will be based on two Lemmas. The first one is the following.

Lemma 2.1. *Let T be a Calderón-Zygmund operator and let $w \in A_p$, $p \geq 1$. Then, there is a constant c depending on n, p and T such that*

$$\|Tf\|_{L^1(w)} \leq c\|w\|_{A_p} \|Mf\|_{L^1(w)}. \tag{2.3}$$

Remark 2.2. This estimate for $w \in A_\infty$ with some constant on the right-hand side depending on w is due to R.R. Coifman [2] (see also [3]). However, the standard proofs of (2.3) do not yield the linear dependence with respect to $\|w\|_{A_p}$.

Proof of Lemma 2.1. The lemma is just a combination of several known results. The first one is the sharp good-λ inequality proved by S. Buckley [1]:

$$|\{x \in Q : T^*f > 2\alpha, Mf < \gamma\alpha\}| \leq c_1 e^{-c_2/\gamma} |Q|, \tag{2.4}$$

where T^* is the maximal singular integral operator, Q is any cube in the Whitney decomposition of $\{T^*f > \alpha\}$, and c_1, c_2 depend only on T and n. The second one is the following sharp A_∞ property of A_p weights due to R. Fefferman and J. Pipher [7] (see Lemma 3.6 along with the subsequent remark on page 359): there is a constant c_3 depending on p and n such that for any cube Q and any subset $E \subset Q$,

$$\frac{|E|}{|Q|} < e^{-c_3\|w\|_{A_p}} \quad \text{implies} \quad \frac{w(E)}{w(Q)} < \frac{1}{100}. \tag{2.5}$$

Setting now in (2.4) $\gamma = \frac{c'}{\|w\|_{A_p}}$, where c' depends on c_1, c_2 and c_3, and using (2.5), we get

$$w\{x : T^*f > 2\alpha, Mf < c'\alpha/\|w\|_{A_p}\} \leq \frac{1}{100} w\{T^*f > \alpha\},$$

which easily gives (2.3). □

The second lemma is based on an application of Rubio the Francia's algorithm to produce special weights with appropriate properties.

Lemma 2.3. *Let $1 < s < \infty$, and let v be a weight. Then there exists a nonnegative sublinear operator R satisfying the following properties:*

(i) $h \leq R(h)$;

(ii) $\|R(h)\|_{L^s(v)} \leq 2\|h\|_{L^s(v)}$;

(iii) $R(h)v^{1/s} \in A_1$ *with*

$$\|R(h)v^{1/s}\|_{A_1} \leq cs'.$$

Proof. We consider first the operator

$$S(f) = \frac{M(f\, v^{1/s})}{v^{1/s}}$$

Since $\|M\|_{L^s} \sim s'$, we have

$$\|S(f)\|_{L^s(v)} \leq cs'\|f\|_{L^s(v)}.$$

Now, define the Rubio de Francia operator R by

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{(\|S\|_{L^s(v)})^k}.$$

In the standard way one can check that R satisfies the properties (i), (ii) and (iii). □

We are now ready to give the proof of the main Lemma.

Proof of Lemma 1.2. By duality we have,

$$\left\| \frac{Tf}{M_r w} \right\|_{L^p(M_r w)} = \sup_{\|h\|_{L^{p'}(M_r w)}=1} \int_{\mathbb{R}^n} |Tf| h\, dx. \tag{2.6}$$

Next, by Lemma 2.3 with $s = p'$ and $v = M_r w$, there exists an operator R such that

(i) $h \leq R(h)$;
(ii) $\|R(h)\|_{L^{p'}(M_r w)} \leq 2\|h\|_{L^{p'}(M_r w)}$;
(iii) $\|R(h)(M_r w)^{1/p'}\|_{A_1} \leq cp$.

Using property (iii) along with inequalities (2.1) and (2.2), we obtain

$$\begin{aligned} \|R(h)\|_{A_3} &= \|R(h)(M_r w)^{1/p'}\big((M_r w)^{1/2p'}\big)^{-2}\|_{A_3} \\ &\leq \|R(h)(M_r w)^{1/p'}\|_{A_1}\|(M_r w)^{1/2p'}\|_{A_1}^2 \\ &\leq cp. \end{aligned}$$

Therefore, by Lemma 2.1 and by properties (i) and (ii),

$$\begin{aligned} \int_{\mathbb{R}^n} |Tf| h\, dx &\leq \int_{\mathbb{R}^n} |Tf| R(h)\, dx \leq c\|R(h)\|_{A_3} \int_{\mathbb{R}^n} M(f)R(h)\, dx \\ &\leq cp \left\| \frac{Mf}{M_r w} \right\|_{L^p(M_r w)} \|h\|_{L^{p'}(M_r w)}, \end{aligned}$$

which along with (2.6) completes the proof. □

3. Proof of Theorem 1.1

As we mentioned in the Introduction, the proof of Theorem 1.1 follows the same lines as the corresponding proof of inequalities (1.3) and (1.4) in [8] (of course, taking into account this time Lemma 1.2). Hence, we just outline briefly the main ideas used in the proof.

First, using the duality argument and some standard estimates for the maximal operator, one can show that Lemma 1.2 implies

$$\|Tf\|_{L^p(w)} \le c\, pp'\Big(\frac{1}{r-1}\Big)^{1-1/pr}\|f\|_{L^p(M_r w)}, \tag{3.1}$$

where $1<r<2, p>1$ and $c=c(n,T)$.

Setting $r=r_w=1+\frac{1}{2^{n+1}\|w\|_{A_1}}$ in (3.1) and using that

$$M_{r_w}w(x)\le 2\,\|w\|_{A_1}\,w(x) \ \text{ a.e.} \tag{3.2}$$

(see [8, Lemma 3.1]), we obtain easily (1.5).

In order to prove (1.6), we follow the proof of Theorem 1.6 in [10]. By the classical Calderón-Zygmund decomposition, we have a family of pairwise disjoint cubes $\{Q_j\}$ such that $\lambda<|f|_{Q_j}\le 2^n\lambda$. Let $\Omega=\cup_j Q_j$, and $\widetilde{\Omega}=\cup_j 2Q_j$. Next, let $f=g+b$, where $g=\sum_j f_{Q_j}\chi_{Q_j}(x)+f(x)\chi_{\Omega^c}(x)$. Then

$$\begin{aligned}&w\{x\in\mathbb{R}^n:|Tf(x)|>\lambda\}\le w(\widetilde{\Omega})+w\{x\in(\widetilde{\Omega})^c:|Tb(x)|>\lambda/2\}\\&+w\{x\in(\widetilde{\Omega})^c:|Tg(x)|>\lambda/2\}\equiv I+II+III.\end{aligned}$$

The first two terms are bounded by $\frac{c\|w\|_{A_1}}{\lambda}\|f\|_{L^1(w)}$ (see [10, p. 303]). Next, by Chebyshev's inequality and (3.1),

$$III\le c(pp')^p\Big(\frac{1}{r-1}\Big)^{p-1/r}\frac{1}{\lambda}\int_{\mathbb{R}^n}|g|\,M_r(w\chi_{(\widetilde{\Omega})^c})dx.$$

Using the argument from [10, p. 303], we obtain

$$\int_{\mathbb{R}^n}|g|M_r(w\chi_{(\widetilde{\Omega})^c})dx\le c\int_{\mathbb{R}^n}|f|M_r w dx.$$

Combining two previous estimates with (3.2) and setting $r=1+1/2^{n+1}\|w\|_{A_1}$, we get

$$III\le\frac{c(pp'\|w\|_{A_1})^p}{\lambda}\int_{\mathbb{R}^n}|f|wdx.$$

Setting here $p=1+\frac{1}{\log(1+\|w\|_{A_1})}$ gives

$$III\le\frac{c\|w\|_{A_1}(1+\log\|w\|_{A_1})}{\lambda}\int_{\mathbb{R}^n}|f|wdx.$$

Combining this with estimates for I and II completes the proof.

4. Proof of Corollary 1.3

We shall need the following lemma proved in [5].

Lemma 4.1. *Let $1 < q < \infty$ and let $w \in A_q$. Then there exists a nonnegative sublinear operator D bounded on $L^{q'}(w)$ such that for any nonnegative $h \in L^{q'}(w)$:*
(a) $h \le D(h)$;
(b) $\|D(h)\|_{L^{q'}(w)} \le 2\,\|h\|_{L^{q'}(w)}$;
(c) $D(h)\cdot w \in A_1$ *with*

$$\|D(h)\cdot w\|_{A_1} \le c\, q\, \|w\|_{A_q},$$

where the constant c depends on n.

Proof of Corollary 1.3. For $\alpha > 0$ we set $\Omega_\alpha = \{|Tf| > \alpha\}$ and let $\varphi(t) = t(1+\log t)$.

Applying Lemma 4.1 with $q = p$, we get a sublinear operator D satisfying properties (a), (b) and (c). Using these properties and inequality (1.6), we obtain

$$\begin{aligned}\int_{\Omega_\alpha} h\, w dx \quad &\le \quad \int_{\Omega_t} D(h)\, w\, dx \le \frac{c}{\alpha}\varphi(\|D(h)\, w\|_{A_1})\|f\|_{L^1(D(h)w)}\\ &\le \quad \frac{c}{t}\,\varphi(\|w\|_{A_p})\|f\|_{L^p(w)}\|h\|_{L^{p'}(w)}.\end{aligned}$$

Taking the supremum over all h with $\|h\|_{L^{p'}(w)} = 1$ completes the proof. □

Proof of Corollary 1.4. Applying (1.8) and using that $\|\sigma\|_{A_{p'}} = \|w\|_{A_p}^{\frac{1}{p-1}}$, we get

$$\|T^* f\|_{L^{p',\infty}(\sigma)} \le c\|w\|_{A_p}^{\frac{1}{p-1}}(1+\log\|w\|_{A_p})\|f\|_{L^{p'}(\sigma)},$$

where T^* is the adjoint operator. From this, by duality we obtain

$$\|Tf\|_{L^p(w)} \le c\|w\|_{A_p}^{\frac{1}{p-1}}(1+\log\|w\|_{A_p})\|f/\sigma\|_{L^{p,1}(\sigma)},$$

where $L^{p,1}(\sigma)$ is the standard weighted Lorentz space. Setting here $f = \sigma\chi_E$, where E is any measurable set, completes the proof. □

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