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LOGARITHMIC COMBINATORIAL DIFFERENTIALS

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Abstract. Given a morphism $X \to S$ of fine log schemes, we develop a geometric description of the sheaves of higher-order differentials $\Omega^n_{X/S}$ for $n > 1$, as well as a definition of the de Rham complex in terms of this description.

Introduction

Given a smooth morphism $X \to S$ of schemes, it is standard to define $\Omega^1_{X/S} := I/I^2$, where I is the ideal sheaf of the diagonal in $X \times_S X$. One normally then defines $\Omega^q_{X/S} := \bigwedge^q \Omega^1_{X/S}$ for $q > 1$. On the other hand, in [1], Breen and Messing give an alternate definition of $\Omega^q_{X/S}$ extending the geometric definition of $\Omega^1_{X/S}$. This paper was inspired by similar definitions introduced by A. Kock in his study of synthetic differential geometry [3], which in turn was an attempt to transpose the methods in algebraic geometry, due to Grothendieck and others, of studying the concept of infinitesimally close points to the setting of C^∞-manifolds.

For simplicity, let us assume that 2 is invertible on S. Let $\Delta^n_{X/S} := X \times_S X \times_S \cdots \times_S X$ be the $n+1$-fold product, with the factors indexed from 0 to n. For $0 \le i, j \le n$, let I_{ij} be ideal of $\mathscr{O}_{\Delta^n_{X/S}}$ defining the partial diagonal $\{(x_0, \dots, x_n) \in \Delta^n_{X/S} : x_i = x_j\}$. Now let $\Delta^{(n)}_{X/S}$ denote the closed subscheme of $\Delta^n_{X/S}$ defined by $\sum_{0 \le i,j \le n} I^2_{ij}$, and $\tilde{I}_{ij}$ the image of I_{ij} in $\mathscr{O}_{\Delta^{(n)}_{X/S}}$. Then

$$\prod_{i=1}^n \tilde{I}_{i-1,i} = \bigcap_{i=1}^n \tilde{I}_{i-1,i} = \prod_{i=1}^n \tilde{I}_{0i} = \bigcap_{i=1}^n \tilde{I}_{0i} = \bigcap_{0 \le i,j \le n} \tilde{I}_{ij},$$

and this common ideal, considered as an $\mathscr{O}_X$-module via any of the $n+1$ projections $\Delta^{(n)}_{X/S} \to X$, is canonically isomorphic to $\Omega^n_{X/S}$. (In the general case, this construction instead gives the nth antisymmetric power of $\Omega^1_{X/S}$.)

Our first observation is that in the general case, we can fix this discrepancy by starting with the divided power envelope $D(n)$ of the diagonal in $\Delta^n_{X/S}$. In other words, if we let $\Delta^{[n]}_{X/S}$ be the closed subscheme of $D(n)$ defined by $\sum_{0 \le i,j \le n} \bar{I}^{[2]}_{ij}$, and $\tilde{I}_{ij}$ the image of $\bar{I}_{ij}$ in $\mathscr{O}_{\Delta^{[n]}_{X/S}}$, then the five ideals above are once again equal, and are canonically isomorphic to $\Omega^n_{X/S}$. (In [1], Breen and Messing corrected the discrepancy by expanding $\sum_{0 \le i,j \le n} I^2_{ij}$ in a non-symmetric way.)

Log geometry provides a convenient language for discussing topics related to compactification and singularities. Recall that a pre-log scheme X is a scheme X equipped

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with a sheaf of commutative monoids $\mathscr{M}_X$ and a morphism $\alpha_X : \mathscr{M}_X \to \mathscr{O}_X^\times$, where $\mathscr{O}_X^\times$ is the multiplicative monoid of $\mathscr{O}_X$. (Note that we use additive notation for $\mathscr{M}_X$, thus considering $m \in \mathscr{M}_X$ to be a logarithm of $\alpha(m)$, and considering α to be an exponentation map.) This is a log scheme if the induced morphism $\alpha_X^{-1}(\mathscr{O}_X^*) \to \mathscr{O}_X^*$ is an isomorphism. A log scheme is called *fine* if locally the log structure is induced by a pre-log structure $P \to \mathscr{O}_X^\times$ where P is the constant sheaf of a finitely-generated integral monoid. Given a morphism $X \to S$ of log scheme, Kato [2] defines a universal sheaf of relative log differentials $\Omega^1_{X/S}$ with a log derivation $(d, d\log) : (\mathscr{O}_X, \mathscr{M}_X) \to \Omega^1_{X/S}$. This means that $d : \mathscr{O}_X \to \Omega^1_{X/S}$ is an $\mathscr{O}_S$-derivation, $d\log : \mathscr{M}_X \to \Omega^1_{X/S}$ is an additive map annihilating the image of $\mathscr{M}_S$, and for $m \in \mathscr{M}_X$, we have

$$d\alpha(m) = \alpha(m)\, d\log m.$$

For example, suppose X is a smooth scheme over a field k, and D is a divisor with normal crossings on X. Let $Y := X \setminus D$, with open immersion $i : Y \to X$. We then define $\mathscr{M}_X := i_* \mathscr{O}_Y^* \cap \mathscr{O}_X$, with α_X the natural inclusion map. This defines a log scheme, and the sheaf of log differentials $\Omega^1_{X/k}$ is exactly the classical sheaf $\Omega^1_{X/k}(\log D)$ of differentials with log poles along D.

Our aim in this paper is to extend Breen and Messing's theory to give an intrinsic geometric description of $\wedge^n \Omega^1_{X/S}$ for $n > 1$ in the case of log schemes. Thus, consider a morphism $X \to S$ of fine log schemes. (Note that we do not require this morphism to be log smooth.) Again, let $\Delta^n_{X/S} := X \times_S \cdots \times_S X$ be the $n+1$-fold product. Then there exists a right universal log scheme $D(n)$ with an *exact* closed immersion $X \to D(n)$ defined by a PD ideal on $D(n)$, and a morphism $D(n) \to \Delta^n_{X/S}$, factoring the diagonal morphism $X \to \Delta^n_{X/S}$ [2]. Again, let $\Delta^{[n]}_{X/S}$ be the closed subscheme of $D(n)$ defined by the ideal $\sum_{0 \le i,j \le n} \bar{I}_{ij}^{[2]}$, where $\bar{I}_{ij}$ is the ideal of the partial diagonal $\{x_i = x_j\}$ in $\Delta(n)$, and $\tilde{I}_{ij}$ the image of $\bar{I}_{ij}$ in $\mathscr{O}_{\Delta^{[n]}_{X/S}}$. Then we will prove that in this more general case, once again the five ideals above are equal and are canonically isomorphic to $\Omega^n_{X/S}$. The proof we give here is an improvement on the proof given in [1].

In terms of this description, the de Rham complex becomes particularly simple, in the form of an Alexander-Spaniel complex. First, for $m, n \ge 0$, consider $\Delta^m_{X/S}$ as a scheme over X via the last projection, and $\Delta^n_{X/S}$ as a scheme over X via the first projection. Then we have a morphism

$$\begin{aligned} \Delta^{m+n}_{X/S} &\to \Delta^m_{X/S} \times_X \Delta^n_{X/S}, \\ (x_0, \ldots, x_m, \ldots, x_{m+n}) &\mapsto ((x_0, \ldots, x_m), (x_m, \ldots, x_{m+n})). \end{aligned}$$

This induces a map $\Delta^{[m+n]}_{X/S} \to \Delta^{[m]}_{X/S} \times_X \Delta^{[n]}_{X/S}$, which in turn induces the wedge product. Similarly, given $n \ge 0$ and $0 \le i \le n+1$, define $d_i : \Delta^{n+1}_{X/S} \to \Delta^n_{X/S}$ to be the map which forgets the ith component. This induces maps $d_i : \Delta^{[n+1]}_{X/S} \to \Delta^{[n]}_{X/S}$, and the differential $d : \Omega^n_{X/S} \to \Omega^{n+1}_{X/S}$ is induced by

$$d_0^* - d_1^* + \cdots + (-1)^{n+1} d_{n+1}^* : \mathscr{O}_{\Delta^{[n]}_{X/S}} \to \mathscr{O}_{\Delta^{[n+1]}_{X/S}}.$$

Finally, suppose $X \to S$ is log smooth. We observe that each $\Delta^{[n]}_{X/S}$ is an object in the log crystalline site of X over S, and each d_i is a morphism in this site. Therefore, given a crystal E on this site, which corresponds to a module with quasi-nilpotent connection (E_X, ∇), we have transition maps $\theta_{d_i} : E_{\Delta^{[n]}_{X/S}} \to E_{\Delta^{[n+1]}_{X/S}}$. Here $E_{\Delta^{[n]}_{X/S}} \simeq E_X \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n]}_{X/S}}$ via the isomorphism $\theta_{\pi_0} : \pi_0^* E_X \to E_{\Delta^{[n]}_{X/S}}$. We will show that the differential

$$\nabla : E_X \otimes_{\mathscr{O}_X} \Omega^n_{X/S} \to E_X \otimes_{\mathscr{O}_X} \Omega^{n+1}_{X/S}$$

in the de Rham complex of (E_X, ∇) is induced by

$$\theta_{d_0} - \theta_{d_1} + \cdots + (-1)^{n+1}\theta_{d_{n+1}} : E_X \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n]}_{X/S}} \to E_X \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n+1]}_{X/S}}.$$

1. Combinatorial Differentials

1.1. Local Construction. We begin in this section with a simplified situation: suppose A is a ring, B an A-algebra, and $Q \to P$ a morphism of finitely-generated integral monoids with compatible maps $Q \to A$ and $P \to B$. Let $S := \operatorname{Spec} A$, $X := \operatorname{Spec} B$, with the log structures induced by Q and P, respectively. Now let $\Delta^n_{X/S} := X \times_S X \times_S \cdots \times_S X$ be the $n+1$-fold product, with the factors indexed from 0 to n. In other words, $\Delta^n_{X/S} = \operatorname{Spec}(B_n)$, where $B_n := B \otimes_A \cdots \otimes_A B$, with log structure induced by P_n, the quotient of $P \oplus P \oplus \cdots \oplus P$ by the congruence generated by

$$(q, 0, \cdots, 0) \equiv (0, q, \cdots, 0) \equiv \cdots \equiv (0, 0, \cdots, q)$$

for $q \in Q$. Then P_n^{gp} is the quotient of $P^{\mathrm{gp}} \oplus P^{\mathrm{gp}} \oplus \cdots \oplus P^{\mathrm{gp}}$ by $\{(q_0, q_1, \ldots, q_n) \in Q^{\mathrm{gp}} \oplus \cdots \oplus Q^{\mathrm{gp}} : q_0 + \cdots + q_n = 0\}$.

Now the diagonal map $X \to \Delta^n_{X/S}$ corresponds to the product map $B_n \to B$, and it has a chart given by the sum map $P_n \to P$. Now let

$$P'_n := \{(p_0, p_1, \ldots, p_n) \in P_n^{\mathrm{gp}} : p_0 + p_1 + \ldots + p_n \in P\},$$

$B'_n := B_n \otimes_{\mathbb{Z}[P_n]} \mathbb{Z}[P'_n]$, and $Z_n := \operatorname{Spec} B'_n$ with the log structure induced by P'_n. (For $p \in P$, we will use the notation e^p for the corresponding element of $\mathbb{Z}[P]$, in order to avoid confusion between addition in P and addition in $\mathbb{Z}[P]$.) Then the map $X \to Z_n$ corresponding to the sum map $P'_n \to P$ is an exact closed immersion, and the map $Z_n \to \Delta^n_{X/S}$ corresponding to the inclusion $P_n \hookrightarrow P'_n$ is log étale. Therefore, Z_n may be used as the basis for constructing the log infinitesimal neighborhoods and the divided power envelope of X in $\Delta^n_{X/S}$ [2]. (Recall that a map $g : Q \to P$ of integral monoids is *exact* if $(g^{\mathrm{gp}})^{-1}(P) = Q$, and a morphism $X \to S$ of fine log schemes is exact if for every point $x \in X$ with image $s \in S$, $\mathscr{M}_{S,s} \to \mathscr{M}_{X,x}$ is exact. A log closed immersion $f : X \to Y$ is exact if and only if it is strict, i.e. $f^*\mathscr{M}_Y \to \mathscr{M}_X$ is an isomorphism, where $f^*\mathscr{M}_Y$ is the log structure induced by $f^{-1}\mathscr{M}_Y$.)

For notation, let $\pi_i^* : P \to P'_n$ be the ith inclusion map, corresponding to the ith projection $\pi_i : Z_n \to X$. Now for each pair i, j with $0 \le i, j \le n$, we have a closed immersion $m_{ij} : Z_{n-1} \to Z_n$ corresponding to the map $\mu_{ij} : B'_n \to B'_{n-1}$,

$$(y_0 \otimes \cdots \otimes y_i \otimes \cdots \otimes y_j \otimes \cdots \otimes y_n) \otimes e^{(p_0, \ldots, p_i, \ldots, p_j, \ldots, p_n)} \mapsto$$
$$(y_0 \otimes \cdots \otimes y_i y_j \otimes \cdots \otimes \hat{y}_j \otimes \cdots \otimes y_n) \otimes e^{(p_0, \ldots, p_i + p_j, \ldots, \hat{p}_j, \ldots, p_n)}.$$

Let $I_{ij} \subseteq \mathscr{O}_{Z_n}$ be the ideal sheaf defining this closed immersion, and $\Delta^{(n)}_{X/S}$ the closed subscheme of Z_n defined by $\sum_{0\le i,j\le n} I_{ij}^2$. It is easy to see that I_{ij} is generated by elements of the form

$$\delta^{i,j}p := 1 \otimes (e^{\pi_j^* p - \pi_i^* p} - 1) \in B'_n$$

for $p \in P^{\mathrm{gp}}$ and

$$d^{i,j}y := (\pi_j^* y - \pi_i^* y) \otimes 1$$

for $y \in B$. Let $\tilde{I}_{ij}$ be the image of I_{ij} in $\mathscr{O}_{\Delta^{(n)}_{X/S}}$.

We first note the following for future reference:

Lemma 1.1. *Let $0 \le i, j, k, \ell \le n$.*

(1) *Assume $i < j$ and $k < \ell$. Then $\mu_{ij}(I_{k\ell}) = 0$ if $i = k$ and $j = \ell$; otherwise, $\mu_{ij}(I_{k\ell}) = I_{k'\ell'}$, where*

$$k' = \begin{cases} k, & k < j; \\ i, & k = j; \\ k-1, & k > j, \end{cases}$$

and similarly for ℓ'. Hence μ_{ij} gives a well-defined map $\mathscr{O}_{\Delta^{(n)}_{X/S}} \to \mathscr{O}_{\Delta^{(n-1)}_{X/S}}$, and the same is true with $\tilde{I}_{k\ell}$ and $\tilde{I}_{k'\ell'}$ in place of $I_{k\ell}$ and $I_{k'\ell'}$.

(2) *$I_{i\ell} \subseteq I_{ij} + I_{j\ell}$.*

Proof. The first statement follows from the fact that μ_{ij} acts the same on the generators $d^{k,\ell}y$ and $\delta^{k,\ell}p$ of $I_{k\ell}$.

For the second statement, note that $d^{i,\ell}y = d^{i,j}y + d^{j,\ell}y$ for $y \in B$. Similarly, since $1 + \delta^{i,j}p = 1 \otimes e^{\pi_j^* p - \pi_i^* p}$, for $p \in P^{\mathrm{gp}}$ we have

$$1 + \delta^{i,\ell}p = (1 + \delta^{i,j}p)(1 + \delta^{j,\ell}p).$$

Therefore, $\delta^{i,\ell}p = \delta^{i,j}p + \delta^{j,\ell}p + (\delta^{i,j}p)(\delta^{j,\ell}p) \in J_{ij} + J_{j\ell}$ also. $\square$

Let $\Omega^{(n)}_{X/S}$ be the nth antisymmetric product of $\Omega^1_{X/S}$. We first define a map $\bigcap_{i=1}^n \tilde{I}_{0i} \to \Omega^{(n)}_{X/S}$.

Proposition 1.2. *There exists a unique A-linear map $\Psi_n : B'_n \to \Gamma(X, \Omega^{(n)}_{X/S})$ such that for $y_0, \ldots, y_n \in B$, $(p_0, \ldots, p_n) \in P'_n$, we have*

$$\Psi_n[(y_0 \otimes \cdots \otimes y_n) \otimes e^{(p_0,\ldots,p_n)}] = \\ y_0 \alpha(p_0 + p_1 + \cdots + p_n)(dy_1 + y_1\, d\log p_1) \tilde{\wedge} \cdots \tilde{\wedge} (dy_n + y_n\, d\log p_n).$$

(Here $\tilde{\wedge}$ denotes the product in the antisymmetric product algebra $\Omega^{(\cdot)}_{X/S}$.)

Proof. The uniqueness is clear. To see the map is well-defined, we have several things to check:

- The above expression is A-multilinear in the variables $y_0, y_1, \ldots, y_n$.
  This is clear from the A-linearity of d.
- The expression above is independent of the choice of $p_0, \ldots, p_n \in P^{\mathrm{gp}}$.
  This follows from the fact that $p_0 + \cdots + p_n \in P$ is well-defined, and the fact that $d\log$ induces a well-defined map $P^{\mathrm{gp}}/Q^{\mathrm{gp}} \to \Omega^1_{X/S}$.

- For $p' \in P$,
$$\Psi_n[(y_0 \otimes \cdots \otimes y_i\alpha(p') \otimes \cdots \otimes y_n) \otimes e^{(p_0,\ldots,p_i,\ldots,p_n)}] =$$
$$\Psi_n[(y_0 \otimes \cdots \otimes y_i \otimes \cdots \otimes y_n) \otimes e^{(p_0,\ldots,p_i+p',\ldots,p_n)}].$$
For $i = 0$, this is clear. Otherwise, for $i > 0$, this follows from the formula
$$d(y_i\alpha(p')) = y_i d(\alpha(p')) + \alpha(p')dy_i = \alpha(p')[dy_i + y_i\, d\log p'].$$

□

Remark 1.3. In the case of trivial log structure, i.e. $P = 0$, the formula for Ψ_n reduces to
$$\Psi_n(y_0 \otimes y_1 \otimes \cdots \otimes y_n) = y_0\, dy_1 \tilde{\wedge} \cdots \tilde{\wedge}\, dy_n.$$
This is the isomorphism commonly used in synthetic differential geometry, for example in [3].

Also note that if in fact $p_0, \ldots, p_n \in P$, then
$$(y_0 \otimes y_1 \otimes \cdots \otimes y_n) \otimes e^{(p_0,p_1,\ldots,p_n)} = (y_0\alpha(p_0) \otimes y_1\alpha(p_1) \otimes \cdots \otimes y_n\alpha(p_n)) \otimes 1,$$
and in this case the formula for Ψ_n agrees with
$$y_0\alpha(p_0)\, d(y_1\alpha(p_1)) \tilde{\wedge} \cdots \tilde{\wedge}\, d(y_n\alpha(p_n)).$$
Thus we may view the given formula for Ψ_n as a natural generalization of the simpler formula from the case of trivial log structure.

Proposition 1.4. *The map Ψ_n annihilates I_{ij}^2 for each pair $0 \le i, j \le n$.*

Proof. We first check the case $i = 0$. In this case, since $d(y_j y) = y_j\, dy + y\, dy_j$, it is straightforward to calculate that for $x \in B'_n$, $y \in B$, we have
$$\Psi_n(x\, d^{0,j}y) = (-1)^{j-1} dy \,\tilde{\wedge}\, \Psi_{n-1}(\mu_{0,j}x).$$
Therefore, if $x \in I_{0j}$, then $\Psi_n(x\, d^{0,j}y) = 0$. Similarly, for $p \in P^{\mathrm{gp}}$,
$$\Psi_n(x\, \delta^{0,j}p) = (-1)^{j-1}\, d\log p \,\tilde{\wedge}\, \Psi_{n-1}(\mu_{0,j}x),$$
so again if $x \in I_{0j}$, then $\Psi_n(x\, \delta^{0,j}p) = 0$.

Now for the general case, by symmetry assume $i < j$. We observe that $d^{i,j}y = d^{0,j}y - d^{0,i}y$. Thus, if $y, y' \in B$, then
$$(d^{i,j}y)(d^{i,j}y') \equiv -[(d^{0,i}y)(d^{0,j}y') + (d^{0,i}y')(d^{0,j}y)] \pmod{J_{0i}^2 + J_{0j}^2}. \tag{1.1}$$
However, since $\mu_{0j}(x(d^{0,i}y')) = (\mu_{0j}x)(d^{0,i}y)$ for $x \in B'_n$, we have
$$\begin{aligned}\Psi_n(x(d^{0,i}y)(d^{0,j}y')) &= (-1)^{j-1} dy' \,\tilde{\wedge}\, \Psi_{n-1}((\mu_{0j}x)(d^{0,i}y)) \\ &= (-1)^{i+j} dy' \,\tilde{\wedge}\, dy \,\tilde{\wedge}\, \Psi_{n-2}(\mu_{0i}\mu_{0j}x).\end{aligned}$$
Therefore, for $x \in B'_n$,
$$\Psi_n(x(d^{i,j}y)(d^{i,j}y')) = (-1)^{i+j+1}(dy' \,\tilde{\wedge}\, dy + dy \,\tilde{\wedge}\, dy') \,\tilde{\wedge}\, \Psi_{n-2}(\mu_{0i}\mu_{0j}x) = 0.$$
Similarly, since $1 + \delta^{i,j}p = 1 \otimes e^{\pi_j^* p - \pi_i^* p}$, we have $1 + \delta^{0,j}p = (1 + \delta^{0,i}p)(1 + \delta^{i,j}p)$. Multiplying both sides by $1 - \delta^{0,i}p$, this implies
$$1 + \delta^{i,j}p \equiv (1 + \delta^{0,j}p)(1 - \delta^{0,i}p) \pmod{J_{0i}^2 + J_{0j}^2},$$

so $\delta^{i,j}p \equiv \delta^{0,j}p - \delta^{0,i}p - (\delta^{0,j}p)(\delta^{0,i}p)$. Therefore,

$$(\delta^{i,j}p)(d^{i,j}y) \equiv -[(\delta^{0,i}p)(d^{0,j}y) + (d^{0,i}y)(\delta^{0,j}p)] \pmod{J_{0i}^2 + J_{0j}^2} \tag{1.2}$$

and

$$(\delta^{i,j}p)(\delta^{i,j}p') \equiv -[(\delta^{0,i}p)(\delta^{0,j}p') + (\delta^{0,i}p')(\delta^{0,j}p)] \pmod{J_{0i}^2 + J_{0j}^2}. \tag{1.3}$$

From these formulas, the proof that Ψ annihilates $x(\delta^{i,j}p)(d^{i,j}y)$ and $x(\delta^{i,j}p)(\delta^{i,j}p')$ proceeds as before. □

Therefore, since $\Omega^{(n)}_{X/S}$ is a quasi-coherent $\mathscr{O}_X$-module, Ψ_n induces a map $\Psi_n : \mathscr{O}_{\Delta^{(n)}_{X/S}} \to \Omega^{(n)}_{X/S}$, which we will restrict to the ideal $\bigcap_{j=1}^n \tilde{I}_{0j}$ of $\mathscr{O}_{\Delta^{(n)}_{X/S}}$. We now turn to defining a map in the other direction.

Proposition 1.5. *For each i with $0 < i \leq n$, there is a unique B-linear map $\phi_i : \Omega^1_{X/S} \to \tilde{I}_{0i}$ such that $\phi_i(dy) = d^{0,i}y$ for $y \in B$ and $\phi_i(d\log p) = \delta^{0,i}p$ for $p \in P^{\mathrm{gp}}$. (Here we consider $\tilde{I}_{0i}$ to be a B-module via π_0^*.)*

Proof. By the universal property of $\Omega^1_{X/S}$, we need only check that $(D,\delta) : (B, P^{gp}) \to \tilde{I}_{0i}$ defined by $Dy = d^{0,i}y$ and $\delta p = \delta^{0,i}p$ is a log derivation over A. However, D is clearly A-linear, and since

$$(d^{0,i}y)(d^{0,i}y') = d^{0,i}(yy') - (\pi_0^*y)d^{0,i}y' - (\pi_0^*y')d^{0,i}y \in I_{0i}^2,$$

D is also a derivation. Similarly, for $p \in P$, we have $d^{0,i}(\alpha(p)) = (\pi_0^*\alpha(p))\delta^{0,i}p$. Finally, to see that δ is additive, since $1 + \delta^{0,i}p = 1 \otimes e^{\pi_i^*p - \pi_0^*p}$, we have

$$1 + \delta^{0,i}(p+p') = (1+\delta^{0,i}p)(1+\delta^{0,i}p') = 1 + \delta^{0,i}p + \delta^{0,i}p' + (\delta^{0,i}p)(\delta^{0,i}p').$$

Therefore, $\delta^{0,i}(p+p') \equiv \delta^{0,i}p + \delta^{0,i}p' \pmod{J_{0i}^2}$. □

Proposition 1.6. *There is a unique map $\Phi_n : \Omega^{(n)}_{X/S} \to \prod_{j=1}^n \tilde{J}_{0j}$ such that for $\omega_1, \omega_2, \ldots, \omega_n \in \Omega^1_{X/S}$,*

$$\Phi_n(\omega_1 \mathbin{\tilde{\wedge}} \omega_2 \mathbin{\tilde{\wedge}} \cdots \mathbin{\tilde{\wedge}} \omega_n) = \phi_1(\omega_1)\phi_2(\omega_2)\cdots\phi_n(\omega_n).$$

Proof. Since the formula above is clearly multilinear in $\omega_1, \ldots, \omega_n$, we need only check it is antisymmetric. We claim that in fact, for $\omega, \tau \in \Omega^1_{X/S}$, $\phi_i(\omega)\phi_j(\tau) + \phi_i(\tau)\phi_j(\omega) = 0$ in $\mathscr{O}_{\Delta^{(n)}_{X/S}}$. To see this, we refer again to the formulas (1.1) through (1.3). Thus, if $\omega = dy$ and $\tau = dy'$, then by (1.1),

$$(d^{0,i}y)(d^{0,j}y') + (d^{0,i}y')(d^{0,j}y) \equiv -(d^{i,j}y)(d^{i,j}y') \pmod{J_{0i}^2 + J_{0j}^2},$$

so $\phi_i(\omega)\phi_j(\tau) + \phi_i(\tau)\phi_j(\omega) \in J_{0i}^2 + J_{0j}^2 + J_{ij}^2$. Similarly, for the cases $\omega = d\log p$, $\tau = dy$ and $\omega = d\log p$, $\tau = d\log p'$, we use the corresponding formulas (1.2) and (1.3). □

We now show the two maps defined above are inverses.

Theorem 1.7. (1) *The composition*

$$\Omega^{(n)}_{X/S} \xrightarrow{\Phi_n} \prod_{j=1}^{n} \tilde{J}_{0j} \xrightarrow{\Psi_n} \Omega^{(n)}_{X/S}$$

is the identity on $\Omega^{(n)}_{X/S}$.

(2) *The composition*

$$\bigcap_{j=1}^{n} \tilde{J}_{0j} \xrightarrow{\Psi_n} \Omega^{(n)}_{X/S} \xrightarrow{\Phi_n} \prod_{j=1}^{n} \tilde{J}_{0j} \hookrightarrow \bigcap_{j=1}^{n} \tilde{J}_{0j}$$

is the identity map on $\bigcap_{j=1}^{n} \tilde{J}_{0j}$.

Proof. For the first composition, it suffices to check for

$$\omega = dy_1 \tilde{\wedge} \cdots \tilde{\wedge} dy_i \tilde{\wedge} d\log p_{i+1} \tilde{\wedge} \cdots \tilde{\wedge} d\log p_n,$$

for $y_1, \ldots, y_i \in B$, $p_{i+1}, \ldots, p_n \in P^{\mathrm{gp}}$. However,

$$\Phi_n(\omega) = \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} (y_{0S} \otimes y_{1S} \otimes \cdots \otimes y_{iS} \otimes 1 \otimes \cdots \otimes 1) \otimes \\ e^{(p_{0S}, 0, \ldots, 0, p_{i+1,S}, \ldots, p_{nS})},$$

where:

- $y_{0S} = \prod_{1 \le j \le i, j \in S} y_j$;
- $y_{jS} = y_j$ if $j \notin S$ and $y_{jS} = 1$ if $j \in S$, $1 \le j \le i$;
- $p_{0S} = -\sum_{i < j \le n, j \notin S} p_j$;
- $p_{jS} = p_j$ if $j \notin S$ and $p_{jS} = 0$ if $j \in S$, $i < j \le n$.

Therefore,

$$\Psi_n(\Phi_n(\omega)) = \sum_S y_{0S}\, dy_{1S} \tilde{\wedge} \cdots \tilde{\wedge} dy_{iS} \tilde{\wedge} d\log p_{i+1,S} \tilde{\wedge} \cdots \tilde{\wedge} d\log p_{nS}.$$

However, if $S \neq \emptyset$, then either $y_{jS} = 1$ or $p_{jS} = 0$ for some $j \in S$, so the corresponding term is zero. On the other hand, for $S = \emptyset$, the corresponding term is exactly ω.

For the second composition, let $x = (y_0 \otimes \cdots \otimes y_n) \otimes e^{(p_0, \ldots, p_n)} \in B'_n$. Then since $\pi_i^* y_i \equiv \pi_0^* y_i \pmod{J_{0i}}$, we calculate that

$$\begin{aligned} \phi_i(dy_i + y_i\, d\log p_i) &\equiv \pi_i^* y_i \otimes 1 - \pi_0^* y_i \otimes 1 + \pi_i^* y_i \otimes (e^{\pi_i^* p_i - \pi_0^* p_i} - 1) \\ &= \pi_i^* y_i \otimes e^{\pi_i^* p_i - \pi_0^* p_i} - \pi_0^* y_i \otimes 1 \pmod{J_{0i}^2}. \end{aligned}$$

From this we see that

$$\Phi_n \circ \Psi_n = \prod_{j=1}^{n} (\mathrm{id} - M_j)$$

on $\mathscr{O}_{\Delta^{(n)}_{X/S}}$, where

$$M_j[(y_0 \otimes y_1 \otimes \cdots \otimes y_j \otimes \cdots \otimes y_n) \otimes e^{(p_0, p_1, \ldots, p_j, \ldots, p_n)}] = \\ (y_0 y_j \otimes y_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes y_n) \otimes e^{(p_0 + p_j, p_1, \ldots, 0, \ldots, p_n)}.$$

However, M_j factors through μ_{0j}, so this implies that $\Phi_n \circ \Psi_n = \mathrm{id}$ on $\bigcap_{j=1}^{n} \tilde{I}_{0j}$. $\square$

Corollary 1.8. *We have*

$$\bigcap_{j=1}^{n} \tilde{J}_{0j} = \prod_{j=1}^{n} \tilde{J}_{0j} = \bigcap_{0\le i,j\le n} \tilde{J}_{ij} \simeq \Omega^{(n)}_{X/S}.$$

Proof. From the theorem, the inclusion map $\prod_{j=1}^n \tilde{J}_{0j} \hookrightarrow \bigcap_{j=1}^n \tilde{J}_{0j}$ must be surjective, so it is the identity map and the two ideals are equal. Thus Φ_n and Ψ_n are inverse isomorphisms between this common ideal and $\Omega^{(n)}_{X/S}$. Now clearly, $\bigcap_{0\le i,j\le n} \tilde{J}_{ij} \subseteq \bigcap_{j=1}^n \tilde{J}_{0j}$. On the other hand, if $0 \le i, j \le n$, then $\tilde{J}_{0j} \subseteq \tilde{J}_{0i} + \tilde{J}_{ij}$, so $\tilde{J}_{0i}\tilde{J}_{0j} \subseteq \tilde{J}_{ij}$. Therefore, $\prod_{j=1}^n \tilde{J}_{0j} \subseteq \bigcap_{0\le i,j\le n} \tilde{J}_{ij}$ also. □

We now present another formulation of this ideal which is more useful in certain situations. First, we note that

$$\begin{aligned}
&\tilde{J}_{01} = \tilde{J}_{01};\\
&\tilde{J}_{02} \subseteq \tilde{J}_{01} + \tilde{J}_{12};\\
&\tilde{J}_{03} \subseteq \tilde{J}_{01} + \tilde{J}_{12} + \tilde{J}_{23};\\
&\quad\vdots\\
&\tilde{J}_{0n} \subseteq \tilde{J}_{01} + \tilde{J}_{12} + \cdots + \tilde{J}_{n-1,n}.
\end{aligned}$$

Therefore, $\prod_{j=1}^n \tilde{J}_{0j} \subseteq \prod_{j=1}^n \tilde{J}_{j-1,j}$. Similarly, since $\tilde{J}_{j-1,j} \subseteq \tilde{J}_{0,j-1} + \tilde{J}_{0,j}$, the reverse inclusion also holds, and $\prod_{j=1}^n \tilde{J}_{0j} = \prod_{j=1}^n \tilde{J}_{j-1,j}$.

In fact, since $d^{0,j}y \equiv d^{j-1,j}y \pmod{J_{0,j-1}}$ and $\delta^{0,j}p \equiv \delta^{j-1,j}p \pmod{J_{0,j-1}}$, while $J_{0,j-1} \subseteq J_{01} + J_{12} + \cdots + J_{j-2,j-1}$, we see that

$$\Phi_n(\omega_1 \tilde{\wedge} \cdots \tilde{\wedge}\, \omega_n) = \psi_1(\omega_1) \cdots \psi_n(\omega_n),$$

where $\psi_i(dy) = d^{i-1,i}y$ and $\psi_i(d\log p) = \delta^{i-1,i}p$. From this we may calculate that

$$\Phi_n \circ \Psi_n = (\mathrm{id} - M_n') \circ \cdots \circ (\mathrm{id} - M_2') \circ (\mathrm{id} - M_1').$$

Here

$$\begin{aligned}
&M_j'[(y_0 \otimes \cdots \otimes y_{j-1} \otimes y_j \otimes \cdots \otimes y_n) \otimes e^{(p_0,\ldots,p_{j-1},p_j,\ldots,p_n)}]\\
&\quad=(y_0 \otimes \cdots \otimes y_{j-1}y_j \otimes 1 \otimes \cdots \otimes y_n) \otimes e^{(p_0,\ldots,p_{j-1}+p_j,0,\ldots,p_n)}.
\end{aligned}$$

However, M_j' factors through $\mu_{j-1,j}$, hence $\Phi_n \circ \Psi_n = \mathrm{id}$ on $\bigcap_{j=1}^n \tilde{J}_{j-1,j}$. From this we conclude that

$$\bigcap_{j=1}^{n} \tilde{J}_{j-1,j} = \prod_{j=1}^{n} \tilde{J}_{j-1,j} = \bigcap_{j=1}^{n} \tilde{J}_{0j} = \prod_{j=1}^{n} \tilde{J}_{0j} = \bigcap_{0\le i,j\le n} \tilde{J}_{ij}.$$

1.2. Globalization. Now let $X \to S$ be an arbitrary morphism of fine log schemes, and let $\Delta^n_{X/S} := X \times_S X \times_S \cdots \times_S X$ be the $n+1$-fold product. Let $\tilde{\Delta}^n_{X/S}$ be the log formal neighborhood of the diagonal immersion $\Delta : X \to \Delta^n_{X/S}$. Then for $0 \le i < j \le n$, we have a closed immersion $m_{ij} : \Delta^{n-1}_{X/S} \to \Delta^n_{X/S}$ defined by

$$\begin{aligned}
&m_{ij}(x_0, \ldots, x_i, \ldots, x_{j-1}, x_j, \ldots, x_{n-1}) =\\
&\qquad(x_0, \ldots, x_i, \ldots, x_{j-1}, x_i, x_j, \ldots, x_{n-1}).
\end{aligned}$$

This induces a closed immersion $\tilde{\Delta}^{n-1}_{X/S} \to \tilde{\Delta}^n_{X/S}$. Let J_{ij} be the ideal of $\mathscr{O}_{\tilde{\Delta}^n_{X/S}}$ defining this closed immersion, and let $\Delta^{(n)}_{X/S}$ be the closed subscheme of $\tilde{\Delta}^n_{X/S}$ defined by $\sum_{0\le i<j\le n} J_{ij}^2$. Finally, let $\tilde{J}_{ij}$ be the image of J_{ij} in $\mathscr{O}_{\Delta^{(n)}_{X/S}}$.

Theorem 1.9. *We have*

$$\bigcap_{j=1}^n \tilde{J}_{j-1,j} = \prod_{j=1}^n \tilde{J}_{j-1,j} = \bigcap_{j=1}^n \tilde{J}_{0j} = \prod_{j=1}^n \tilde{J}_{0j} = \bigcap_{0\le i,j\le n} \tilde{J}_{ij},$$

and this ideal is canonically isomorphic to $\Omega^{(n)}_{X/S}$.

Proof. Since the construction above is local with respect to both X and S, in proving the first statement we may assume that X and S are affine and that we have a chart $(P \to \mathscr{O}_X, Q \to \mathscr{O}_S, Q \to P)$ of the morphism $X \to S$. Then note that in the construction of the previous section, the ideal J of the diagonal immersion $X \to \Delta^n_{X/S}$ satisfies

$$J \subseteq J_{01} + J_{12} + \cdots + J_{n-1,n}.$$

Therefore, $J^{n+1} \subseteq \sum_{0\le i,j\le n} J_{ij}^2$, and in fact we could have used the nth log infinitesimal neighborhood of the diagonal in place of Z_n. The same holds true for the global construction above. Now it is easy to see that the global construction reduces to the local construction of the last section in this case. From this we immediately see the equality of the five ideals.

Now to establish an isomorphism between $\prod_j \tilde{J}_{0j}$ and $\Omega^{(n)}_{X/S}$, a similar proof to the proof of 1.5 shows that there are unique maps $\phi_i : \Omega^1_{X/S} \to \tilde{J}_{0i}$ such that $\phi_i(dy) = \pi_i^* y - \pi_0^* y$ for $y \in \mathscr{O}_X$, and $\phi_i(d\log m) = \alpha(\pi_i^* m - \pi_0^* m) - 1$ for $m \in \mathscr{M}_X^{\mathrm{gp}}$. (Here, since $X \to \tilde{\Delta}^n_{X/S}$ is exact and $\pi_i^* m - \pi_0^* m$ pulls back to 0 in $\mathscr{M}_X^{\mathrm{gp}}$, we must have $\pi_i^* m - \pi_0^* m \in \mathscr{M}_{\tilde{\Delta}^n_{X/S}}$.) Therefore, there is a unique map $\Phi_n : \Omega^{(n)}_{X/S} \to \prod_{j=1}^n \tilde{J}_{0j}$ such that

$$\Phi_n(\omega_1 \tilde{\wedge} \cdots \tilde{\wedge}\, \omega_n) = \phi_1(\omega_1) \cdots \phi_n(\omega_n)$$

for $\omega_1, \dots, \omega_n \in \Omega^1_{X/S}$. (The map exists locally by the previous section, and the uniqueness allows us to glue the local maps.) By the previous section, Φ_n is locally an isomorphism, so Φ_n gives a global isomorphism $\Omega^{(n)}_{X/S} \xrightarrow{\sim} \prod_{j=1}^n \tilde{J}_{0j}$. □

1.3. The Divided Power Envelope. In this section, let $D(n)$ denote the log PD envelope of the diagonal in $\Delta^n_{X/S}$. As before we get closed immersions $m_{ij} : D(n-1) \to D(n)$. Let $\bar{J}_{ij} \subseteq \mathscr{O}_{D(n)}$ be the PD ideal corresponding to m_{ij}, $\Delta^{[n]}_{X/S}$ be the closed subscheme of $D(n)$ defined by $\sum_{i,j} \bar{J}_{ij}^{[2]}$, and $\tilde{J}_{ij}$ be the ideal of $\mathscr{O}_{\Delta^{[n]}_{X/S}}$ corresponding to $\bar{J}_{ij}$.

Theorem 1.10.

$$\bigcap_{j=1}^n \tilde{J}_{0j} = \prod_{j=1}^n \tilde{J}_{0j} = \bigcap_{j=1}^n \tilde{J}_{j-1,j} = \prod_{j=1}^n \tilde{J}_{j-1,j} = \bigcap_{0\le i,j\le n} \tilde{J}_{ij},$$

and this ideal is canonically isomorphic to $\Omega^n_{X/S}$.

Proof. First, observe that by the universal property of the PD envelope, $D(n) \simeq D(1) \times_X \cdots \times_X D(1)$, the product of n factors of $D(1)$, each considered as a scheme over X via the projection π_0. Therefore, $\Delta^{[1]}_{X/S} \times_X \cdots \times_X \Delta^{[1]}_{X/S}$ is isomorphic to the closed subscheme of $D(n)$ corresponding to $\sum_j \bar{J}^{[2]}_{0j}$. Also, it is easy to see that $\Delta^{[1]}_{X/S} \simeq \Delta^{(1)}_{X/S}$.

Therefore, to see that the previous map Ψ_n induces a map $\mathscr{O}_{\Delta^{[n]}_{X/S}} \to \Omega^n_{X/S}$, it suffices to check that for $1 \le i, j \le n$, $\Psi_n[x \cdot (d^{i,j} y)^{[2]}] = \Psi_n[x \cdot (\delta^{i,j} p)^{[2]}] = 0$ for $x \in \mathscr{O}_{D(n)}$, $y \in \mathscr{O}_X$, $p \in \mathscr{M}^{\mathrm{gp}}_X$. However, since $d^{i,j} y = d^{0,j} y - d^{0,i} y$, we have

$$\begin{aligned}(d^{i,j} y)^{[2]} &= (d^{0,j} y)^{[2]} + (d^{0,i} y)^{[2]} - (d^{0,i} y)(d^{0,j} y) \\ &\equiv -(d^{0,i} y)(d^{0,j} y) \pmod{\bar{J}^{[2]}_{0i} + \bar{J}^{[2]}_{0j}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi_n(x \cdot (d^{i,j} y)^{[2]}) &= -\Psi_n(x(d^{0,i} y)(d^{0,j} y)) \\ &= (-1)^{i+j} dy \wedge dy \wedge \Psi_{n-2}(\mu_{0i}\mu_{0j} x) = 0.\end{aligned}$$

Similarly,

$$(\delta^{i,j} p)^{[2]} \equiv -(\delta^{0,i} p)(\delta^{0,j} p) \pmod{\bar{J}^{[2]}_{0i} + \bar{J}^{[2]}_{0j}}.$$

Therefore,

$$\begin{aligned}\Psi_n(x \cdot (\delta^{i,j} p)^{[2]}) &= -\Psi_n(x(\delta^{0,i} p)(\delta^{0,j} p)) \\ &= (-1)^{i+j}\, d\log p \wedge d\log p \wedge \Psi_{n-2}(\mu_{0i}\mu_{0j} x) = 0.\end{aligned}$$

Similarly, we already know $\Phi_n : \Omega^{(n)}_{X/S} \to \prod_{j=1}^n \tilde{J}_{0j}$ is antisymmetric. Therefore, to check Φ_n induces a map $\Omega^n_{X/S} \to \prod_{j=1}^n \tilde{J}_{0j}$, it suffices to check that it annihilates $dy \mathbin{\tilde{\wedge}} dy \mathbin{\tilde{\wedge}} \omega$ and $d\log p \mathbin{\tilde{\wedge}} d\log p \mathbin{\tilde{\wedge}} \omega$. However, from the above, we see that in fact

$$(d^{0,1} y)(d^{0,2} y) \equiv -(d^{1,2} y)^{[2]} \pmod{\bar{J}^{[2]}_{01} + \bar{J}^{[2]}_{02}},$$

so $\phi_1(dy)\phi_2(dy) \in \bar{J}^{[2]}_{01} + \bar{J}^{[2]}_{02} + \bar{J}^{[2]}_{12}$, and this is zero in $\mathscr{O}_{\Delta^{[n]}_{X/S}}$. The proof that $\phi_1(d\log p)\phi_2(d\log p) = 0$ in $\mathscr{O}_{\Delta^{[n]}_{X/S}}$ is similar.

Now since Ψ_n and Φ_n were induced from inverse maps, they are inverse isomorphisms, and the equality of the ideals follows as before. □

2. The de Rham Complex

We now describe the de Rham complex $\Omega^{\cdot}_{X/S}$ in terms of our characterization of $\Omega^n_{X/S}$. We begin with the wedge product: thus, let $m, n > 0$. Then we have a map

$$\begin{aligned}\Delta^{m+n}_{X/S} &\to \Delta^m_{X/S} \times_X \Delta^n_{X/S}, \\ (x_0, \ldots, x_m, \ldots, x_{m+n}) &\mapsto ((x_0, \ldots, x_m), (x_m, \ldots, x_{m+n})).\end{aligned}$$

Here we consider $\Delta^m_{X/S}$ as a scheme over X via the last projection π_m, and $\Delta^n_{X/S}$ as a scheme over X via the first projection π_0. This induces a map $D(m+n) \to D(m) \times_X D(n)$, which in turn induces a map

$$s_{mn} : \Delta^{[m+n]}_{X/S} \to \Delta^{[m]}_{X/S} \times_X \Delta^{[n]}_{X/S}.$$

Locally, this map is also induced by the "smashing" map

$$\begin{aligned} B'_m \otimes_B B'_n \to & B'_{m+n}, \\ [(y_0 \otimes \cdots \otimes y_m) \otimes e^{(p_0,\ldots,p_m)}] \otimes [(y'_0 \otimes \cdots \otimes y'_n) \otimes e^{(p'_0,\ldots,p'_n)}] \mapsto & \\ (y_0 \otimes \cdots \otimes y_m y'_0 \otimes \cdots \otimes y'_n) \otimes e^{(p_0,\ldots,p_m+p'_0,\ldots,p'_n)}. & \end{aligned}$$

Remark 2.1. Although we also have a map $D(m) \times_X D(n) \to D(m+n)$, we do not get an induced map $\Delta^{[m]}_{X/S} \times_X \Delta^{[n]}_{X/S} \to \Delta^{[m+n]}_{X/S}$ in general. For example, the pullback of $\bar{J}^{[2]}_{0,m+n}$ does not correspond to anything from $\sum_{0\le i,j\le m} \bar{J}^{[2]}_{ij}$ or $\sum_{0\le i,j\le n} \bar{J}^{[2]}_{ij}$.

Proposition 2.2. *We have a commutative diagram*

$$\begin{array}{ccc} \bigcap_{i=1}^m \tilde{J}_{i-1,i} \otimes_{\mathscr{O}_X} \bigcap_{j=1}^n \tilde{J}_{j-1,j} & \xrightarrow{s^*_{mn}} & \bigcap_{j=1}^{m+n} \tilde{J}_{j-1,j} \\ {\scriptstyle \Psi_m\otimes\Psi_n}\downarrow \simeq & & {\scriptstyle \Psi_{m+n}}\downarrow \simeq \\ \Omega^m_{X/S} \otimes_{\mathscr{O}_X} \Omega^n_{X/S} & \xrightarrow{\wedge} & \Omega^{m+n}_{X/S}. \end{array}$$

Proof. Our first task is to verify that s^*_{mn} actually induces a map as in the top row. To see this, note that

$$s_{mn} \circ m_{j-1,j} = \begin{cases} (m_{j-1,j}, \mathrm{id}) \circ s_{m-1,n}, & j \le m; \\ (\mathrm{id}, m_{j-m-1,j-m}) \circ s_{m,n-1}, & j > m. \end{cases}$$

Therefore, converting to dual statements in terms of s^*_{mn} and $m^*_{j-1,j}$, we see that the image of s^*_{mn} is annihilated by each $m^*_{j-1,j}$ and is thus in each kernel $\tilde{J}_{j-1,j}$.

Now to check the commutativity, we first reverse the vertical arrows and replace them by $\Phi_m \otimes \Phi_n$ and Φ_{m+n}, respectively. Now, from the fact that

$$\Phi_n(\omega_1 \wedge \cdots \wedge \omega_n) = \psi_1(\omega_1) \cdots \psi_n(\omega_n),$$

where $\psi_j(dy) = d^{j-1,j} y$ and $\psi_j(d\log p) = \delta^{j-1,j} p$, the commutativity is clear. □

We now turn to the differential map in the de Rham complex; thus, fix $n \ge 0$. Then for $0 \le j \le n+1$, we have maps

$$\Delta^{n+1}_{X/S} \to \Delta^n_{X/S},\ (x_0, \ldots, x_j, \ldots, x_{n+1}) \mapsto (x_0, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

These induce maps $D(n+1) \to D(n)$, which in turn induce maps

$$d_j : \Delta^{[n+1]}_{X/S} \to \Delta^{[n]}_{X/S}.$$

Locally, these maps are also induced by the insertion maps

$$\begin{aligned} B'_n &\to B'_{n+1}, \\ (y_0 \otimes \cdots \otimes y_n) \otimes e^{(p_0,\ldots,p_n)} &\mapsto (y_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes y_n) \otimes e^{(p_0,\ldots,0,\ldots,p_n)}, \end{aligned}$$

with insertion in the jth position.

Proposition 2.3. *We have a commutative diagram*

$$\begin{array}{ccc} \bigcap_{j=1}^{n} \tilde{J}_{j-1,j} & \xrightarrow{d_0^* - d_1^* + \cdots + (-1)^{n+1} d_{n+1}^*} & \bigcap_{j=1}^{n+1} \tilde{J}_{j-1,j} \\ \Psi_n \downarrow \simeq & & \Psi_{n+1} \downarrow \simeq \\ \Omega^n_{X/S} & \xrightarrow{d} & \Omega^{n+1}_{X/S}. \end{array}$$

Proof. Let $e_n := d_0^* - d_1^* + \cdots + (-1)^{n+1} d_{n+1}^* : \mathscr{O}_{\Delta^{[n]}_{X/S}} \to \mathscr{O}_{\Delta^{[n+1]}_{X/S}}$. Again, we first need to check that e_n induces a map as in the top row. To see this, note that

$$d_j \circ m_{i-1,i} = \begin{cases} m_{i-2,i-1} \circ d_j, & j < i-1; \\ \mathrm{id}, & j = i-1 \text{ or } i; \\ m_{i-1,i} \circ d_{j-1}, & j > i. \end{cases}$$

From this, it is easy to check that the image of e_n is annihilated by each $m_{i-1,i}^*$.

Now it follows formally from the appropriate identities that $e_{n+1} \circ e_n = 0$, corresponding to the requirement that $d \circ d = 0$. Furthermore,

$$e_{m+n} \circ s_{mn}^* = s_{m+1,n}^* \circ (e_m \otimes \mathrm{id}) + (-1)^m s_{m,n+1}^* \circ (\mathrm{id} \otimes e_n),$$

which corresponds to the requirement that $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^m \omega \wedge d\tau$. It is easy to see that e_0 agrees with $d : \mathscr{O}_X \to \Omega^1_{X/S}$. Therefore, all that is left is to verify that $e_1(d \log p) = 0$ for $p \in \mathscr{M}_X^{\mathrm{gp}}$. We calculate locally, where $d \log p = 1 \otimes (e^{(-p,p)} - 1) \in B_1'$ for $p \in P^{\mathrm{gp}}$. Thus,

$$e_1(d \log p) = 1 \otimes e^{(0,-p,p)} - 1 \otimes e^{(-p,0,p)} + 1 \otimes e^{(-p,p,0)} - 1 \otimes 1.$$

Now by the definition of Ψ_2, the last three terms are annihilated, and the first gets mapped to $-d \log p \wedge d \log p = 0$. □

Remark 2.4. In the case of trivial log structure, we have an easier proof: we see that Ψ_{n+1} annihilates the image of d_j^* for $j > 0$, while locally,

$$\Psi_{n+1}[d_0^*(y_0 \otimes \cdots \otimes y_n)] = dy_0 \wedge \cdots \wedge dy_n = d[\Psi_n(y_0 \otimes \cdots \otimes y_n)].$$

However, to extend this proof to the case of log schemes, we must verify that

$$d[y_0 \alpha(p_0 + \cdots + p_n)(dy_1 + y_1 \, d \log p_1) \wedge \cdots \wedge (dy_n + y_n \, d \log p_n) = \alpha(p_0 + \cdots + p_n)(dy_0 + y_0 \, d \log p_0) \wedge \cdots \wedge (dy_n + y_n \, d \log p_n).$$

While this can be done, we prefer to give the more conceptual proof above.

Remark 2.5. By taking the corresponding maps on the antisymmetric powers $\Omega^{(\cdot)}_{X/S}$ of $\Omega^1_{X/S}$, we can define a natural complex. However, from the above calculations, we see that we get $d(d \log m) = d \log m \,\tilde{\wedge}\, d \log m$, instead of 0. This illustrates why in defining the logarithmic de Rham complex such that $d(d \log m) = 0$, we need the full alternating product instead of just the antisymmetric product. (This requirement appears in the need to check that $d^2 \alpha(m) = d(\alpha(m) \, d \log m) = 0$.)

3. Coefficients

In this section, we will assume that X is log smooth over S, and (E, ∇) is an $\mathscr{O}_X$-module with quasi-nilpotent integrable connection. Then this corresponds to a crystal E of $\mathscr{O}_{X/S}$-modules on the log crystalline site $(X/S)_{cris}$. Recall that an object of $(X/S)_{cris}$ is a tuple (U, T, i, δ) where U is an open subscheme of X, $i : U \to T$ is an *exact* log closed immersion, and δ is a PD structure on the ideal of i. (For convenience of notation, we often use T to represent this object.) Then a morphism $g : T_1 \to T_2$ in this site is a morphism respecting the closed immersions and the PD structures, and a covering $(U_\lambda, T_\lambda, i_\lambda, \delta_\lambda)_{\lambda \in \Lambda}$ of T is a family such that (T_λ) is a Zariski open covering of T. Giving a sheaf E on this site is then equivalent to giving a sheaf E_T on T for each object T of $(X/S)_{cris}$, along with transition maps $\theta_g : g^{-1}E_{T_2} \to E_{T_1}$ for each morphism $g : T_1 \to T_2$ in the site, satisfying the compatibility relation

$$\theta_{hg} = \theta_g \circ g^{-1}\theta_h$$

for the composition of $g : T_1 \to T_2$, $h : T_2 \to T_3$. We define $\mathscr{O}_{X/S}$ to be the sheaf with $(\mathscr{O}_{X/S})_T := \mathscr{O}_T$, and a sheaf E of $\mathscr{O}_{X/S}$-modules is a *crystal* if for each morphism $g : T_1 \to T_2$ in $(X/S)_{cris}$, the induced transition map $\theta_g : g^*E_{T_2} \to E_{T_1}$ is an isomorphism. For more details, see [2].

We note that by construction, each $\Delta^{[n]}_{X/S}$ is an object of $(X/S)_{cris}$, and each $d_j : \Delta^{[n+1]}_{X/S} \to \Delta^{[n]}_{X/S}$ is a morphism in this site. We thus get transition maps

$$\theta_{d_j} : d_j^* E_{\Delta^{[n]}_{X/S}} \xrightarrow{\sim} E_{\Delta^{[n+1]}_{X/S}}.$$

Here we will consider $E_{\Delta^{[n]}_{X/S}}$ as being identified with $E \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n]}_{X/S}}$ via $\theta_{\pi_0} : \pi_0^* E_X \xrightarrow{\sim} E_{\Delta^{[n]}_{X/S}}$.

Also, since the map

$$inc \circ \Phi_n : \Omega^n_{X/S} \xrightarrow{\sim} \bigcap_{j=1}^{n} \tilde{J}_{j-1,j} \hookrightarrow \mathscr{O}_{\Delta^{[n]}_{X/S}}$$

is a split injection (with splitting Ψ_n), so is the map $\mathrm{id} \otimes (inc \circ \Phi_n) : E \otimes_{\mathscr{O}_X} \Omega^n_{X/S} \to E \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n]}_{X/S}}$. Furthermore, we see that the image is equal to the intersection of the kernels of the transition maps $\theta_{m_{j-1,j}} : E_{\Delta^{[n]}_{X/S}} \to E_{\Delta^{[n+1]}_{X/S}}$. We will treat $\mathrm{id} \otimes \Phi_n$ as identifying $E \otimes_{\mathscr{O}_X} \Omega^n_{X/S}$ with this submodule of $E \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n]}_{X/S}}$.

We now give a characterization of the de Rham complex with coefficients in E, in terms of the transition maps θ_{d_j} and the above identifications.

Proposition 3.1. *We have a commutative diagram*

$$\begin{array}{ccccccc}
\xrightarrow{\nabla} & E \otimes_{\mathscr{O}_X} \Omega^n_{X/S} & & \xrightarrow{\nabla} & & E \otimes_{\mathscr{O}_X} \Omega^{n+1}_{X/S} & \xrightarrow{\nabla} \\
& \big\downarrow & & & & \big\downarrow & \\
\longrightarrow & E \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n]}_{X/S}} & & \longrightarrow & & E \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[n+1]}_{X/S}} & \longrightarrow \\
& {\scriptstyle \theta_{\pi_0}} \big\downarrow \simeq & & & & {\scriptstyle \theta_{\pi_0}} \big\downarrow \simeq & \\
\longrightarrow & E_{\Delta^{[n]}_{X/S}} & & \xrightarrow{\theta_{d_0} - \theta_{d_1} + \cdots + (-1)^{n+1}\theta_{d_{n+1}}} & & E_{\Delta^{[n+1]}_{X/S}} & \longrightarrow .
\end{array}$$

Proof. Let $e \in E$, $\omega \in \Omega^n_{X/S} \subseteq \mathscr{O}_{\Delta^{[n]}_{X/S}}$. If $j > 0$, then since the composition of $d_j : \Delta^{[n+1]}_{X/S} \to \Delta^{[n]}_{X/S}$ and $\pi_0 : \Delta^{[n]}_{X/S} \to X$ is equal to $\pi_0 : \Delta^{[n+1]}_{X/S} \to X$, we see that $\theta_{d_j}(e \otimes \omega) = e \otimes d_j^* \omega$. Therefore, all we need to do to finish the proof is to show that $\theta_{d_0}(e \otimes \omega) = e \otimes d_0^* \omega + \theta_{s_{1n}}(\nabla e \otimes \omega)$. Then since $\pi_0 \circ s_{1n} = \pi_0$, $\theta_{s_{1n}}(\nabla e \otimes \omega) = \nabla e \wedge \omega$, and it immediately follows that for $\omega \in \Omega^n_{X/S}$,

$$(\theta_{d_0} - \theta_{d_1} + \cdots \pm \theta_{d_{n+1}})(e \otimes \omega) = \nabla e \wedge \omega + e \otimes d\omega = \nabla(e \otimes \omega).$$

However, by definition,

$$\nabla = \theta_{\pi_1} - \theta_{\pi_0} : E \to E \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \hookrightarrow E \otimes_{\mathscr{O}_X} \mathscr{O}_{\Delta^{[1]}_{X/S}}.$$

Hence by the linearity of θ_{π_1}, $\theta_{\pi_1}(e \otimes \omega) = e \otimes \omega + (\nabla e)\omega$ for $e \in E, \omega \in \mathscr{O}_{\Delta^{[1]}_{X/S}}$. Now considering the map $(\pi_0, \pi_1) : \Delta^{[n]}_{X/S} \to \Delta^{[1]}_{X/S}$, we must have $\theta_{\pi_1} = \theta_{(\pi_0,\pi_1)} \circ (\pi_0, \pi_1)^* \theta_{\pi_1}$, so

$$\theta_{\pi_1}(e \otimes \omega) = e \otimes \omega + [\theta_{(\pi_0,\pi_1)}(\nabla e)]\omega$$

for $e \in E, \omega \in \mathscr{O}_{\Delta^{[n]}_{X/S}}$. But since $\pi_0 \circ d_0 = \pi_1$, we now get

$$\begin{aligned}\theta_{d_0}(e \otimes \omega) = \theta_{\pi_1}(e \otimes d_0^*\omega) &= e \otimes d_0^*\omega + [\theta_{(\pi_0,\pi_1)}(\nabla e)]d_0^*\omega \\ &= e \otimes d_0^*\omega + \theta_{s_{1n}}(\nabla e \otimes \omega).\end{aligned}$$

□

Remark 3.2. It is easy to see, independently of the above calculation, that $\sum_j (-1)^j \theta_{d_j}$ induces maps $E \otimes_{\mathscr{O}_X} \Omega^n_{X/S} \to E \otimes_{\mathscr{O}_X} \Omega^{n+1}_{X/S}$ forming the differential maps in a complex. We thus have an alternate proof that the standard formula $\nabla(e \otimes \omega) = \nabla e \wedge \omega + e \otimes d\omega$ gives a well-defined complex, and in fact we see in this way that this is a natural generalization of the usual de Rham complex $(\Omega^{\cdot}_{X/S}, d)$.

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